CHAPTER 5: RESEARCH METHODOLOGY

5.1 ALGORITHM FOR THESIS WORK

Start

- Literature Review
- Research objective
- Selection of Problem
- Finite Element Analysis
- Calculation of Stiffness
- Calculation of Stress
- Software Development
- Result & Discussion
- Conclusion
- References
5.2 ALGORITHM OF SOFTWARE

Start

Data Input:

Number Of Material

Name Of Material

Poisson’s ratio

Modulus of Elasticity

Length Of Beam

Width Of Beam

Depth Of Beam

Type Of Beam

Type Of Load

Load Value
1. Discritization of Depth of The Beam
2. Calculation of Modulus of Elasticity(E) for Composite Beam
3. Calculation of Possion’s ratio (µ) for Composite Beam
4. Computation of Stiffness
5. Calculation of \( \sum EI \) for Composite Beam
6. Computation of Deflection of Beam
7. Computation of Stress
8. Proceed for Next Element
9. Result Analysis/Checking
   - If Yes Print Result
   - Result
5.3 CONCEPT OF METHODS AVAILABLE

For problems involving complex shapes, material properties & complicated boundary conditions, it is difficult & in many cases intractable to obtain analytical solution that satisfies the governing differential equation. Hence for most of the practical problems the engineer resorts to numerical methods that provides approximate but acceptable solution. The three methods that are used are as follows-

1. Functional Approximation
2. Finite difference method
3. Finite element method

In Functional Approximation Raly-Ritz, Galerkin & collocation method are used which are based on the trial & error method so the calculated result is not more accurate while in finite difference method the original body or the system is discritized by a mesh of nodal points so the calculated result is accurate from the theoretical point of view but not from practical point of view. The calculated result from this method is more accurate as compared to other method.

5.3.1 WHY FINITE ELEMENT ANALYSIS

FEM gives more accurate result as compaired to other method. By using FEM, the analysis of any part or element can be done, because analysis of any part or element can be done before manufacturing so it will save to the manufacturing cost. By using this method the failure of any component can also be protected. it will also save to time and Material.

5.4 FINITE ELEMENT ANALYSIS BASIC STEPS -

Basically the steps of the Finite element analysis will be discussed here -

5.4.1 DISCRETIZATION & PRE –PROCESSING OF FINITE ELEMENT MODEL

As a first step in the analysis [97], the given solid or structure is to be described into finite elements. The steps requires knowledge of the physical behaviour of the solid or structure to decide on the type of analysis and elements to be used to arrive at the finite element model. In addition decision has to be made in the shape of elements to be used (higher or lower order elements), the number of elements and the pattern of the finite element mesh.
After the Discretization the nodes are numbered keeping in view the minimum bandwidth requirement discussed. Graphics based pre-processor are available in many package programs to automatically generate the mesh & number the nodes & elements.

5.4.2 COMPUTATION OF ELEMENT PROPERTIES
The stress-displacement matrix [B], element stiffness matrix [k] & nodal load vector {Q} are computed for each element.

5.4.3 ASSEMBLAGE OF ELEMENTS
The direct stiffness method is used to constitute the structure or global stiffness matrix [K] & nodal load vector {P}. The connectivity relation between element & global degree of freedom is used to compute the contribution from an element to the global stiffness matrix {K}.

5.4.4 SOLUTION OF EQUATION OF EQUILIBRIUM
The linear simultaneous equation of equilibrium [K] {r}={P} are solved for the nodal displacements of the structure or solid. Gauss elimination procedure or the variation of it such as the [L] [D] [L]T decomposition described is used for this purpose. The solution for the Nodal displacement for the cantilever beam, with five element is given below-

\[
\{r\}^T = \{-0.002,-0.00267,0.002,-0.00267,-0.004,-
-0.01067,0.004,-0.01067,-0.006,-0.024,0.006,-0.024,-0.008,-0.04267,0.008,-0.04267,-
-0.010,-0.06667,0.010,-0.06667\}
\]

5.4.5. COMPUTATION OF STRESS & POST-PROCESSING OF RESULTS
The stresses at any point in the element can be computed using the equation \{\sigma\} = [C] [B] \{d\}. Although one would like to get the stresses evaluated at the nodal points, they appear to be the worst sampling points. Barlow shown that for two dimensional isoparametric elements the Gauss points are the optimal sampling points for stress computation.

