Chapter 2

Feedback Queueing Models with Intermittently Available Server

The classical literature on queues deals with queueing systems in which the service is provided instantaneously. However, there are queueing problems, like telephone services in which the operator (service channel) is not always instantaneously available i.e. there are interruptions in the process of taking a unit into service immediately after the previous service is completed. Even for reasons such as fatigue or untimely call by the boss, the service channel, if it is manually operated may not be instantaneously available. These interruptions in service do have definite influence on various parameters of a queueing system such as waiting line, waiting time, busy period of the system etc. Various models can be thought of that deal with queueing situations in which the service facility provides service with interruptions. White and Christie (1958) seem to be first to consider a single channel queueing system in which the service is interrupted at random intervals. Agarwal (1965) considers a queueing system in which the interruption in the service, although random, does not occur during the period of service but occurs only after the service in hand is completed. The service so provided has been named as intermittently available service. Later on Sharda (1968), Gaur (1973) and Jai Singh (1976) considered many queueing situations in which the server is intermittently available.

Further, in order to provide a better service facility, an arriving unit is given another chance of rejoining the system after getting the service, if required. Queues in which every unit after being served, either immediately joins the system again with certain probability or leaves the system permanently with certain probability are called queues with feedback. Finch (1959) introduced the concept of feedback explicitly through his paper ‘Cyclic Queues with Feedback’
where he considered that all the channels are arranged in series and a unit has a certain probability $p_j$ to return to the $j^{th}$ phase after the completion of its service at the last phase. In model A of this chapter, a queueing problem with intermittently available server is considered with the further provision that units are given another chance of service, if required.

In literature a good number of queueing problems have been solved, where the arrivals and departures occur singly. There are many queueing models wherein arrivals occur singly or in batches and departures occur in batches. Bailey (1954) and Jaiswal (1960a, 1960b) have successfully solved the queueing problems in case departures occur in batches of fixed size. But the assumption of service in batches of variable size is more realistic in various practical situations such as service of (i) passengers in buses and lifts (ii) spectators in cinema halls (iii) tourists by a guide (iv) clothes in a washing machine etc.

Sharda (1968) investigated a queueing problem with intermittently available server wherein arrivals and departures occur in batches of variable size. Goswami et al. (2006) analyzed discrete-time single-server infinite (finite) buffer bulk-service queues in which the inter-arrival time of successive arrivals and service times of batches are assumed to be independent and geometrically distributed.

Garg and Srivastava (2006) obtained transient solution for the exact number of arrivals & departures of an instantaneous service channel queueing system in which units are occurring singly and are served in batches of variable size with a further provision that a batch of units are given another chance of service, if required. Al-khedhairi and Tadj (2007) investigated the queueing process of a bulk service queueing system under Bernoulli schedule. The queueing process is studied both in discrete time and in continuous time. Indra and Vijay (2008) obtained transient solution of a two-state bulk arrival markovian queueing model with intermittently
available server. In model B of this chapter, a queueing problem with intermittently available server is considered in which units are occurring singly and are served in batches of variable size with a further provision that a batch of units are given another chance of service, if required.

**Model A**

In this model, a queueing problem with intermittently available server is considered with the further provision that units are given another chance of service, if required. Various practical situations correspond to this queueing model, for example (i) service of customers in a hair cutting saloon (ii) check up of patients in OPD of a private clinic (iii) service at an electric repair shop (iv) service of customers on query counter of certain organization etc.

Transient as well as steady state solutions and the distribution of the interval during which the server is busy are obtained. Some special cases of interest are also obtained. Some illustrations are discussed and their numerical solutions are obtained. Thereafter various probabilities are plotted and compared graphically.

The queueing system investigated in this model is described by the following assumptions:

(i) Arrivals are Poisson with parameter \( \lambda \).

(ii) When there are units in the queue, availability time of intermittent server is exponentially distributed with parameter \( v \). The server is available to an empty queue and its availability is also exponentially distributed with parameter \( v_0 \).

(iii) Service time is exponentially distributed with parameter \( \mu \).

(iv) Units are taken for service in their order of arrival.
(v) The probability of rejoining the system is $p$ and that of leaving the system is $q$ for the units getting service first time, so that $p + q = 1$. However the units will have to leave the system definitely after getting service for the second time.

(vi) The probability that the unit joins the service channel for the first time is assumed to be $c_1$ and that for the second time is $c_2$, so that $c_1 + c_2 = 1$.

(vii) The stochastic processes involved, viz

a. arrival of units

b. departure of units

c. availability of the server

are statistically independent.

Definitions

\[ P^{(k)}_{n,B}(t) = \text{Probability that there are } n \text{ units in the system at time } t \text{ and the next unit is to depart for the first time or second time according as } k = 0 \text{ or } 1 \text{ and the server is busy in relation to the queue, i.e. either a unit is being served or else one is to be taken just then.} \]

\[ P^{(k)}_{n,F}(t) = \text{Probability that there are } n \text{ units in the system at time } t \text{ and the next unit is to depart for the first time or second time according as } k = 0 \text{ or } 1 \text{ and the server is free in relation to the queue, i.e. neither a unit is being served nor is any to be taken at that instant.} \]

\[ P_n(t) = \text{Probability that there are } n \text{ units in the system at time } t. \]

\[ P_{n,B}(t) = P^{(0)}_{n,B}(t) + P^{(1)}_{n,B}(t), \quad n \geq 1; \quad Q_{0,B}(t) = Q^{(0)}_{0,B}(t). \]
The difference differential equations describing the system are

\[ \frac{d}{dt} P_{n,F}(t) = -(\lambda + \mu) P_{n,F}(t) + \lambda P_{n-1,F}(t) + v P_{n,F}(t) + \lambda Q_{0,F}(t) \delta_{n,1}, \quad n \geq 1 \]  

\[ \frac{d}{dt} P_{n,B}(t) = -(\lambda + \mu) P_{n,B}(t) + \lambda P_{n-1,B}(t) + v P_{n,F}(t) \]  

\[ \frac{d}{dt} Q_{0,B}(t) = -\lambda Q_{0,B}(t) + v P_{0,F}(t) \]  

\[ \frac{d}{dt} P_{n,F}(t) = -(\lambda + v) P_{n,F}(t) + \lambda P_{n-1,F}(t) + \mu c_1 \{ q P_{n+1,F}(t) + P_{n+1,B}(t) \} + \mu c_1 p P_{n,B}(t) \]  

\[ \frac{d}{dt} P_{n,F}(t) = -(\lambda + v) P_{n,F}(t) + \lambda P_{n-1,F}(t) + \mu c_2 \{ q P_{n+1,F}(t) + P_{n+1,B}(t) \} + \mu c_2 p P_{n,B}(t) \]  

\[ \frac{d}{dt} P_{0,F}(t) = -(\lambda + v_0) P_{0,F}(t) + \mu \{ q P_{1,F}(t) + P_{1,B}(t) \} \]  

where \( \delta_{n,1} = 1 \) for \( n = 1 \),

\[ = 0, \quad \text{otherwise.} \]

The steady state difference equations describing the systems are

\[ (\rho + 1) P_{n,B}^{(0)} = \rho P_{n-1,B}^{(0)} + \left( \frac{v}{\mu} \right) P_{n,F}^{(0)} + \rho Q_{0,B}^{(0)} \delta_{n,1}, \quad n \geq 1 \]  

\[ (\rho + 1) P_{n,B}^{(1)} = \rho P_{n-1,B}^{(1)} + \left( \frac{v}{\mu} \right) P_{n,F}^{(1)} \]
\[ \rho Q_{0,B}^{(0)} = \left( \frac{v_0}{\mu} \right) P_{0,F}^{(0)} \]  

(2A.9)

\[ (\rho + \frac{v}{\mu}) P_{n,F}^{(0)} = \rho P_{n-1,F}^{(0)} + c_1 (q P_{n+1,B}^{(0)} + P_{n+1,B}^{(1)}) + c_1 p P_{n,B}^{(0)} (1 - \delta_{n,1}) \quad , n \geq 1 \]

(2A.10)

\[ (\rho + \frac{v}{\mu}) P_{n,F}^{(1)} = \rho P_{n-1,F}^{(1)} + c_2 (q P_{n+1,B}^{(0)} + P_{n+1,B}^{(1)}) + (c_1 \delta_{n,1} + c_2) p P_{n,B}^{(0)} \quad , n \geq 1 \]

(2A.11)

\[ (\rho + \frac{v_0}{\mu}) P_{0,F}^{(0)} = q P_{1,B}^{(0)} + P_{1,B}^{(1)} \]

(2A.12)

where \( \rho = \frac{\lambda}{\mu} \)

Taking the Laplace Transformation \( \overline{P}_n(s) = \int_0^\infty e^{-st} P_n(t) dt \quad \text{Re} \quad s > 0 \)

of (2A.1)-(2A.6) and dividing both sides by \( \mu \)

\[ (\rho + 1 + \frac{s}{\mu}) \overline{P}_{n,B}^{(0)}(s) = \rho \overline{P}_{n-1,B}^{(0)}(s) + \left( \frac{v}{\mu} \right) \overline{P}_{n,F}^{(0)}(s) + \rho \overline{Q}_{0,B}^{(0)}(s) \delta_{n,1} \quad , n \geq 1 \]

(2A.13)

\[ (\rho + 1 + \frac{s}{\mu}) \overline{P}_{n,B}^{(1)}(s) = \rho \overline{P}_{n-1,B}^{(1)}(s) + \left( \frac{v}{\mu} \right) \overline{P}_{n,F}^{(1)}(s) \quad , n \geq 1 \]

(2A.14)

\[ (\rho + \frac{s}{\mu}) \overline{Q}_{0,B}^{(0)}(s) = \left( \frac{v_0}{\mu} \right) P_{0,F}^{(0)}(s) \]

(2A.15)

\[ (\rho + \frac{v}{\mu} + \frac{s}{\mu}) \overline{P}_{n,F}^{(0)}(s) = \rho \overline{P}_{n-1,F}^{(0)}(s) + c_1 \left\{ q \overline{P}_{n+1,B}^{(0)}(s) + \overline{P}_{n+1,B}^{(1)}(s) \right\} + c_1 p \overline{P}_{n,B}^{(0)}(1 - \delta_{n,1}) \quad , n \geq 1 \]

(2A.16)

\[ (\rho + \frac{v}{\mu} + \frac{s}{\mu}) \overline{P}_{n,F}^{(1)}(s) = \rho \overline{P}_{n-1,F}^{(1)}(s) + c_2 \left\{ q \overline{P}_{n+1,B}^{(0)}(s) + \overline{P}_{n+1,B}^{(1)}(s) \right\} + (c_1 \delta_{n,1} + c_2) p \overline{P}_{n,B}^{(0)}(s) \quad , n \geq 1 \]

(2A.17)

\[ (\rho + \frac{v_0}{\mu} + \frac{s}{\mu}) \overline{P}_{0,F}^{(0)}(s) = \frac{1}{\mu} + q \overline{P}_{1,B}^{(0)}(s) + \overline{P}_{1,B}^{(1)}(s) \]

(2A.18)
Definitions

\[ P_b^{(k)}(z,t) = \sum_{n=1}^{\infty} P_{n,B}^{(k)}(t)Z^n + Q_{0,B}^{(0)}(t) \] , \quad \[ P_f^{(k)}(z,t) = \sum_{n=0}^{\infty} P_{n,F}^{(k)}(t)Z^n \]

\[ P_b(z,t) = P_b^{(0)}(z,t) + P_b^{(1)}(z,t) \] , \quad \[ P_f(z,t) = P_f^{(0)}(z,t) + P_f^{(1)}(z,t) \]

\[ P(z,t) = P_b(z,t) + P_f(z,t) \]

and

\[ \overline{P}_b^{(k)}(z,s) = \int_0^\infty e^{-\mu t} P_b^{(k)}(z,t)dt \] , \quad \[ \overline{P}_f^{(k)}(z,s) = \int_0^\infty e^{-\mu t} P_f^{(k)}(z,t)dt \]

\[ \overline{P}(z,s) = \int_0^\infty e^{-\mu t} P(z,t)dt \] , \quad \text{all for } k = 0 \text{ or } 1 \text{ with } |z| \leq 1

Steady State Solution of the Problem

Using \( Ef(x) = f(x+1) \), equations (2A.10), (2A.11), (2A.7) and (2A.8) give (for \( n \geq 2 \))

\[ (c_1 q E^2 + c_1 p E) P_{n,B}^{(0)} + c_1 E^2 P_{n,B}^{(1)} - \left( \rho + \frac{v}{\mu} \right) E - \rho \right] P_{n,F}^{(0)} = 0 \] \hspace{1cm} (2A.19)

\[ (c_2 q E^2 + c_2 p E) P_{n,B}^{(0)} + c_2 E^2 P_{n,B}^{(1)} - \left( \rho + \frac{v}{\mu} \right) E - \rho \right] P_{n,F}^{(1)} = 0 \] \hspace{1cm} (2A.20)

\[ - \left( \rho + 1 \right) E - \rho \right] P_{n,B}^{(0)} + \left( \frac{v}{\mu} \right) E \right] P_{n,F}^{(0)} = 0 \] \hspace{1cm} (2A.21)

\[ - \left( \rho + 1 \right) E - \rho \right] P_{n,B}^{(1)} + \left( \frac{v}{\mu} \right) E \right] P_{n,F}^{(1)} = 0 \] \hspace{1cm} (2A.22)

To have a solution of above system of equations, we must have

\[ (E-1) \left( \rho + 1 \right) E - \rho \right] \left[ \rho + \frac{v}{\mu} \right] E - \rho \right] \left[ \frac{v}{\mu} \right] (1-c_1 p) E^2 - \rho \left( \rho + \left( \rho + \frac{v}{\mu} \right) E^2 \right] = 0 \] \hspace{1cm} (2A.23)

The values of \( P_{n,B}^{(0)} \), \( P_{n,B}^{(1)} \), \( P_{n,F}^{(0)} \) and \( P_{n,F}^{(1)} \) are given by
\[ P_{n,B}^{(0)} = \sum_{i=0}^{4} a_i z_i^n \quad \text{and} \quad P_{n,B}^{(1)} = \sum_{i=0}^{4} b_i z_i^n \]

\[ P_{n,F}^{(0)} = \sum_{i=0}^{4} c_i z_i^n \quad \text{and} \quad P_{n,F}^{(1)} = \sum_{i=0}^{4} d_i z_i^n \quad \text{for} \quad n \geq 2 \]

where \( a_i, b_i, c_i, d_i \) (\( i = 0, 1, 2, 3, 4 \)) are arbitrary constants to be evaluated and \( z_0, z_1, z_2, z_3 \) & \( z_4 \) are the roots of \((2A.23)\) with \( z_0 = 1; z_1 = \frac{\rho}{\left( \rho + \frac{v}{\mu} \right)}; z_2 = \frac{\rho}{\left( \rho + 1 \right)} \). The other two roots \( z_3 \) and \( z_4 \) depend on the parameters \( \lambda, \mu, v, c_1 \) & \( p \) and are evaluated from the equation

\[
\frac{v}{\mu} (1 - c_1 p) E^2 - p \left[ \rho + \left( 1 + \frac{v}{\mu} \right) \right] E + \rho^2 = 0 \]

after putting the values of \( v, \mu, \rho, c_1 \) and \( p \) in this quadratic equation. After evaluating \( z_3 \) and \( z_4 \), we find if any of these is \( \geq 1 \). In case \( z_i \geq 1 \), take \( a_i = b_i = c_i = d_i = 0 \) for \( i = 3, 4 \). The roots \( z_1 \) and \( z_2 \) are always \(< 1\) whatever the values of various parameters. To have convergence of the solution any root \( \geq 1 \) must be rejected. Thus rejecting \( z_0 \), we have

\[ P_{n,B}^{(0)} = \sum_{i=1}^{4} a_i z_i^n \quad \text{and} \quad P_{n,B}^{(1)} = \sum_{i=1}^{4} b_i z_i^n \]

\[ P_{n,F}^{(0)} = \sum_{i=1}^{4} c_i z_i^n \quad \text{and} \quad P_{n,F}^{(1)} = \sum_{i=1}^{4} d_i z_i^n \quad \text{for} \quad n \geq 2 \]

From equations \((2A.7), (2A.8), (2A.10) \& (2A.11)\) putting \( n = 2 \) and then putting \( n = 1 \) in \((2A.7)\), we get the probabilities \( P_{1,B}^{(0)}, P_{1,B}^{(1)}, P_{1,F}^{(0)}, P_{1,F}^{(1)} \) and \( Q_{0,B}^{(0)} \) respectively in terms of unknown constants \( a_i, b_i, c_i, d_i \) (\( i = 1, 2, 3, 4 \)) and probability \( P_{0,F}^{(0)} \). Substituting above values in equation \((2A.12)\), in equations \{\((2A.8), (2A.10) \& (2A.11)\) for \( n = 1 \}\) and in equations \{\((2A.7), (2A.8), (2A.10) \& (2A.11)\) for \( n = 3, 4, 5 \}\), we get sixteen relations between sixteen unknown constants.
from these relations, sixteen unknown constants
\( a_i', \ b_i', \ c_i', \ d_i' \ (i = 1,2,3,4) \) and probability \( P^{(0)}_{0,F} \). Further \( P^{(0)}_{0,F} \) can be obtained by
the relation
\[ \sum_{n=1}^{\infty} P_{n,B}^{(0)} + \sum_{n=0}^{\infty} P_{n,F}^{(0)} + Q^{(0)}_{0,B} = 1. \]
Hence by using the values of \( a_i', \ b_i', \ c_i', \ d_i' \ (i = 1,2,3,4) \) and \( P^{(0)}_{0,F} \), the probabilities \( Q^{(0)}_{0,B}, P^{(0)}_{n,B}, P^{(0)}_{n,F} \) and \( P^{(1)}_{n,B}, P^{(1)}_{n,F} \) are completely known for various values of \( n \).

**Laplace Transformation of Probability Generating Function of transient-state queue length probabilities**

\[
\overline{P}(z,s) = \frac{z \left( -\frac{z}{\mu} \left( \frac{B + \frac{v}{\mu}}{A + \frac{v}{\mu} c_2 p(z-1)} \right) + \left( z - 1 \right) \left( qA + \frac{v}{\mu} c_2 p(z-1) \right) \left( \frac{(v-v_0)}{\mu} \overline{F}_{0,F}^{(0)}(s) + C \overline{Q}_{0,B}^{(0)}(s) \right) \right) + z(z-1) \frac{v}{\mu} \left( B + \frac{v}{\mu} \right) \left( c_2 q - c_1 p(z-1) \right) \overline{F}_{1,B}^{(0)}(s) + c_1 c_2^{(1)}(s) }{\left( A - \frac{v}{\mu} c_2 \right) \left( A - \frac{v}{\mu} c_1 (q + pz) \right) - \left( \frac{v}{\mu} \right)^2 c_1 c_2 (q + pz) }, \quad \rho = \lambda/\mu < 1; |z| \leq 1 \quad (2A.24)
\]

where
\[ B = \left\{ -\rho z + \left( \rho + 1 + \frac{s}{\mu} \right) \right\} , \quad C = \left\{ -\rho z + \left( \rho + \frac{v}{\mu} + \frac{s}{\mu} \right) \right\} \]
and
\[ A = z.B.C = \left\{ -\rho z^2 + \left( \rho + 1 + \frac{s}{\mu} \right) z \right\} \left\{ -\rho z + \left( \rho + \frac{v}{\mu} + \frac{s}{\mu} \right) \right\} \]

Let
\[ D = K_1(z) K_2(z) - \left( \frac{v}{\mu} \right)^2 c_1 c_2 (q + pz) \]

where
\[ K_1(z) = \left\{ -\rho z^2 + \left( \rho + 1 + \frac{s}{\mu} \right) z \right\} \left\{ -\rho z + \left( \rho + \frac{v}{\mu} + \frac{s}{\mu} \right) \right\} - c_2 \left( \frac{v}{\mu} \right) \]
\[ K_2(z) = \left\{ -\rho z^2 + \left( \rho + 1 + \frac{s}{\mu} \right) z \right\} \left\{ -\rho z + \left( \rho + \frac{v}{\mu} + \frac{s}{\mu} \right) \right\} - c_1 \left( \frac{v}{\mu} \right) (q + pz) \]
Chapter 2 - Model A

Obviously $K_1(z) \ast K_2(z)$ have two zeroes inside the unit circle.

Let $f(z) = K_1(z) \ast K_2(z)$ and $g(z) = \left(\frac{v}{\mu}\right)^2 c_1 c_2 (q + pz)$

Now $|f(z)| = |K_1(z) \ast K_2(z)| = |K_1(z)| \cdot |K_2(z)|$

$$= \left[\left\{-\rho z^2 + \left(\rho + 1 + \frac{s}{\mu}\right)z\right\}\left\{-\rho z + \left(\rho + \frac{v}{\mu} + \frac{s}{\mu}\right)\right\} - c_2\left(\frac{v}{\mu}\right)\right]$$

$$\cdot \left[\left\{-\rho z^2 + \left(\rho + 1 + \frac{s}{\mu}\right)z\right\}\left\{-\rho z + \left(\rho + \frac{v}{\mu} + \frac{s}{\mu}\right)\right\} - c_1\left(\frac{v}{\mu}\right)(q + pz)\right]$$

$$\geq \left(\frac{v}{\mu}\right)^2 c_1 c_2 \left|\frac{s}{\mu} = \zeta + i\eta , |z| = 1\right.$$
On using L’Hospital’s rule, it can be shown that $\overline{P}(0, s) = \overline{P}_{0,F}^{(0)}(s) + \overline{Q}_{0,B}^{(0)}(s) = \overline{P}_{0}^{(0)}(s)$.

**Special Cases**

(1) When the server is available to an empty queue with the same mean availability time as is to a non-empty queue then putting $v_0 = v$ in (2A.24), generating function for queue length probabilities is

$$
\overline{P}(z, s) = \frac{z}{\mu} \left( B + \frac{v}{\mu} \right) \left[ A + \frac{v}{\mu} c_z p(z-1) \right] + (z-1) C \left( qA + \frac{v}{\mu} c_z p z \right) \overline{Q}_{0,B}^{(0)}(s) 
+ z(z-1) \frac{v}{\mu} p \left( B + \frac{v}{\mu} \right) \left[ c_z q - c_1 p z \right] \overline{P}_{1,B}^{(0)}(s) + c_z \overline{P}_{1,B}^{(1)}(s)

, \rho = \lambda/\mu < 1; |z| \leq 1 \quad (2A.25)
$$

(2) When the server is never available to an empty queue then putting $v_0 = 0$ and $\overline{Q}_{0,B}^{(0)}(s) = 0$ in (2A.24)

$$
\overline{P}(z, s) = \frac{z}{\mu} \left( B + \frac{v}{\mu} \right) \left[ A + \frac{v}{\mu} c_z p(z-1) \right] + (z-1) \left( qA + \frac{v}{\mu} c_z p z \right) \overline{P}_{0,F}^{(0)}(s) 
+ z(z-1) \frac{v}{\mu} p \left( B + \frac{v}{\mu} \right) \left[ c_z q - c_1 p z \right] \overline{P}_{1,B}^{(0)}(s) + c_z \overline{P}_{1,B}^{(1)}(s)

, \rho = \lambda/\mu < 1; |z| \leq 1 \quad (2A.26)
$$

Now for the two zeroes of the denominator, the numerator must vanish and these two equations along with equation (2A.18) determine three unknowns $\overline{P}_{0,F}^{(0)}(s), \overline{P}_{0,B}^{(0)}(s)$ and $\overline{P}_{1,B}^{(0)}(s)$. Hence the generating function $\overline{P}(z, s)$ is completely known.
Busy Period Distribution

The probability density function for the busy period distribution is given by

\[
\frac{d}{dt} \left\{ P_{0,0}^{(0)}(t) + Q_{0,0}^{(0)}(t) \right\}
\]

and can be obtained using the equations (2A.1) and (2A.4) for \( n \geq 2 \) and the equations (2A.2) and (2A.5) for \( n \geq 1 \) along with the equations

\[
\frac{d}{dt} P_{1,B}^{(0)}(t) = -(\lambda + \mu) P_{1,B}^{(0)}(t) + v P_{1,F}^{(0)}(t)
\]

(2A.27)

\[
\frac{d}{dt} Q_{0,B}^{(0)}(t) = 0
\]