5.5 NATURAL CO-ORDINATE SYSTEM
In finite element formulation, the natural coordinate system has been found to be quite effective in formulating the element properties. The natural coordinate system is a local system in which a point within an element will be expressed by dimensionless set of numbers whose magnitude never exceeds unity. Moreover, these system will be so defined that the nodal points will have unit magnitude will be so defined that the nodal point will have unit magnitude or zero, or a convenient set of fractions. This type of expressing the coordinates also facilitates the integration to compute to compute element stiffness.

5.6 SHAPE FUNCTION (INTERPOLATION FUNCTION)

In finite element analysis using the displacement model, we assume the variation of displacement within an element. In general, in higher mathematics, it is necessary in many situation to deal with function whose analytical form is either total unknown or else is of such a nature that the function cannot easily be subjected to such operation as may be required. In either case, it is desirable to replace the given function by another function which can be more easily handled. This operation of replacing or representing a given function by simpler one is known as interpolation in a broad sense. In finite element literature it is also referred to as ‘shape function’. There are two type of interpolation function (i) Lagrange interpolation and (ii) hermitian interpolated. In the Lagrange interpolation, which is widely used in practice, the assumed function takes on the same values as the given function at specified point. In the hermitian type of function, the slope of the function at function also takes the same value as the given function at specified points. In this section attention will be restricted to Lagrange interpolation or shape function.

5.6.1 SHAPE FUNCTION FOR SECOND ORDER RECTANGULAR ELEMENT

In natural coordinates, the variation of displacement can be expressed by

\[ u = a_1 + a_2 r + a_3 s + a_4 r^2 + a_5 rs + a_6 s^2 + a_7 r^2 s + a_8 r s^2 \]  

(5.1)

In the above expression the cubic terms \( r^3 \) and \( s^3 \) are omitted and ‘geometric invariance’, described in sec.3.3.2, is maintained by the above choice of the terms.

The nodal displacement \( \{u\} \) can be obtained by substituting the coordinates for the nodes as

\[
\{K_u\} = \begin{bmatrix}
1 & -1 & -1 & 1 & 1 & -1 & 1 & 1 & -1 & -1
1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & 1
\end{bmatrix} \begin{bmatrix}
\alpha_1 \\
\alpha_2 \\
\end{bmatrix}
\]
\[\{ \alpha \} = [A]^{-1}\{d_0\}\]  \hspace{1cm} (5.2)

Where \([A]^{-1}\) can be shown to be equal be to

\[
\begin{pmatrix}
-1 & -1 & -1 & -1 & 2 & 2 & 2 & 2 \\
0 & 0 & 0 & 0 & 0 & 2 & 0 & -2 \\
0 & 0 & 0 & 0 & -2 & 0 & 2 & 0 \\
1 & 1 & 1 & 1 & -2 & 0 & -2 & 0 \\
1 & -1 & 1 & -1 & 0 & 0 & 0 & 0
\end{pmatrix}
\]  \hspace{1cm} (5.3)

Thus,

\[
\{N_2\}^T = \{\phi_2\}^T [A]^{-1}
\]

\[
\{\phi_2\}^T = [1 \ r \ s \ r^2 \ rs \ s^2 \ r^2 s \ rs^2]
\]  \hspace{1cm} (5.4)

And \([A]^{-1}\) is given by Eq.3.57
\{N_2\} = \begin{bmatrix} 1/4(1-r) (1-s) (-r-s-1), & 1/4(1+r) (1-s) (r-s-1) \\ 1/4 (1+r) (1+s) (r+s-1), & 1/4 (1-r) (1+s) (-r+s-1) \\ 1/2(1+r) (1-r) (1-s), & 1/2(1+r) (1+s) (1-s) \\ 1/2(1+r) (1-r) (1+s), & 1/2(1+r) (1+s) (1-s) \end{bmatrix}

(5.5)

Or can be expressed in concise form as
\{N_2\} = [N_1 \ N_2 \ N_3 \ N_4 \ N_5 \ N_6 \ N_7 \ N_8]

(5.6)

Where each element of is given in Eq.3.59 (a).

The shape function for the element is thus given by

\[
[N] = \begin{bmatrix} \{N_2\}^T & \{0\} \\ 0 & \{N_2\}^T \end{bmatrix}
\]

5.7 ISOPARAMETRIC ELEMENT

For the analysis of the structural problems of complex shapes involving curved boundaries or surfaces, simple triangular or rectangular elements are no longer sufficient. This has led to the development of elements of more arbitrary shape & are called Isoparametric Elements. These elements are widely used in two three dimensional stress analysis & plates & shell problems.