(2A.28)

\[
\frac{d}{dt} P_{1,F}^{(0)}(t) = -(\lambda + v) P_{1,F}^{(0)}(t) + \mu c_1 \left\{ qP_{2,B}^{(0)}(t) + P_{2,B}^{(1)}(t) \right\}
\]

(2A.29)

\[
\frac{d}{dt} P_{0,F}^{(0)}(t) = \mu \left\{ qP_{1,B}^{(0)}(t) + P_{1,B}^{(1)}(t) \right\}
\]

(2A.30)

Initially \( P_{1,F}^{(0)}(0) = 1 \)

Define

\[
G_B^{(k)}(z,t) = \sum_{n=1}^{\infty} P_{n,B}^{(k)}(t)Z^n, \quad G_F^{(k)}(z,t) = \sum_{n=1}^{\infty} P_{n,F}^{(k)}(t)Z^n
\]

\[
G_B(z,t) = G_B^{(0)}(z,t) + G_B^{(1)}(z,t) \quad , \quad G_F(z,t) = G_F^{(0)}(z,t) + G_F^{(1)}(z,t)
\]

\[
G^{(k)}(z,t) = G_B^{(k)}(z,t) + G_F^{(k)}(z,t) \quad , \quad G(z,t) = G^{(0)}(z,t) + G^{(1)}(z,t)
\]

and \( G(z,t) = G_B(z,t) + G_F(z,t) \), all for \( k = 0 \) or \( 1 \) with \( |z| \leq 1 \)
Laplace Transformation of probability generating function for the busy period distribution

\[
\overline{G}(z,s) = \frac{z^2}{\mu} \left( B + \frac{v}{\mu} \right) \left\{ zBC + \frac{v}{\mu} c_z p(z-1) \right\} - z \left( B + \frac{v}{\mu} \right) \left\{ qA + \frac{v}{\mu} c_1 p^2 z(z-1) \right\} \overline{P}_{1,B}^{(0)}(s) + A \overline{P}_{1,B}^{(1)}(s) \\
\left\{ A - \frac{v}{\mu} c_z \right\} \left\{ A - \frac{v}{\mu} c_1 (q + pz) \right\} - \left( \frac{v}{\mu} \right)^2 c_1 c_z (q + pz) \\
, \quad \rho = \lambda / \mu < 1; \quad |z| \leq 1 \quad (2A.31)
\]

The denominator of (2A.31) is same as that of (2A.24) and hence numerator of (2A.31) must vanish for those two zeroes of the denominator of (2A.24). These two equations determine the two unknowns \( \overline{P}_{1,B}^{(0)}(s) \) and \( \overline{P}_{1,B}^{(1)}(s) \). Using Laplace Inverse of \( \overline{P}_{1,B}^{(0)}(s) \) and \( \overline{P}_{1,B}^{(1)}(s) \) in (2A.28) and (2A.30), \( \frac{d}{dt} \{ P_{0,F}^{(0)}(t) + Q_{0,B}^{(0)}(t) \} \) can be obtained.
**NUMERICAL SOLUTION AND GRAPHICAL REPRESENTATION**

The numerical results are generated using MATLAB programming following the work of Bunday (1986). Various probabilities are listed in Table 2A.1 for the case when

\[
\rho = \frac{\lambda}{\mu} = 0.3, \eta = \frac{\nu}{\mu} = 0.7, \eta_0 = \frac{\nu_0}{\mu} = 0.5, c_1 = 0.8, q = 0.75
\]

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Various probabilities are plotted Vs time for the data from Table 2A.1 through Figs. 2A.1 to 2A.7. Probability plot of $Q^{(0)}_{0,B}$ and $P^{(0)}_{0,F}$ with respect to time (average service times) is shown by Fig. 2A.1. It is clear from the graph that probability $Q^{(0)}_{0,B}$ increases rapidly in the starting moments, then decreases slowly and finally becomes almost steady from the initial value zero at time $t=0$. Probability $P^{(0)}_{0,F}$ decreases rapidly in the starting moments and then becomes almost steady from the initial value $=1$ at time $t=0$.

Fig. 2A.2 shows relative change in probabilities $P^{(0)}_{1,B}$, $P^{(1)}_{1,B}$, $P^{(0)}_{1,F}$ and $P^{(1)}_{1,F}$ with respect to time (average service times). Probabilities $P^{(0)}_{1,B}$ and $P^{(0)}_{1,F}$ increase rapidly in the starting moments, then decrease to some extent and finally attain some steady values. Probabilities $P^{(1)}_{1,B}$ and $P^{(1)}_{1,F}$ increase in the starting moments, then decrease to a small extent and become
steady for higher values of $t$. In Fig. 2A.3, probabilities $P_{2,B}^{(0)}$, $P_{2,B}^{(1)}$, $P_{2,F}^{(0)}$ and $P_{2,F}^{(1)}$ are plotted versus time.

![Fig. 2A.2](image1)

![Fig. 2A.3](image2)
Figs. 2A.4 and 2A.5 show that the probabilities $P_{3,B}^{(0)}$, $P_{3,B}^{(1)}$, $P_{3,F}^{(0)}$, $P_{3,F}^{(1)}$, $P_{4,B}^{(0)}$, $P_{4,B}^{(1)}$, and $P_{4,F}^{(1)}$ increase for some time and become almost constant for higher values of $t$. 

![Fig. 2A.4](image1)

![Fig. 2A.5](image2)
Comparison among four probabilities when server is busy and the unit at the head of the queue is to join the server for the first time i.e. among \( P_{1,B}^{(0)} \), \( P_{2,B}^{(0)} \), \( P_{3,B}^{(0)} \) and \( P_{4,B}^{(0)} \) is done through Fig. 2A.6. It is apparent that probability decreases as \( n \) (number of units in the system) increases for the case under study.

Fig. 2A.7 illustrates relationship among four probabilities when server is free and the unit at the head of the queue is to join the server for the first time i.e. among \( P_{1,F}^{(0)} \), \( P_{2,F}^{(0)} \), \( P_{3,F}^{(0)} \) and \( P_{4,F}^{(0)} \). It shows that probability decreases as \( n \) (number of units in the system) increases. It is also seen that \( P_{1,F}^{(0)} \) increases rapidly in the starting moments, then decreases to some extent and finally approach to a steady value. Other three probabilities increase in the initial moments and attain some steady values for high values of \( t \).
Chapter 2-Model A

Fig. 2A.7

Probabilities $P_{ij}^{(n)}$, $P_{ij}^{(m)}$, $P_{ij}^{(n)}$ and $P_{ij}^{(n)}$ Vs time

$p=0.3$, $\eta=0.7$, $\eta_0=0.5$, $c_1=0.8$, $q=0.75$
Various probabilities are listed in Table 2A.2 for the case when

\[ \rho = \frac{\lambda}{\mu} = 0.8, \eta = \frac{\nu}{\mu} = 0.7, \eta_0 = \frac{\nu_0}{\mu} = 0.5, k = 3, c_1 = 0.8, q = 0.75 \]

### Table 2A.2

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Table 2A.1 are also plotted for the data of Table 2A.2 through Figs. 2A.8 to 2A.14.

To study the behavior of various probabilities in case when $\rho$ takes a comparatively high value, the graphs plotted for Table 2A.1 are also plotted for the data of Table 2A.2 through Figs. 2A.8 to 2A.14.
Chapter 2 - Model A

Probabilities $Q_{B}^{(0)}$ and $P_{B}^{(0)}$ Vs time

Fig. 2A.8

Probabilities $P_{B}^{(0)}$, $P_{B}^{(1)}$, $P_{F}^{(0)}$, and $P_{F}^{(1)}$ Vs time

Fig. 2A.9
Chapter 2 - Model A

Probabilities $P_{2,B}^{(0)}, P_{2,B}^{(1)}, P_{2,F}^{(0)}, P_{2,F}^{(1)}$ vs time

\[ \rho = 0.8, \eta = 0.7, \eta_0 = 0.5, c_1 = 0.8, q = 0.75 \]

Fig. 2A.10

Probabilities $P_{3,B}^{(0)}, P_{3,B}^{(1)}, P_{3,F}^{(0)}, P_{3,F}^{(1)}$ vs time

\[ \rho = 0.8, \eta = 0.7, \eta_0 = 0.5, c_1 = 0.8, q = 0.75 \]

Fig. 2A.11
Chapter 2 - Model A

Probabilities $P_{4,B}^{(0)}$, $P_{4,B}^{(1)}$, $P_{4,F}^{(0)}$, and $P_{4,F}^{(1)}$ Vs time

Fig. 2A.12

Probabilities $P_{4,B}^{(0)}$, $P_{2,B}^{(0)}$, $P_{3,B}^{(0)}$, and $P_{4,B}^{(0)}$ Vs time

Fig. 2A.13
To study the effect of various parameters on different probabilities of the model, the data of various probabilities is generated for different values of \( \rho \) keeping other parameters constant. The set of values that \( \rho \) took is \{0.2, 0.3, 0.5, 0.8\}. The other parameters were fixed at \( \eta = 0.7, \eta_0 = 0.5, c_1 = 0.8, q = 0.75 \). Figs. 2A.15 and 2A.16 show probability plots of \( Q_{0,B}^{(0)} \) and \( P_{0,F}^{(0)} \) against time for different values of \( \rho \). From the Fig. 2A.15 it is concluded that as \( \rho \) increases \( Q_{0,B}^{(0)} \) decreases. So more the traffic intensity i.e. more customers are arriving per unit average service time less is the probability of server being busy with the empty queue. Fig. 2A.16 shows that probability \( P_{0,F}^{(0)} \) decreases rapidly in the starting moments and becomes almost steady for higher values of \( t \). It is also observed that more the traffic intensity less is the probability of zero units in the system.
Chapter 2 - Model A

$Q_{0,B}^{(0)}$ Vs time for different values of traffic intensity $\rho(=\lambda/\mu)$

Fig. 2A.15

$P_{0,F}^{(0)}$ Vs time for different values of traffic intensity $\rho(=\lambda/\mu)$

Fig. 2A.16
Behavior of the probabilities $P_{1,B}^{(0)}$, $P_{2,B}^{(0)}$ and $P_{4,B}^{(0)}$ with respect to time and changing $\rho$ is apparent from Figs. 2A.17 to 2A.19.

![Diagram](image-url)
Chapter 2 - Model A

$P_{2,B}^{(0)}$ vs time for different values of traffic intensity $\rho(=\lambda/\mu)$

![Graph showing $P_{2,B}^{(0)}$ vs time for different values of traffic intensity $\rho(=\lambda/\mu)$ with parameters $\gamma=0.7$, $\eta=0.5$, $c_1=0.8$, $q=0.75$.]

$P_{4,B}^{(0)}$ vs time for different values of traffic intensity $\rho(=\lambda/\mu)$

![Graph showing $P_{4,B}^{(0)}$ vs time for different values of traffic intensity $\rho(=\lambda/\mu)$ with parameters $\gamma=0.7$, $\eta=0.5$, $c_1=0.8$, $q=0.75$.]
Study of the effect of changing $\eta$ and $\eta_0$ values is done by generating data of various probabilities for different values of $\eta$ and $\eta_0$ keeping other parameters constant. The set of values that $\eta$ and $\eta_0$ took is $\{0.3, 0.2\}, \{0.7, 0.5\}, \{0.9, 0.7\}$. The other parameters were fixed at $\rho = 0.5, c_1 = 0.8, q = 0.75$. In Fig. 2A.20, the probability $Q_{0,B}^{(0)}$ is plotted against time for different values of $\eta$ and $\eta_0$. From the figure it is clear that as $\eta$ and $\eta_0$ increases $Q_{0,B}^{(0)}$ also increases. So more is the availability rate per unit average service time, more is the probability of server being busy with the empty queue.

Fig. 2A.21 shows probability plot of $P_{0,F}^{(0)}$ with respect to $t$ (average service times) for a set of $\eta$ and $\eta_0$ values. In all these three cases probability $P_{0,F}^{(0)}$ decreases rapidly in the
starting moments and then becomes almost steady for higher values of \( t \). The figure shows that in the starting moments more the value of \( \eta \) and \( \eta_0 \) less is the probability \( P_{0,F}^{(0)} \) which shows that due to high availability server remains in busy state most of the times. After some time the trend reverses which shows that in cases of higher value of server availability rate, “n=0” state is achieved more often than the cases when availability is comparatively low.

From Figs. 2A.22 to 2A.24, it is observed that behavior of \( P_{1,B}^{(0)} \), \( P_{2,B}^{(0)} \) and \( P_{4,B}^{(0)} \) is similar to each other with changing \( \eta \) and \( \eta_0 \). All the probabilities increase with increasing \( \eta \) and \( \eta_0 \).
Chapter 2 - Model A

\( P_{1,B}(0) \) Vs time for different values of \( \eta = \frac{v}{\mu} \)

Fig. 2A.22

\( P_{2,B}(0) \) Vs time for different values of \( \eta = \frac{v}{\mu} \)

Fig. 2A.23
Analysis of the graphs plotted for different probabilities suggests that all the probabilities tend to attain some steady values for high values of time $t$ (average service times).
Model B

Like model A, this model also considers a queueing problem with intermittently available server with feedback facility but with a difference that here units are served in batches of variable size. Some of the practical situations to which our model corresponds are (i) service of tourists by tourist-guide (ii) training of yoga-learners by yoga teachers at yoga-centers.

The transient-state queue length probabilities and the distribution of the interval during which the server is busy are obtained. Some special cases of interest are also obtained. Some illustrations are discussed and their numerical solutions are obtained. Thereafter various probabilities are plotted and compared graphically.

The feedback queueing system investigated in this model is described by the following assumptions in addition to the assumptions (i), (ii) and (iv) given in model A of this chapter:

(i) Service time distribution of each batch is exponential with parameter $\mu$.

(ii) The capacity of the server is a random variable. The size of the batch is determined at the beginning of each service and is either equal to the total number of units or to the capacity of the server determined afresh before each service whichever is less. The probability that the server can serve $\gamma$ units is $b_\gamma$ so that $\sum_{\gamma=1}^{K} b_\gamma = 1$, where $K$ is the max capacity of the server.

(iii) The probability of rejoining the system is $p$ and that of leaving the system is $q$ for the batch of units getting service first time, so that $p+q=1$. However the batch will have to leave the system definitely after getting second service.

(iv) The probability that the batch joins the service channel for the first time is assumed to be $c_1$ and that for the second time is $c_2$, so that $c_1 + c_2 = 1$. 
(v) If the queue length is greater than the capacity of the server but the batch at the head of the queue to be served for the second time consists of units less than the capacity of the server, then only that batch consisting of these units to be served for the second time will join the server.

(vi) The stochastic processes involved, viz

a. arrival of units
b. departure of batches
c. availability of the server

are statistically independent.

2. Definitions

\[ P_{n,B}^{(k)}(t) = \text{Probability that there are } n \text{ units in the system at time } t \text{ and the next batch of units is to depart for the first time or second time according as } k=0 \text{ or } 1 \text{ and the server is busy in relation to the queue, i.e. either a batch of units is being served or else one is to be taken just then.} \quad n \geq 1 \]

\[ Q_{0,B}^{(0)}(t) = \text{Probability that there are zero units in the system at time } t \text{ and the next batch of units is to depart for the first time and the server is busy with the empty queue.} \]

\[ P_{n,F}^{(k)}(t) = \text{Probability that there are } n \text{ units in the system at time } t \text{ and the next batch of units is to depart for the first time or second time according as } k=0 \text{ or } 1 \text{ and the server is free in relation to the queue, i.e. neither a batch of units is being served nor is any to be taken at that instant.} \quad n \geq 0 \]

\[ P_n(t) = \text{Probability that there are } n \text{ units in the system at time } t. \quad n \geq 0 \]

\[ P_{n,B}(t) = P_{n,B}^{(0)}(t) + P_{n,B}^{(1)}(t) \quad n \geq 1 \quad ; \quad Q_{0,B}(t) = Q_{0,B}^{(0)}(t) \]
\[ P_{n,F}(t) = P_{n,F}^{(0)}(t) + P_{n,F}^{(1)}(t) \quad n \geq 0 ; \]

\[ P_n(t) = P_{n,B}(t) + Q_{n,B}(t) + P_{n,F}(t) \quad n \geq 0 ; \]

Initially the system starts when there are no units in the system and the server is free in relation to the queue,

i.e. \[ P_{0,F}^{(0)}(0) = 1 \]

Also \[ P_{0,B}^{(0)}(t) = P_{0,F}^{(1)}(t) = Q_{0,B}^{(1)}(t) = 0 \quad t \geq 0 ; \]

**The difference differential equations describing the system are**

\[
\frac{d}{dt} P_{n,B}^{(0)}(t) = - (\lambda + \mu) P_{n,B}^{(0)}(t) + \lambda \left( P_{n-1,B}^{(0)}(t) + Q_{0,B}^{(0)}(t) \delta_{n,1} \right) + \nu P_{n,F}^{(0)}(t), \quad n \geq 1 \tag{2B.1}
\]

\[
\frac{d}{dt} P_{n,B}^{(1)}(t) = - (\lambda + \mu) P_{n,B}^{(1)}(t) + \lambda P_{n-1,B}^{(1)}(t) + \nu P_{n,F}^{(1)}(t), \quad n \geq 1 \tag{2B.2}
\]

\[
\frac{d}{dt} Q_{0,B}^{(0)}(t) = - \lambda Q_{0,B}^{(0)}(t) + \nu_0 P_{0,F}^{(0)}(t) \tag{2B.3}
\]

\[
\frac{d}{dt} P_{n,F}^{(0)}(t) = - (\lambda + \nu) P_{n,F}^{(0)}(t) + \lambda P_{n-1,F}^{(0)}(t)
\]

\[
+ \mu c_1 \sum_{\gamma=1}^{K} b_{\gamma} \left( \xi P_{n+\gamma,B}^{(0)}(t) + P_{n+\gamma,B}^{(1)}(t) \right), \quad 0 < n \leq K \tag{2B.4}
\]

\[
+ \mu c_1 p \sum_{\gamma=1}^{n-1} b_{\gamma} P_{n,B}^{(0)}(t)
\]
\[
\frac{d}{dt} P_{n,F}^{(0)}(t) = - (\lambda + v) P_{n,F}^{(0)}(t) + \lambda P_{n-1,F}^{(0)}(t) \\
+ \mu c_1 \sum_{\gamma=1}^{K} b_{\gamma} \left\{ qP_{n+\gamma,B}^{(0)}(t) + P_{n+\gamma,B}^{(1)}(t) \right\} , n > K \tag{2B.5}
\]

\[
\frac{d}{dt} P_{n,F}^{(1)}(t) = - (\lambda + v) P_{n,F}^{(1)}(t) + \lambda P_{n-1,F}^{(1)}(t) \\
+ \mu c_2 \sum_{\gamma=1}^{K} b_{\gamma} \left\{ qP_{n+\gamma,B}^{(0)}(t) + P_{n+\gamma,B}^{(1)}(t) \right\} + \mu c_2 p \sum_{\gamma=n}^{n-1} b_{\gamma} P_{n,B}^{(0)}(t) , 0 < n \leq K \tag{2B.6}
\]

\[
\frac{d}{dt} P_{n,F}^{(1)}(t) = - (\lambda + v) P_{n,F}^{(1)}(t) + \lambda P_{n-1,F}^{(1)}(t) \\
+ \mu c_2 \sum_{\gamma=1}^{K} b_{\gamma} \left\{ qP_{n+\gamma,B}^{(0)}(t) + P_{n+\gamma,B}^{(1)}(t) \right\} + \mu c_2 p P_{n,B}^{(0)}(t) , n > K \tag{2B.7}
\]

\[
\frac{d}{dt} P_{n,F}^{(0)}(t) = - (\lambda + v_n) P_{n,F}^{(0)}(t) \\
+ \mu \sum_{\gamma=1}^{K} \left( \sum_{i=1}^{K} b_{\gamma} \right) \left\{ qP_{\gamma,B}^{(0)}(t) + P_{\gamma,B}^{(1)}(t) \right\} \tag{2B.8}
\]

where \( \delta_{n,1} = 1 \) for \( n = 1 \) and \( \delta_{n,1} = 0 \), otherwise.

Taking the Laplace Transformation \( \mathcal{L}_s P_n(t) = \int_0^\infty e^{-st} P_n(t) dt \quad \text{Re } s > 0 \)

of (2B.1)-(2B.8) and dividing both sides by \( \mu \)

\[
(\rho + 1 + \frac{s}{\mu}) \mathcal{L}_s P_{n,B}^{(0)}(s) = \rho \mathcal{L}_s \left[ P_{n-1,B}^{(0)}(s) + \mathcal{L}_s P_{n,B}^{(0)}(s) \delta_{n,1} \right] + \left( \frac{v}{\mu} \right) \mathcal{L}_s P_{n,F}^{(0)}(s) , n \geq 1 \tag{2B.9}
\]
\[
\begin{align*}
\left( \rho + \frac{1}{\mu} \right) \overline{P}_{n,B}^{(1)}(s) &= \rho \overline{P}_{n-1,B}^{(1)}(s) + \left( \frac{v}{\mu} \right) \overline{P}_{n,F}^{(1)}(s) \\
&\quad , n \geq 1 \\
\left( \rho + \frac{s}{\mu} \right) \overline{Q}_{0,B}^{(0)}(s) &= \left( \frac{v_0}{\mu} \right) \overline{P}_{0,F}^{(0)}(s) \\
\left( \rho + \frac{v}{\mu} + \frac{s}{\mu} \right) \overline{P}_{n,F}^{(0)}(s) &= \rho \overline{P}_{n-1,F}^{(0)}(s) \\
&\quad + c_1 \sum_{\gamma=1}^{K} b_\gamma \left\{ q \overline{P}_{n+\gamma,B}^{(0)}(s) + \overline{P}_{n+\gamma,B}^{(1)}(s) \right\} \\
&\quad , 0 < n \leq K \\
&\quad + c_1 p \left( \sum_{\gamma=1}^{n-1} b_\gamma \right) \overline{P}_{n,B}^{(0)}(s) \\
\left( \rho + \frac{v}{\mu} + \frac{s}{\mu} \right) \overline{P}_{n,F}^{(1)}(s) &= \rho \overline{P}_{n-1,F}^{(1)}(s) \\
&\quad + c_2 \sum_{\gamma=1}^{K} b_\gamma \left\{ q \overline{P}_{n+\gamma,B}^{(0)}(s) + \overline{P}_{n+\gamma,B}^{(1)}(s) \right\} \\
&\quad , 0 < n \leq K \\
&\quad + p \left( \sum_{\gamma=n}^{K} b_\gamma \right) \overline{P}_{n,B}^{(0)}(s) + c_2 p \left( \sum_{\gamma=1}^{n-1} b_\gamma \right) \overline{P}_{n,B}^{(0)}(s)
\end{align*}
\]
\[
\left( \frac{\rho + \frac{V_0}{\mu} + \frac{s}{\mu}}{\mu} \right) P_{0,r}^{(0)} (s) = \frac{1}{\mu} + \sum_{\gamma = 1}^{K} \left( \sum_{i=1}^{K} b_i \right) q P_{\gamma,r}^{(0)} (s) + \overline{P}_{\gamma,0}^{(1)} (s) \right) \right] 
\] (2B.16)

**Definitions**

\[
P_B^{(k)} (z,t) = \sum_{n=1}^{\infty} P_{n,b}^{(k)} (t) z^n + Q_{0,b}^{(0)} (t) \quad P_B^{(k)} (z,t) = \sum_{n=0}^{\infty} P_{n,b}^{(k)} (t) z^n
\]

\[
P_B (z,t) = P_B^{(0)} (z,t) + P_B^{(1)} (z,t) \quad P_F (z,t) = P_F^{(0)} (z,t) + P_F^{(1)} (z,t)
\]

\[
P(z,t) = P_B (z,t) + P_F (z,t)
\]

\[
\overline{P}_F^{(k)} (z,s) = \int_0^\infty e^{-st} P_F^{(k)} (z,t) dt \quad \overline{P}(z,s) = \int_0^\infty e^{-st} P(z,t) dt
\]

, all for \( k = 0 \) or \( 1 \) with \( |z| \leq 1 \)

Multiply (2B.9) and (2B.11) by \( z^{n+1} \), sum over all \( n \) and use generating function definitions to get \( \overline{P}_B^{(0)} (z,s) \) in terms of \( \overline{P}_F^{(0)} (z,s) \), \( \overline{P}_{0,F}^{(0)} (s) \) and \( \overline{Q}_{0,b}^{(0)} (s) \). Similarly multiply (2B.10) by \( Z^{n+1} \), sum over all \( n \) and use generating function definitions to get \( \overline{P}_B^{(1)} (z,s) \) in terms of \( \overline{P}_F^{(1)} (z,s) \); multiply (2B.12), (2B.13) and (2B.16) by \( z^{n+1} \), sum over all \( n \) and use generating function definitions to get \( \overline{P}_F^{(0)} (z,s) \) in terms of \( \overline{P}_B^{(0)} (z,s) \), \( \overline{P}_B^{(1)} (z,s) \), \( \overline{P}_{0,F}^{(0)} (s) \), \( \overline{Q}_{0,b}^{(0)} (s) \) & LAPLACE transform of probabilities \( P_{r,b}^{(0)} (t) \) and \( P_{r,b}^{(1)} (t) \) \( r=1,2,...,K \); and multiply (2B.14) and (2B.15) by \( z^{n+1} \), sum over all \( n \) and use generating function definitions to get \( \overline{P}_F^{(1)} (z,s) \) in terms of \( \overline{P}_B^{(0)} (z,s) \), \( \overline{P}_B^{(1)} (z,s) \), \( \overline{Q}_{0,b}^{(0)} (s) \) & LAPLACE transform of probabilities.
Chapter 2-Model B

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\( P_{r,b}^{(0)}(t) \) and \( P_{r,b}^{(1)}(t) \) \( r = 1,2, \ldots, K \). Now solve these to get LAPLACE transform of the probability generating functions.