The concept of isoparametric element is based on the transformation of the parent element in local or natural coordinate system an arbitrary shape in the Cartesian coordinate system. A convenient way of expressing the transformation is to make use of the shape functions of the rectilinear elements in their natural coordinate system & the nodal values of the coordinates. Thus the Cartesian coordinates of a point in an element may be expressed as-
\[ x = N'_1 x_1 + N'_2 x_2 + \ldots + N'_n x_n \]
\[ y = N'_1 y_1 + N'_2 y_2 + \ldots + N'_n y_n \]
\[ z = N'_1 z_1 + N'_2 z_2 + \ldots + N'_n z_n \]
\[ \{ x \} = [N'] \{ x_n \} \]

The shape function \([N']\) used in the above transformation thus help us to define the geometry of the element in the Cartesian coordinate system. If these shape function \([N']\) are the same as the shape functions \([N]\) used to represent the variation of displacement in the element, these elements are called ‘isoparametric’ element.

\[ \{ x \} = [N] \{ x_n \} \]

And in case where the geometry of the element is defined by shape functions of order higher than that for representing the variation of displacements, the elements are called ‘superparametric’. Similarly if more nodes are used to define displacement compared to the nodes used to represent the geometry of the elements then they would be referred to as ‘subparametric’ elements.

**5.8 FOUR-NODED TWO-DIMENSIONAL ELEMENT**

Consider a quadrilateral two dimensional element the parent element is a rectangle mapped into a square in natural coordinates & This in turn is transformed into an arbitrary quadrilateral element with straight boundaries.

\[
\begin{pmatrix}
    x \\
    y
\end{pmatrix} =
\begin{bmatrix}
    N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\
    0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4
\end{bmatrix}
\begin{pmatrix}
    x_1 \\
    y_1 \\
    x_2 \\
    y_2 \\
    x_3 \\
    y_3 \\
    x_4 \\
    y_4
\end{pmatrix}
\]
Where \( N_i \) (i= 1,2,3,4) are given by equation –

\[
\begin{align*}
N_1 &= (1-r) \frac{(1-s)}{4} \\
N_2 &= (1+r) \frac{(1+s)}{4} \\
N_3 &= (1+r) \frac{(1-s)}{4} \\
N_4 &= (1-r) \frac{(1+s)}{4}
\end{align*}
\]

Thus this transformation relates a unit square in \( r \) & \( s \) coordinates to an arbitrary quadrilateral in Cartesian (\( x, y \)) coordinates system whose shape & size are determined by the eight nodal coordinates \( x_1, y_1, x_2, y_2, x_3, y_3, x_4, y_4 \).

\[
\begin{bmatrix}
\delta/\delta r \\
\delta/\delta s
\end{bmatrix} =
\begin{bmatrix}
\delta x/\delta r & \delta y/\delta r & \delta/\delta x \\
\delta x/\delta s & \delta y/\delta s & \delta/\delta y
\end{bmatrix}
\begin{bmatrix}
\delta/\delta x \\
\delta/\delta y
\end{bmatrix}
= [J]
\]

where \([J]\) is the Jacobian matrix. Hence the derivatives with respect to Cartesian coordinate system can be given as-

\[
\begin{bmatrix}
\delta/\delta x \\
\delta/\delta y
\end{bmatrix} = [J]^{-1}
\begin{bmatrix}
\delta/\delta r \\
\delta/\delta s
\end{bmatrix}
\]

\[
x = \sum_{i=1}^{4} N_i x_i \quad \& \quad y = \sum_{i=1}^{4} N_i y_i
\]

& noting that \( N_i \) is a function in \( r, s \) the Jacobian \([J]\) can be evaluated as

\[
[J] =
\begin{bmatrix}
\delta N_1/\delta r & \delta N_2/\delta r & \delta N_3/\delta r & \delta N_4/\delta r \\
\delta N_1/\delta s & \delta N_2/\delta s & \delta N_3/\delta s & \delta N_4/\delta s \\
x_1 & y_1
\end{bmatrix}
\]
\[
\begin{bmatrix}
\frac{\delta N_1}{\delta s} & \frac{\delta N_2}{\delta s} & \frac{\delta N_3}{\delta s} & \frac{\delta N_4}{\delta s} & x_2 & y_2 \\
& & & & x_3 & y_3 \\
& & & & x_4 & y_4 \\
\end{bmatrix}
\]