LAPLACE transform of the probability generating functions is given by

(2B.17) to (2B.21)

\[
\begin{align*}
\mathcal{P}_F^{(0)}(z,s) &= \left( A - \frac{V}{\mu} c_z zH(z^{-1}) \right) \left[ \frac{1}{\mu} + \frac{V - V_0}{\mu} \mathcal{P}_{0,F}^{(0)}(s) + \sum_{r=1}^{K} \left( \sum_{\gamma=r}^{K} b_{\gamma} \right) \mathcal{Q}_{r,b}^{(0)}(s) + \mathcal{P}_{r,b}^{(0)}(s) \right] \\
&\quad - c_z A \mathcal{Q}_H(z^{-1}) + p \left[ \frac{V - V_0}{\mu} \mathcal{P}_{0,F}^{(0)}(s) + (B - 1) \mathcal{Q}_{0,b}^{(0)}(s) \right] \\
&\quad - c_z b \left[ \frac{1}{\mu} + \frac{V - V_0}{\mu} \mathcal{P}_{0,F}^{(0)}(s) + \sum_{r=1}^{K} \left( \sum_{\gamma=r}^{K} b_{\gamma} \right) \mathcal{Q}_{r,b}^{(0)}(s) + \mathcal{P}_{r,b}^{(0)}(s) \right] \mathcal{Q}_{H}^{(0)}(z^{-1}) + p \mathcal{H}(z^{-1}) \\
&\quad , \rho = \lambda/\mu < 1; |z| \leq 1
\end{align*}
\]

(2B.18)

\[
\begin{align*}
\mathcal{P}_F^{(1)}(z,s) &= \left( A - \frac{V}{\mu} c_z zH(z^{-1}) \right) \left[ \frac{1}{\mu} + \frac{V - V_0}{\mu} \mathcal{P}_{0,F}^{(0)}(s) + \sum_{r=1}^{K} \left( \sum_{\gamma=r}^{K} b_{\gamma} \right) \mathcal{Q}_{r,b}^{(0)}(s) + \mathcal{P}_{r,b}^{(0)}(s) \right] \\
&\quad - c_z A \mathcal{Q}_H(z^{-1}) + p \left[ \frac{V - V_0}{\mu} \mathcal{P}_{0,F}^{(0)}(s) + (B - 1) \mathcal{Q}_{0,b}^{(0)}(s) \right] \\
&\quad - zB \left[ c_z A \sum_{r=1}^{K} \left( \sum_{\gamma=r}^{K} b_{\gamma} \right) \mathcal{Q}_{r,b}^{(0)}(s) + \mathcal{P}_{r,b}^{(0)}(s) \right] \mathcal{H}(z^{-1}) + p \mathcal{H}(z^{-1}) \\
&\quad , \rho = \lambda/\mu < 1; |z| \leq 1
\end{align*}
\]
\[
\frac{\nu}{\mu} c_2 z H(z^{-1}) + \left[ \frac{1}{\mu} + \frac{v - \nu_0}{\mu} P_{0,r}^{(0)}(s) + \sum_{r=1}^{K} \sum_{r' = r}^{K} b_{r'} \right] \left\{ qP_{r,b}^{(0)}(s) + \overline{Q}_{r,b}(s) \right\} \\
- zC \left[ A - \frac{\nu}{\mu} c_2 z H(z^{-1}) \right] \left\{ \left( \frac{v - \nu_0}{\mu} P_{0,r}^{(0)}(s) - \overline{Q}_{0,b}(s) \right) + \frac{v}{\mu} c_1 z B \left( qH(z^{-1}) + p \right) \overline{Q}_{0,b}(s) \right\} \\
- \frac{\nu}{\mu} c_1 z \left[ \sum_{r=1}^{K} \sum_{r' = r}^{K} b_{r'} \right] \left\{ qP_{r,b}^{(0)}(s) + \overline{Q}_{r,b}(s) \right\} z' + p \left[ A - \frac{\nu}{\mu} c_2 z H(z^{-1}) \right] \left( \sum_{r=1}^{K} \sum_{r' = r}^{K} b_{r'} \right) \overline{Q}_{r,b}(s) z' \\
= \left\{ A - \frac{\nu}{\mu} c_2 z H(z^{-1}) \right\} \left\{ A - \frac{\nu}{\mu} c_1 z \left( qH(z^{-1}) + p \right) \right\} - \left( \frac{\nu}{\mu} \right)^2 c_1 c_2 z^2 \left( qH(z^{-1}) + p \right) H(z^{-1}) \\
, \rho = \lambda/\mu < 1; |z| \leq 1 \tag{2B.19}
\]

\[
\frac{\nu}{\mu} c_2 z^2 qH(z^{-1}) + \left[ \frac{1}{\mu} + \frac{v - \nu_0}{\mu} P_{0,r}^{(0)}(s) + \sum_{r=1}^{K} \sum_{r' = r}^{K} b_{r'} \right] \left\{ qP_{r,b}^{(0)}(s) + \overline{Q}_{r,b}(s) \right\} \\
- \frac{\nu}{\mu} c_2 z^2 C qH(z^{-1}) + \left[ \frac{v - \nu_0}{\mu} P_{0,r}^{(0)}(s) + (B - 1) \overline{Q}_{0,b}(s) \right] \left\{ qP_{r,b}^{(0)}(s) + \overline{Q}_{r,b}(s) \right\} z' - c_1 p \left[ A - \frac{\nu}{\mu} z qH(z^{-1}) + p \right] \left( \sum_{r=1}^{K} \sum_{r' = r}^{K} b_{r'} \right) \overline{Q}_{r,b}(s) z' \\
= \left\{ A - \frac{\nu}{\mu} c_2 z H(z^{-1}) \right\} \left\{ A - \frac{\nu}{\mu} c_1 z \left( qH(z^{-1}) + p \right) \right\} - \left( \frac{\nu}{\mu} \right)^2 c_1 c_2 z^2 \left( qH(z^{-1}) + p \right) H(z^{-1}) \\
, \rho = \lambda/\mu < 1; |z| \leq 1 \tag{2B.20}
\]
Chapter 2-Model B

where \( H(z^{-1}) = \sum_{j=1}^{K} \frac{b_j}{z^j} \), \( B = \left\{ -\rho z + \left( \rho + 1 + \frac{v}{s} \right) \right\} \), \( C = \left\{ -\rho z + \left( \rho + \frac{v}{\mu} + \frac{s}{\mu} \right) \right\} \)

and \( A = z B C = \left\{ -\rho z^2 + \left( \rho + 1 + \frac{s}{\mu} \right) z \right\} * \left\{ -\rho z + \left( \rho + \frac{v}{\mu} + \frac{s}{\mu} \right) \right\} \)

Let

\[ D = K_1(z)^* K_2(z) - \left( \frac{v}{\mu} \right)^2 c_1 c_2 z^2 \left[ q H(z^{-1}) + p \right] H(z^{-1}) \]

where \( K_1(z) = \left\{ -\rho z^2 + \left( \rho + 1 + \frac{s}{\mu} \right) z \right\} \left\{ -\rho z + \left( \rho + \frac{v}{\mu} + \frac{s}{\mu} \right) \right\} - \frac{v}{\mu} c_2 z H(z^{-1}) \)

\[ K_2(z) = \left\{ -\rho z^2 + \left( \rho + 1 + \frac{s}{\mu} \right) z \right\} \left\{ -\rho z + \left( \rho + \frac{v}{\mu} + \frac{s}{\mu} \right) \right\} - \frac{v}{\mu} c_1 z \left[ q H(z^{-1}) + p \right] \]

Applying Rouche’s Theorem to \( K_1(z) \)

\[ K_1(z) = \left[ \rho^2 z^3 - \left\{ 2 \rho^2 + \left( 1 + \frac{2s}{\mu} + \frac{v}{\mu} \right) \rho \right\} z^2 + \left\{ \rho^2 + \left( 1 + \frac{2s}{\mu} + \frac{v}{\mu} \right) \rho + \left( 1 + \frac{s}{\mu} \left( \frac{s}{\mu} + \frac{v}{\mu} \right) \right\} z \right\} - \frac{v}{\mu} c_2 z H(z^{-1}) \]

Consider

\[ K_{11}(z) = z^k \left[ \rho^2 z^3 - \left\{ 2 \rho^2 + \left( 1 + \frac{2s}{\mu} + \frac{v}{\mu} \right) \rho \right\} z^2 + \left\{ \rho^2 + \left( 1 + \frac{2s}{\mu} + \frac{v}{\mu} \right) \rho + \left( 1 + \frac{s}{\mu} \left( \frac{s}{\mu} + \frac{v}{\mu} \right) \right\} z \right\} \]

And \( K_{12}(z) = \frac{v}{\mu} c_2 z^k H(z^{-1}) \)

Obviously \( K_{11}(z) \) has K zeroes inside the unit circle.

Now \( K_{11}(z) \)

\[ = z^k \left[ \rho^2 z^3 - \left\{ 2 \rho^2 + \left( 1 + \frac{2s}{\mu} + \frac{v}{\mu} \right) \rho \right\} z^2 + \left\{ \rho^2 + \left( 1 + \frac{2s}{\mu} + \frac{v}{\mu} \right) \rho + \left( 1 + \frac{s}{\mu} \left( \frac{s}{\mu} + \frac{v}{\mu} \right) \right\} z \right\} \]
\[
\geq \left( \zeta^2 + \left( \frac{v}{\mu} \right) \right) \text{ for } \frac{s}{\mu} = \zeta + i\eta, |z| = 1
\]

\[
> \left( \frac{v}{\mu} \right) c_2 \geq |K_{12}(z)|
\]

Hence \(|K_{11}(z)| > |K_{12}(z)|\) on \(|z| = 1\)

Since all the conditions of Rouche’s theorem are satisfied, so \(K_{11}(z)\) and \(K_{11}(z) - K_{12}(z)\) has same number of zeroes inside unit circle \(|z| = 1\). But \(K_{11}(z)\) has at least \(K\) zeroes inside the unit circle therefore \(K_{11}(z) - K_{12}(z)\) i.e. \(K(z)\) also has at least \(K\) zeroes inside the unit circle.

Similarly we can prove that \(K_2(z)\) has at least \(K\) zeroes inside the unit circle.

Obviously \(K_1(z)K_2(z)\) has at least \(2K\) zeroes inside the unit circle.

Let \(f(z) = K_1(z)K_2(z)z^{-2K-2}\) and \(g(z) = \left( \frac{v}{\mu} \right)^2 c_1 c_2 z^{2K} \{ qH(z^{-1}) + p \} H(z^{-1}) \)

Now \(|f(z)| = |K_1(z)K_2(z)z^{-2K-2}| = |K_1(z)z^{-K-1}| |K_2(z)z^K|\)

\[
\geq \left( \zeta^2 + \left( \frac{v}{\mu} \right) \right) \left( \zeta^2 + \left( \frac{v}{\mu} \right) c_2 \right) \text{ for } \frac{s}{\mu} = \zeta + i\eta, |z| = 1
\]

\[
> \left( \frac{v}{\mu} \right)^2 c_1 c_2 \geq |g(z)|
\]
Hence $|f(z)| > |g(z)|$ on $|z| = 1$.

Since all the conditions of Rouche’s theorem are satisfied, so denominator $D$ in (2B.17), (2B.18), (2B.19), (2B.20) and (2B.21) has at least $2K$ zeroes inside the unit circle.

Let these zeroes be $z_m (m = 0, 1, ..., 2K - 1)$. Numerators in (2B.17), (2B.18), (2B.19), (2B.20) and (2B.21) must also vanish for these $2K$ zeroes since $\overline{P}_F^{(0)}(z,s), \overline{P}_F^{(1)}(z,s), \overline{P}_B^{(0)}(z,s), \overline{P}_B^{(1)}(z,s)$ and $\overline{P}(z,s)$ are analytical functions of $z$. These $2K$ equations along with equations (2B.11) and (2B.16) will determine the $(2K+2)$ unknowns $\overline{Q}_{0,B}^{(0)}(s), \overline{P}_{0,F}^{(0)}(s), \overline{P}_{r,B}^{(0)}(s)$ and $\overline{P}_{r,B}^{(1)}(s), r = 1, 2, ..., K$. Hence the generating functions $\overline{P}_F^{(0)}(z,s), \overline{P}_F^{(1)}(z,s), \overline{P}_B^{(0)}(z,s), \overline{P}_B^{(1)}(z,s)$ and $\overline{P}(z,s)$ are completely known. $\overline{Q}_{0,B}^{(0)}(s), \overline{P}_{0,F}^{(0)}(s), \overline{P}_{n,B}^{(0)}(s), \overline{P}_{n,B}^{(1)}(s), \overline{P}_{n,F}^{(0)}(s), \overline{P}_{n,F}^{(1)}(s)$ and $\overline{P}_n^{(0)}(s)$ can be obtained by using the following formulae

$$\overline{P}_{n,F}^{(0)}(s) = \frac{1}{n!} \frac{d^{(n)}}{dz^{(n)}} \overline{P}_F^{(0)}(z,s) \quad \text{at} \quad z = 0 \quad \text{for} \quad n=0, 1...$$

$$\overline{P}_{n,F}^{(1)}(s) = \frac{1}{n!} \frac{d^{(n)}}{dz^{(n)}} \overline{P}_F^{(1)}(z,s) \quad \text{at} \quad z = 0 \quad \text{for} \quad n=0, 1...$$

$$\overline{P}_{n,B}^{(0)}(s) = \frac{1}{n!} \frac{d^{(n)}}{dz^{(n)}} \overline{P}_B^{(0)}(z,s) \quad \text{at} \quad z = 0 \quad \text{for} \quad n=1, 2...$$

$$\overline{Q}_{0,B}^{(0)}(s) = \frac{1}{n!} \frac{d^{(n)}}{dz^{(n)}} \overline{P}_B^{(0)}(z,s) \quad \text{at} \quad z = 0$$

$$\overline{P}_{n,B}^{(1)}(s) = \frac{1}{n!} \frac{d^{(n)}}{dz^{(n)}} \overline{P}_B^{(1)}(z,s) \quad \text{at} \quad z = 0 \quad \text{for} \quad n=0, 1...$$

$$\overline{P}_n^{(0)}(s) = \frac{1}{n!} \frac{d^{(n)}}{dz^{(n)}} \overline{P}(z,s) \quad \text{at} \quad z = 0 \quad \text{for} \quad n=0, 1...$$
Hence the generating functions of probabilities when the server is ‘free’ or ‘busy’ are completely determined.

In either case $Q^{(0)}_{0,B}(t)$, $P^{(0)}_{0,F}(t)$, $P^{(0)}_{n,B}(t)$, $P^{(1)}_{n,B}(t)$, $P^{(0)}_{n,F}(t)$, $P^{(1)}_{n,F}(t)$ and $P_n(t)$ can be found by inverting the Laplace transforms $\mathcal{L}\{Q^{(0)}_{0,B}(s)\}$, $\mathcal{L}\{P^{(0)}_{0,F}(s)\}$, $\mathcal{L}\{P^{(0)}_{n,B}(s)\}$, $\mathcal{L}\{P^{(1)}_{n,B}(s)\}$, $\mathcal{L}\{P^{(0)}_{n,F}(s)\}$, $\mathcal{L}\{P^{(1)}_{n,F}(s)\}$ and $\mathcal{L}\{P_n(s)\}$.

Further $\mathcal{L}\{-P(1, s)\} = \frac{1}{s}$, as desired.

And $\mathcal{L}\{-P(0, s)\} = \lim_{z \to 0} P(z, s)$ is of indeterminate form.

On using L’Hospital’s rule, it can be shown that

$$\mathcal{L}\{-P(0, s)\} = \mathcal{L}\{-P^{(0)}_{0,F}(s)\} + \mathcal{L}\{-Q^{(0)}_{0,B}(s)\} = \mathcal{L}\{-P^{(0)}_{0}(s)\}$$

Also $\mathcal{L}\{-P^{(0)}_{F}(0, s)\} = \lim_{z \to 0} P^{(0)}_{F}(z, s)$ is of indeterminate form.

On using L’Hospital’s rule, it can be shown that $\mathcal{L}\{-P^{(0)}_{F}(0, s)\} = \mathcal{L}\{-P^{(0)}_{0,F}(s)\}$

& $\mathcal{L}\{-P^{(0)}_{B}(0, s)\} = \lim_{z \to 0} P^{(0)}_{B}(z, s)$ is also of indeterminate form.

On using L’Hospital’s rule, it can be shown that $\mathcal{L}\{-P^{(0)}_{B}(0, s)\} = \mathcal{L}\{-Q^{(0)}_{0,B}(s)\}$.

Special Cases

1. When the departures are not in batches but only single unit departs then putting $b_\gamma = 1$ when $\gamma = 1$ and $b_\gamma = 0$ otherwise in (2B.17), (2B.18), (2B.19), (2B.20) and (2B.21), we get
\begin{align}
\bar{P}_F^{(0)}(z,s) &= \frac{-c_z \{A - \frac{v}{\mu} c_2 \} \left\{ A - \frac{v}{\mu} (q + pz) \right\} - \left( \frac{v}{\mu} \right)^2 c_c (q + pz)}{
\{ A - \frac{v}{\mu} \} \left\{ A - \frac{v}{\mu} (q + pz) \right\} - \left( \frac{v}{\mu} \right)^2 c_c (q + pz)} + \bar{Q}^{(0)}_{0,0} (s) \biggr|_{\rho = \lambda/\mu < 1; |z| \leq 1} \\
\bar{P}_B^{(0)}(z,s) &= \frac{-c_c (q + pz) A \left\{ \frac{v}{\mu} z \{ A - \frac{v}{\mu} c_2 \} \left\{ A - \frac{v}{\mu} (q + pz) \right\} - \left( \frac{v}{\mu} \right)^2 c_c (q + pz) \right\}}{
\{ A - \frac{v}{\mu} \} \left\{ A - \frac{v}{\mu} (q + pz) \right\} - \left( \frac{v}{\mu} \right)^2 c_c (q + pz)} + \bar{Q}^{(0)}_{0,0} (s) \biggr|_{\rho = \lambda/\mu < 1; |z| \leq 1} \\
\end{align}
\[
\frac{v}{\mu} c_z (q + pz) \left[ \frac{1}{\mu} + \frac{v-v_0}{\mu} \bar{P}_{0,F}^{(0)} (s) + \{q \bar{P}_{1,B}^{(0)} (s) + \bar{P}_{1,B}^{(0)} (s)\} \right] \\
- \frac{v}{\mu} c_z (q + pz) C \left[ \frac{(v-v_0)}{\mu} \bar{P}_{0,F}^{(0)} (s) + (B-1)Q_{0,B}^{(0)} (s) \right] \\
\bar{P}_{B}^{(1)} (z,s) = \left\{ A - \frac{v}{\mu} c_z \right\} \left\{ A - \frac{v}{\mu} c_1 (q + pz) \right\} - \left( \frac{v}{\mu} \right)^2 c_z (q + pz) \\
\text{, } \rho = \lambda \mu < 1; \vert z \vert \leq 1 \quad (2B.25)
\]

\[
\frac{z}{\mu} \left( B + \frac{v}{\mu} \right) \left\{ A + \frac{v}{\mu} c_z (z-1) \right\} \\
+ (z-1) \left\{ q A + \frac{v}{\mu} c_z p z \right\} \left[ \frac{(v-v_0)}{\mu} \bar{P}_{0,F}^{(0)} (s) + C Q_{0,B}^{(0)} (s) \right] \\
+ \frac{v}{\mu} p z (z-1) \left\{ B + \frac{v}{\mu} \right\} \left\{ c_z q - c_z p z \right\} \bar{P}_{1,B}^{(0)} (s) + c_z \bar{P}_{1,B}^{(0)} (s) \right\} \\
\bar{P}_{z,s} = \left\{ A - \frac{v}{\mu} c_z \right\} \left\{ A - \frac{v}{\mu} c_1 (q + pz) \right\} - \left( \frac{v}{\mu} \right)^2 c_z (q + pz) \\
\text{, } \rho = \lambda \mu < 1; \vert z \vert \leq 1 \quad (2B.26)
\]

which coincides with the results of model A of this chapter.

2. When the units are served only once and there is no provision for second service i.e. \( c_z = 0 \)
and \( \bar{P}_{1,B}^{(0)} (t) \) is zero and hence \( \bar{P}_{1,B}^{(0)} (s) \) is also zero, then putting \( p=0, q=1, c_1 = 1 \), and

\[
\bar{P}_{r,B}^{(0)} (s) = \bar{P}_{r,B} (s) \text{ in (2B.17), (2B.19), and (2B.21), we get }
\]
\[
\overline{P}_F(z, s) = \frac{zB \left\{ \frac{1}{\mu} + \frac{v-v_0}{\mu} \overline{P}_{0,F}(s) + \sum_{r=1}^{K} \sum_{y_{r'}} b_{y_{r'}} \overline{P}_{r,b}(s) \right\} - zH(z^{-1}) \left\{ \frac{(v-v_0)}{\mu} \overline{P}_{0,F}(s) + (B-1)\overline{Q}_{0,b}(s) \right\} - zB \left\{ \sum_{r=1}^{K} \frac{b_{y_{r'}}}{z^r} \overline{P}_{r,b}(s)z^r \right\} \left\{ A - \frac{v}{\mu}zH(z^{-1}) \right\}}{A - \frac{v}{\mu}zH(z^{-1})}, \quad \rho = \lambda/\mu < 1; \ |z| \leq 1
\] (2B.27)

\[
\frac{\nu}{\mu} \left\{ \frac{1}{\mu} + \frac{v-v_0}{\mu} \overline{P}_{0,F}(s) + \sum_{r=1}^{K} \sum_{y_{r'}} b_{y_{r'}} \overline{P}_{r,b}(s) \right\} - zC \left\{ \frac{(v-v_0)}{\mu} \overline{P}_{0,F}(s) - \overline{Q}_{0,b}(s) \right\} - \frac{\nu}{\mu}zH(z^{-1}) \left\{ \overline{Q}_{0,b}(s) - \frac{v}{\mu}z \sum_{r=1}^{K} \sum_{y_{r'}} \frac{b_{y_{r'}}}{z^r} \overline{P}_{r,b}(s)z^r \right\} \left\{ A - \frac{v}{\mu}zH(z^{-1}) \right\}
\] (2B.28)

\[
\overline{P}(z, s) = \frac{\frac{z}{\mu} \left\{ B + \frac{\nu}{\mu} \right\} + z \left\{ 1 - H(z^{-1}) \right\} \left\{ \frac{(v-v_0)}{\mu} \overline{P}_{0,F}(s) + C\overline{Q}_{0,b}(s) \right\} - z \left\{ B + \frac{\nu}{\mu} \right\} \sum_{r=1}^{K} \left\{ \sum_{y_{r'}} b_{y_{r'}} \overline{P}_{r,b}(s)z^r - \sum_{y_{r'}} \frac{b_{y_{r'}}}{z^r} \overline{P}_{r,b}(s) \right\} \left\{ A - \frac{v}{\mu}zH(z^{-1}) \right\}}{A - \frac{v}{\mu}zH(z^{-1})}, \quad \rho = \lambda/\mu < 1; \ |z| \leq 1
\] (2B.29)
3. When the server is available to an empty queue with the same mean availability time as is to a non-empty queue then putting \( v_0 = v \) in (2B.17), (2B.18), (2B.19), (2B.20) and (2B.21), we can obtain the corresponding generating functions for queue length probabilities.