Let the inverse of \([J]\) is required-

\[
[J] = \begin{bmatrix}
-(1-s)/4 & +(1-s)/4 & (1+s)/4 & -(1+s)/4 \\
-(1-s)/4 & -(1+s)/4 & (1+s)/4 & +(1-s)/4 \\
\end{bmatrix}
\begin{bmatrix}
x_1 & y_1 \\
x_2 & y_2 \\
x_3 & y_3 \\
x_4 & y_4 \\
\end{bmatrix}
\]
\[ [J]^{-1} = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix} \]

It should be observed here that in order to transform the x \& y coordinates into r \& s coordinates the inverse of [J] must exist. Hence the determinant of the Jacobian [J] must be non zero at every point of (r,s).

\[ [J] = \text{det} [J] = \frac{\delta x}{\delta r} \frac{\delta y}{\delta s} - \frac{\delta x}{\delta s} \frac{\delta y}{\delta r} \neq 0 \]

\[
\begin{pmatrix}
\frac{\delta u}{\delta x} \\
\frac{\delta u}{\delta y} \\
\frac{\delta v}{\delta x} \\
\frac{\delta v}{\delta y}
\end{pmatrix} =
\begin{pmatrix}
J_{11} & J_{12} & 0 & 0 \\
J_{21} & J_{22} & 0 & 0 \\
0 & 0 & J_{11} & J_{12} \\
0 & 0 & J_{21} & J_{22}
\end{pmatrix}
\begin{pmatrix}
\frac{\delta u}{\delta r} \\
\frac{\delta u}{\delta s} \\
\frac{\delta v}{\delta r} \\
\frac{\delta v}{\delta s}
\end{pmatrix}
\]

\[
\{ \varepsilon \} = \begin{pmatrix}
\varepsilon_x \\
\varepsilon_y \\
\gamma_{xy}
\end{pmatrix} =
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 1 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
\frac{\delta u}{\delta x} \\
\frac{\delta u}{\delta y} \\
\frac{\delta v}{\delta x} \\
\frac{\delta v}{\delta y}
\end{pmatrix}
\]
\{\varepsilon\} = \begin{bmatrix} J_{11} & J_{12} & 0 & 0 \\ 0 & 0 & J_{21} & J_{22} \\ J_{21} & J_{22} & J_{11} & J_{12} \end{bmatrix} \begin{bmatrix} \delta u/\delta r \\ \delta u/\delta s \\ \delta v/\delta r \\ \delta v/\delta s \end{bmatrix}

u = \sum_{i=1}^{4} N_i u_i \quad \& \quad v = \sum_{i=1}^{4} N_i v_i

\begin{bmatrix} \delta u/\delta r \\ \delta u/\delta s \\ \delta v/\delta r \\ \delta v/\delta s \end{bmatrix} = \begin{bmatrix} \delta N_1/\delta r & 0 & \delta N_2/\delta r & 0 & \delta N_3/\delta r & 0 & \delta N_4/\delta r & 0 \\ \delta N_1/\delta s & 0 & \delta N_2/\delta s & 0 & \delta N_3/\delta s & 0 & \delta N_4/\delta s & 0 \\ 0 & \delta N_1/\delta r & 0 & \delta N_2/\delta r & 0 & \delta N_3/\delta r & 0 & \delta N_4/\delta r \\ 0 & \delta N_1/\delta s & 0 & \delta N_2/\delta s & 0 & \delta N_3/\delta s & 0 & \delta N_4/\delta s \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{bmatrix}

[k] = \iint [B]^T [C] [B] \, dx \, dy \, dz

In the case of the present example of the two dimensional element it is given by-

\begin{align*}
[k] &= h \iint [B]^T [C] [B] \, dx \, dy \, dz \\
\text{Where } h \text{ is the thickness of the element. It can be shown that the elemental area in the } \\
\text{Cartesian coordinates (x-y) can be expressed in terms of the area in the local coordinates (r-s) as-}
\end{align*}
\[ dx \, dy = [J] \, drds \]

\[ [k] = h \int \left[ B \right]^T [C] [B] [J] \, drds \]

\[ [k] = h \, w_i \, w_j \left[ B \right]^T [C] [B] [J] \]