4. When the server is never available to an empty queue then putting \( v_0 = 0 \) and \( \bar{Q}_{0,B}^{(0)}(s) = 0 \), in (2B.17), (2B.18), (2B.19), (2B.20) and (2B.21), we can obtain the corresponding generating functions for queue length probabilities.

**Busy Period Distribution**

The probability density function for the busy period distribution is given by

\[
\frac{d}{dt} P_{1,F}^{(0)}(t) = - (\lambda + \mu) P_{1,F}^{(0)}(t) + v P_{1,F}^{(0)}(t)
\]

and can be obtained using the equations (2B.1)&( 2B.4) for \( n \geq 2 \), the equations (2B.2)&( 2B.6) for \( n \geq 1 \) and the equations (2B.5)&( 2B.7) along with the equations

\[
\frac{d}{dt} Q_{0,B}^{(0)}(t) = 0
\]

\[
\frac{d}{dt} P_{1,F}^{(0)}(t) = - (\lambda + v) P_{1,F}^{(0)}(t) + \mu c_1 \sum_{y=1}^{K} b_y \left\{ q P_{1+\gamma,B}^{(0)}(t) + P_{1+\gamma,B}^{(1)}(t) \right\}
\]

\[
\frac{d}{dt} P_{0,F}^{(0)}(t) = \mu \sum_{y=1}^{K} \left( \sum_{i=y}^{K} b_i \right) \left\{ q P_{\gamma,B}^{(0)}(t) + P_{\gamma,B}^{(1)}(t) \right\}
\]

Initially \( P_{1,F}^{(0)}(0) = 1 \)
Define

\[
G_B^{(k)}(z,t) = \sum_{n=1}^{\infty} p_{n,B}^{(k)}(t) Z^n, \quad G_F^{(k)}(z,t) = \sum_{n=1}^{\infty} p_{n,F}^{(k)}(t) Z^n
\]

\[
G_B(z,t) = G_B^{(0)}(z,t) + G_B^{(1)}(z,t), \quad G_F(z,t) = G_F^{(0)}(z,t) + G_F^{(1)}(z,t)
\]

\[
G^{(k)}(z,t) = G_B^{(k)}(z,t) + G_F^{(k)}(z,t), \quad G(z,t) = G^{(0)}(z,t) + G^{(1)}(z,t)
\]

And \( G(z,t) = G_B(z,t) + G_F(z,t) \), all for \( k = 0 \) or \( 1 \) with \( |z| \leq 1 \)

Laplace Transformation of probability generating function for the busy period distribution

\[
\frac{z^2}{\mu} \left[ B + \frac{\nu}{\mu} \right] A + \frac{\nu}{\mu} c_2pz \left[ 1 - H(z^{-1}) \right] - \frac{\nu}{\mu} c_1 p^2 z^2 \left[ 1 - H(z^{-1}) \right] \left( B + \frac{\nu}{\mu} \sum_{r=1}^{K} \sum_{\gamma=r}^{K} b_{\gamma} \right) \overline{F}^{(0)}_{r,B}(s) z^r
\]

\[
- z A \left( B + \frac{\nu}{\mu} \sum_{r=1}^{K} \sum_{\gamma=r}^{K} b_{\gamma} \right) \overline{q} \overline{F}^{(0)}_{r,B}(s) + \overline{F}^{(1)}_{r,B}(s) z^r
\]

\[
\overline{G}(z,s) = \frac{A - \frac{\nu}{\mu} c_2 z H(z^{-1}) \left[ A - \frac{\nu}{\mu} c_1 z \left( qH(z^{-1}) + p \right) \right] - \left( \frac{\nu}{\mu} \right)^2 c_1 c_2 z^2 \left( qH(z^{-1}) + p \right) H(z^{-1})}{A - \frac{\nu}{\mu} c_2 z H(z^{-1}) \left[ A - \frac{\nu}{\mu} c_1 z \left( qH(z^{-1}) + p \right) \right] - \left( \frac{\nu}{\mu} \right)^2 c_1 c_2 z^2 \left( qH(z^{-1}) + p \right) H(z^{-1})}
\]

, \( \rho = \lambda/\mu < 1; |z| \leq 1 \) \hspace{1cm} (2B.34)

The denominator of (2B.34) is same as that of (2B.17), (2B.18), (2B.19), (2B.20) and (2B.21). Numerator of (2B.34) must vanish for those \( 2K \) zeroes of the denominator \( D \) in (2B.17), (2B.18), (2B.19), (2B.20) and (2B.21). These \( 2K \) equations determine the \( 2K \) unknowns \( \overline{F}^{(0)}_{r,B}(s) \) and \( \overline{F}^{(1)}_{r,B}(s) \) for \( r=1, 2...K \). Using LAPLACE inverse of \( \overline{F}^{(0)}_{r,B}(s) \) and \( \overline{F}^{(1)}_{r,B}(s) \) (\( r=1,2,...K \)) in (2B.31) and (2B.33), \( \frac{d}{dt} \left( p_{0,B}^{(0)}(t) + Q_{0,B}^{(0)}(t) \right) \) can be obtained.
Special Case

When the departures are not in batches but only single unit departs then putting $b_\gamma = 1$ when

$\gamma = 1$ and $b_\gamma = 0$ otherwise in (2B.34), we get

$$
\overline{G}(z, s) = \frac{\frac{z^2}{\mu} \left[ B + \frac{\nu}{\mu} \right] \left[ A + \frac{\nu}{\mu} c_2 p(z-1) \right] - \frac{\nu}{\mu} c_1 p^2 z^2 (z-1) \left[ B + \frac{\nu}{\mu} \right] \overline{P}_{1,0}(s)}{zA\left[ B + \frac{\nu}{\mu} \right] \left\{ q\overline{P}_{1,0}(s) + \overline{P}_{1,1}(s) \right\} - \left( A - \frac{\nu}{\mu} c_2 \right) \left[ A - \frac{\nu}{\mu} c_1 (q + pz) \right] - \left( \frac{\nu}{\mu} \right)^2 c_1 c_2 (q + pz) } , \rho = \lambda/\mu \leq 1; \ |z| \leq 1
$$

(2B.35)

The result coincides with that of model A of this chapter.
NUMERICAL SOLUTION AND GRAPHICAL REPRESENTATION

The numerical results are generated using MATLAB programming following the work of Bunday (1986). Various probabilities are listed in Table 2B.1 for the case when

\[
\rho \left( \frac{\lambda}{\mu} \right) = 0.3, \eta \left( \frac{\nu}{\mu} \right) = 0.7, \eta_0 \left( \frac{v_0}{\mu} \right) = 0.5, K = 3, c_1 = 0.8, q = 0.75
\]

<p>| ( t ) | ( Q_{0,B}^{(0)} ) | ( P_{0,F}^{(0)} ) | ( P_{1,B}^{(0)} ) | ( P_{1,F}^{(0)} ) | ( P_{1,B}^{(1)} ) | ( P_{1,F}^{(1)} ) | ( P_{2,B}^{(0)} ) | ( P_{2,F}^{(0)} ) | ( P_{2,B}^{(1)} ) | ( P_{2,F}^{(1)} ) | ( P_{3,B}^{(0)} ) | ( P_{3,F}^{(0)} ) | ( P_{3,B}^{(1)} ) | ( P_{3,F}^{(1)} ) | ( P_{4,B}^{(0)} ) | ( P_{4,F}^{(0)} ) | ( P_{4,B}^{(1)} ) | ( P_{4,F}^{(1)} ) |
| 0 | 0.000 | 1.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| 1 | 0.294 | 0.469 | 0.072 | 0.001 | 0.124 | 0.006 | 0.010 | 0.000 | 0.018 | 0.001 | 0.001 | 0.000 | 0.002 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| 2 | 0.369 | 0.276 | 0.126 | 0.006 | 0.115 | 0.021 | 0.032 | 0.002 | 0.032 | 0.006 | 0.006 | 0.000 | 0.006 | 0.001 | 0.001 | 0.000 | 0.000 | 0.000 |
| 3 | 0.376 | 0.215 | 0.138 | 0.012 | 0.096 | 0.031 | 0.045 | 0.006 | 0.036 | 0.012 | 0.012 | 0.002 | 0.010 | 0.003 | 0.003 | 0.000 | 0.002 | 0.001 |
| 4 | 0.367 | 0.196 | 0.135 | 0.017 | 0.086 | 0.036 | 0.049 | 0.010 | 0.037 | 0.017 | 0.016 | 0.004 | 0.013 | 0.005 | 0.004 | 0.001 | 0.004 | 0.001 |
| 5 | 0.355 | 0.191 | 0.130 | 0.019 | 0.083 | 0.038 | 0.050 | 0.013 | 0.037 | 0.020 | 0.018 | 0.005 | 0.015 | 0.007 | 0.006 | 0.002 | 0.005 | 0.002 |
| 6 | 0.345 | 0.188 | 0.126 | 0.020 | 0.082 | 0.038 | 0.050 | 0.015 | 0.038 | 0.021 | 0.020 | 0.007 | 0.017 | 0.008 | 0.007 | 0.002 | 0.006 | 0.002 |
| 7 | 0.336 | 0.187 | 0.124 | 0.020 | 0.082 | 0.038 | 0.050 | 0.016 | 0.039 | 0.022 | 0.021 | 0.008 | 0.018 | 0.009 | 0.008 | 0.003 | 0.007 | 0.003 |
| 8 | 0.329 | 0.185 | 0.122 | 0.020 | 0.083 | 0.038 | 0.050 | 0.016 | 0.040 | 0.022 | 0.021 | 0.008 | 0.019 | 0.009 | 0.009 | 0.003 | 0.008 | 0.003 |
| 9 | 0.324 | 0.184 | 0.120 | 0.020 | 0.083 | 0.037 | 0.050 | 0.017 | 0.041 | 0.022 | 0.022 | 0.009 | 0.020 | 0.010 | 0.016 | 0.004 | 0.009 | 0.004 |
| 10 | 0.319 | 0.183 | 0.119 | 0.020 | 0.083 | 0.037 | 0.050 | 0.017 | 0.042 | 0.022 | 0.022 | 0.009 | 0.021 | 0.010 | 0.010 | 0.004 | 0.010 | 0.004 |
| 11 | 0.315 | 0.182 | 0.118 | 0.020 | 0.083 | 0.037 | 0.050 | 0.017 | 0.042 | 0.022 | 0.023 | 0.009 | 0.022 | 0.010 | 0.011 | 0.004 | 0.011 | 0.004 |
| 12 | 0.312 | 0.181 | 0.117 | 0.020 | 0.083 | 0.037 | 0.050 | 0.017 | 0.043 | 0.023 | 0.023 | 0.009 | 0.022 | 0.010 | 0.011 | 0.004 | 0.011 | 0.004 |
| 13 | 0.309 | 0.180 | 0.117 | 0.020 | 0.083 | 0.037 | 0.050 | 0.017 | 0.043 | 0.023 | 0.024 | 0.010 | 0.023 | 0.011 | 0.011 | 0.004 | 0.012 | 0.004 |
| 14 | 0.306 | 0.179 | 0.116 | 0.020 | 0.083 | 0.036 | 0.050 | 0.017 | 0.044 | 0.023 | 0.024 | 0.010 | 0.023 | 0.011 | 0.012 | 0.005 | 0.012 | 0.005 |
| 15 | 0.304 | 0.179 | 0.115 | 0.020 | 0.083 | 0.036 | 0.050 | 0.017 | 0.044 | 0.023 | 0.024 | 0.010 | 0.024 | 0.011 | 0.012 | 0.005 | 0.013 | 0.005 |
| 16 | 0.302 | 0.178 | 0.115 | 0.020 | 0.083 | 0.036 | 0.050 | 0.017 | 0.044 | 0.023 | 0.024 | 0.010 | 0.024 | 0.011 | 0.012 | 0.005 | 0.013 | 0.005 |</p>
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In Figs. 2B.1 to 2B.8, various probabilities are plotted Vs time for the data from Table 2B.1. Fig. 2B.1 shows plot of probabilities $Q_{0,B}^{(0)}$ and $P_{0,F}^{(0)}$ with respect to time (average service times). It is clear from the graph that probability $Q_{0,B}^{(0)}$ increases rapidly in the starting moments, then decreases slowly and then becomes almost steady from the initial value zero at time $t=0$. $P_{0,F}^{(0)}$ decreases rapidly in the starting moments and then becomes almost steady from the initial value one at time $t=0$.

![Probabilities $Q_{0,B}^{(0)}$ and $P_{0,F}^{(0)}$ Vs time](image)

Fig. 2B.2 shows that probabilities $P_{1,B}^{(0)}$, $P_{1,B}^{(1)}$, $P_{1,F}^{(0)}$ and $P_{1,F}^{(1)}$ also behave in the manner similar to $Q_{0,B}^{(0)}$. In Fig. 2B.3 probabilities $P_{2,B}^{(0)}$, $P_{2,B}^{(1)}$, $P_{2,F}^{(0)}$ and $P_{2,F}^{(1)}$ are plotted versus time. All these four probabilities firstly increase and then finally settle down to some steady values.
Probabilities $P_{1,B}^{(0)}$, $P_{1,B}^{(1)}$, $P_{1,F}^{(0)}$ and $P_{1,F}^{(1)}$ Vs time

$\rho = 0.3, \eta = 0.7, \eta_a = 0.5, K = 3, c_i = 0.8, q = 0.75$

Fig. 2B.2

Probabilities $P_{2,B}^{(0)}$, $P_{2,B}^{(1)}$, $P_{2,F}^{(0)}$ and $P_{2,F}^{(1)}$ Vs time

$\rho = 0.3, \eta = 0.7, \eta_a = 0.5, K = 3, c_i = 0.8, q = 0.75$

Fig. 2B.3
Fig. 2B.4 shows that the probabilities $P_{3,B}^{(0)}$, $P_{3,B}^{(1)}$, $P_{3,F}^{(0)}$, $P_{3,F}^{(1)}$, $P_{4,B}^{(0)}$, $P_{4,B}^{(1)}$, $P_{4,F}^{(0)}$ and $P_{4,F}^{(1)}$ increase rapidly for some time and then increase gradually for some more time and finally these become almost constant for higher values of $t$.

![Graph showing probabilities vs time](image)

Comparison among four probabilities when server is busy and the unit at the head of the queue is to join the server for the first time i.e. among $P_{1,B}^{(0)}$, $P_{2,B}^{(0)}$, $P_{3,B}^{(0)}$ and $P_{4,B}^{(0)}$ is done through Fig. 2B.5. It is apparent that probability decreases as $n$ (number of units in the system) increases for the case under study. It is also seen that all the four probabilities increase rapidly in the starting moments, after rapid increase probability $P_{1,B}^{(0)}$ decreases to some extent and all show slight variability before finally approaching to some steady
values. Fig. 2B.6 shows nearly similar type of relationship among probabilities $P_{1,B}^{(1)}$, $P_{2,B}^{(1)}$, $P_{3,B}^{(1)}$ and $P_{4,B}^{(1)}$ as shown among $P_{1,B}^{(0)}$, $P_{2,B}^{(0)}$, $P_{3,B}^{(0)}$ and $P_{4,B}^{(0)}$ through Fig. 2B.5.

---

**Fig. 2B.5**

Probabilities $P_{1,B}^{(0)}$, $P_{2,B}^{(0)}$, $P_{3,B}^{(0)}$ and $P_{4,B}^{(0)}$ Vs time

$\rho = 0.3, \eta = 0.7, \eta_h = 0.5, K = 3, c_1 = 0.8, q = 0.75$
Fig. 2B.7 illustrates the relationship among $P_{1,F}^{(0)}$, $P_{2,F}^{(0)}$, $P_{3,F}^{(0)}$ and $P_{4,F}^{(0)}$. It shows that probability decreases as $n$ (number of units in the system) increases. It is also seen that $P_{3,F}^{(0)}$ increases rapidly in the starting moments, then decreases sharply to some extent and then show slight variability before finally approaching to a steady value. Other three probabilities increases for some time and finally attain some constant values for higher values of $t$. Fig. 2B.8 shows similar type of relationship among probabilities $P_{1,F}^{(1)}$, $P_{2,F}^{(1)}$, $P_{3,F}^{(1)}$ and $P_{4,F}^{(1)}$ when plotted against time as is shown by Fig. 2B.7 among $P_{1,F}^{(0)}$, $P_{2,F}^{(0)}$, $P_{3,F}^{(0)}$ and $P_{4,F}^{(0)}$ with a difference that decrease in $P_{1,F}^{(1)}$ is not that much pronounced as is there in case of $P_{1,F}^{(0)}$ (Fig. 2B.7).
Probabilities $P_{i,F}^n(0)$, $P_{i,F}^n(1)$, $P_{i,F}^n(2)$ and $P_{i,F}^n(3)$ Vs time

$\rho = 0.3, \eta = 0.7, \eta_0 = 0.5, K = 3, c_i = 0.8, q = 0.75$

Fig. 2B.7

Probabilities $P_{i,F}^n(0)$, $P_{i,F}^n(1)$, $P_{i,F}^n(2)$ and $P_{i,F}^n(3)$ Vs time

$\rho = 0.3, \eta = 0.7, \eta_0 = 0.5, K = 3, c_i = 0.8, q = 0.75$

Fig. 2B.8
Various probabilities are listed in Table 2B.2 for the case when

\[
\rho = \frac{\lambda}{\mu} = 0.8, \eta = \frac{V}{\mu} = 0.7, \eta_0 = \frac{V_0}{\mu} = 0.5, K = 3, c_1 = 0.8, q = 0.75
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Some of the graphs plotted for the data of Table 2B.1 are also plotted for the data of Table 2B.2 to study the behavior of various probabilities when $\rho$ takes a comparatively high value. In Figs. 2B.9 to 2B.13, various probabilities are plotted Vs time for the data from Table 2B.2.
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Fig. 2B.9

Probabilities $Q_{0,B}^{(0)}$ and $P_{0,F}^{(0)}$ Vs time

Fig. 2B.10

Probabilities $P_{0,B}^{(0)}$, $P_{0,F}^{(0)}$, $P_{1,B}^{(0)}$, and $P_{1,F}^{(0)}$ Vs time
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Probabilities $p_{3,B}^0$, $p_{3,B}^1$, $p_{3,F}^0$ and $p_{3,F}^1$ Vs time

\[ \rho = 0.8, \eta = 0.7, \eta_i = 0.5, k = 3, \epsilon = 0.8, q = 0.75 \]

Fig. 2B.11

Probabilities $p_{1,B}^0$, $p_{2,B}^0$, $p_{3,B}^0$ and $p_{4,B}^0$ Vs time

\[ \rho = 0.8, \eta = 0.7, \eta_i = 0.5, k = 3, \epsilon = 0.8, q = 0.75 \]

Fig. 2B.12
To study the effect of various parameters on different probabilities of the model, the data of various probabilities is generated for different values of $\rho$ keeping other parameters constant. The set of values that $\rho$ took is \{0.2, 0.3, 0.5, 0.8\}. The other parameters were fixed at $\eta = 0.7$, $\eta_0 = 0.5$, $K = 3$, $c_1 = 0.8$, $q = 0.75$. In Fig. 2B.18 the probability $Q_{0,B}^{(0)}$ is plotted against time for different values of $\rho$. From the figure it is concluded that as $\rho$ increases $Q_{0,B}^{(0)}$ decreases. So more the traffic intensity i.e. more customers are arriving per unit average service time less is the probability of server being busy with the empty queue.
Fig. 2B.15 shows plot of probability $P_0^{(0)}$ with respect to time (average service times) for the set of $\rho$ values. It is clear from the graph that in all the four cases probability $P_0^{(0)}$ decreases rapidly in the starting moments and then becomes almost steady for higher values of $t$. The figure shows that more the traffic intensity less is the probability of zero units in the system.

Behavior of the probabilities $P_{1,B}^{(0)}$, $P_{2,B}^{(0)}$, and $P_{3,B}^{(0)}$ with respect to time and changing $\rho$ is apparent from Figs. 2B.16 to 2B.18.
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Figure 2B.15

$P_{0,F}^{(0)}$ Vs time for different values of traffic intensity $\rho(=\lambda/\mu)$

Figure 2B.16

$P_{1,B}^{(0)}$ Vs time for different values of traffic intensity $\rho(=\lambda/\mu)$
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$P_{3,B}^{(0)}$ Vs time for different values of traffic intensity $\rho(=\lambda/\mu)$

![Graph of $P_{3,B}^{(0)}$](image)

Fig. 2B.17

$P_{4,B}^{(0)}$ Vs time for different values of traffic intensity $\rho(=\lambda/\mu)$

![Graph of $P_{4,B}^{(0)}$](image)

Fig. 2B.18
Study of the effect of changing $\eta$ and $\eta_0$ values is done by generating data of various probabilities for different values of $\eta$ and $\eta_0$ keeping other parameters constant. The set of values that $\eta$ and $\eta_0$ took is $\{(0.3, 0.2), (0.7, 0.5), (0.9, 0.7)\}$. The other parameters were fixed at $\rho = 0.5, K = 3, c_1 = 0.8, q = 0.75$. In Fig. 2B.19 the probability $Q_{0,B}^{(0)}$ is plotted against time for different values of $\eta$ and $\eta_0$. From the figure it is clear that as $\eta$ and $\eta_0$ increases $Q_{0,B}^{(0)}$ also increases. So more is the availability rate per unit average service time, more is the probability of server being busy with the empty queue.

Fig. 2B.20 shows probability plot of $P_{0,F}^{(0)}$ with respect to $t$ (average service times) for a set of $\eta$ and $\eta_0$ values. It is clear from the graph that in all the three cases probability $P_{0,F}^{(0)}$ decreases rapidly in the starting moments and then becomes almost steady for higher values of $t$. The figure shows that in the starting moments more the value of $\eta$ and $\eta_0$ less is the probability $P_{0,F}^{(0)}$ which shows that due to high availability server remains in busy state most of the times. After some time the trend reverses which shows that in cases of higher value of server availability rate, $n=0$ state is achieved more often than the cases when availability is comparatively low. Fig 2B.21 shows that probability $P_{1,B}^{(0)}$ increases with increasing $\eta$ and $\eta_0$. 
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**Fig. 2B.19**

**$Q_{0,B}^{(0)}$ Vs time for different values of $\eta = \frac{v}{\mu}$**

- $\rho = 0.5, K = 3, c_1 = 0.8, q = 0.75$

**Fig. 2B.20**

**$P_{0,F}^{(0)}$ Vs time for different values of $\eta = \frac{v}{\mu}$**

- $\rho = 0.5, K = 3, c_1 = 0.8, q = 0.75$
Effect of maximum batch size $K$ on $Q_{0,B}^{(0)}$ and $P_{0,F}^{(0)}$ is studied by plotting these probabilities against time for different $K$. Probability plot of $Q_{0,B}^{(0)}$ against $t$ (average service times) for $K=1$ and $K=3$ is shown by Fig. 2B.22. It is interpreted from the figure that probability $Q_{0,B}^{(0)}$ take almost same values for different $K$ in the starting moments. Afterwards it becomes more for high value of $K$ which can be explained as $n=0$ state is more probable when maximum capacity of the server is high.

From Fig. 2B.23 which is probability plot of $P_{0,F}^{(0)}$ against time for $K=1$ and $K=3$, it is shown that probability $P_{0,F}^{(0)}$ also take almost same values for different $K$ in the starting moments. Afterwards it becomes more for high value of $K$ which can also be explained as $n=0$ state is more probable when maximum capacity of the server is high.
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**Figure 2B.22**

$Q_{0,B}^{(0)}$ vs time for different values of maximum batch size $K$

- $\rho = 0.2, \eta = 0.7, \eta_o = 0.5, c_i = 0.8, q = 0.75$

**Figure 2B.23**

$P_{0,F}^{(0)}$ vs time for different values of maximum batch size $K$

- $\rho = 0.2, \eta = 0.7, \eta_o = 0.5, c_i = 0.8, q = 0.75$
After analyzing the graphs plotted for different probabilities it is interpreted that all the probabilities tend to attain some steady values for high values of time $t$ (average service times).