Where \( K \) = stiffness value

\( w_i, w_j \) = Gauss weights

\( h \) = Thickness of the Element

\( B \) = Strain-Displacement Matrix

\( J \) = Jacobian

5.9 CONSTITUTIVE MATRIX

To determine the stresses in the members of a structure or in a deformable solid it is necessary to know the components of stress as a function of the components of strain & vice versa. We assume that material is elastic & obey’s the hook’s law According to hook’s law the six components of stress may be expressed as a linear function of six component of strain.

\[
\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\sigma_z \\
\tau_{xy} \\
\tau_{yz} \\
\tau_{zx}
\end{bmatrix}
= 
\begin{bmatrix}
C_{11} & C_{12} & \ldots & C_{16} \\
C_{21} & C_{22} & \ldots & C_{26} \\
C_{31} & C_{32} & \ldots & C_{36} \\
C_{41} & C_{42} & \ldots & C_{46} \\
C_{51} & C_{52} & \ldots & C_{56} \\
C_{61} & C_{62} & \ldots & C_{66}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\varepsilon_z \\
\gamma_{xy} \\
\gamma_{yz} \\
\gamma_{zx}
\end{bmatrix}
\]

So

\[
C = \frac{E}{(1+\mu)(1-2\mu)} \begin{bmatrix}
(1-\mu) & \mu & 0 \\
\mu & (1-\mu) & 0 \\
0 & 0 & 0
\end{bmatrix}
\]
Solid or three-dimensional elements enable the solution of problems for a general three-dimensional stress analysis. There are many problems such as concrete dams, stress distribution in soil & rocks, ring beams, pressure vessels, pipe intersections, stresses around openings, machine components, etc. where three-dimensional stress analysis is required. For such problems Finite Element analysis is required. For such problems finite element analysis provides a powerful tool for getting numerical solution.

Fig. shows typical elements belonging to Tetrahedron, triangular prism & Hexahedron family of elements. A brief description of these elements will be given in this chapter.

The eight-noded isoparametric element is one of the simplest and its performance in situations where bending is involved can be improved by the addition of incompatible modes similar to the quadrilateral elements discussed in this chapter. A detailed derivation of the element properties is presented here. In case of 3D analysis of curved solids, twenty-noded isoparametric elements has been found to be useful and the element properties are described in this chapter.

5.10.1 THREE – DIMENSIONAL SOLID ELEMENTS-

Three-dimensional solid elements can be broadly grouped under tetrahedral, triangular prism & hexahedral family of elements [1, 2, 3, 4].

5.10.2 TETRAHEDRAL ELEMENTS-

The simplest element of the tetrahedral family is a four-noded tetrahedron shown in fig.

\[
\{N_3\}^T = [L_1, L_2, L_3, L_4]
\]

Where \(L_i = V_i / V\)

Figure shows a Ten-noded tetrahedron. The shape function for this element is given by,

\[
\{N_3\}^T = [L_1 (2L - 1) L_2 (2L - 1) L_3 (2L - 1) L_4 (2L - 1)]
\]

The strain variation is linear within the element.
Two other types of eight and twenty noded tetrahedral elements are shown in fig. The main disadvantages of tetrahedral family of elements are:

i) It requires small & costly subdivision &

ii) the division of a space volume into individual tetrahedron sometimes presents difficulties of visualization & could lead to errors in nodal numbering & element connectivity in data preparation.

5.10.3 TRIANGULAR PRISM ISOPARAMETRIC ELEMENTS

The simplest triangular prism element is a six noded element shown in fig. The polynomial function in natural curvilinear coordinates r, s & t describing the geometry & the variation of displacement over the element is

\[ \{ \Phi_3 \}^T = [1 \ r \ s \ t \ rs \ st] \]

The six nodes form a solid bounded by two triangular and three quadrilateral faces and the sides are non-intersecting. This element is compatible with eight noded isoparametric hexahedral element.

When quadratic variation is required, fifteen noded triangular prism can be used. The fifteen nodes form a solid bounded by two curved or straight sided triangular faces and three curved or straight sided quadrilateral faces. Neither the faces nor the edges should intersect each other.

\[ \{ \Phi_3 \}^T = [1 \ r \ s \ t \ rs \ st \ rt \ rst \ r^2 s \ r^2 t \ s^2 t \ rt^2 st^2] \]

This element is compatible with twenty noded isoparametric hexahedral element.

Figure shows a twelve noded triangular prism element similar to the fifteen noded element but with no mid-side nodes along the thickness direction t. The twelve nodes form a solid bounded by two curved or straight sided triangular faces and three quadrilateral faces two edges of which must be straight. The polynomial function defining the geometry and displacement variation is the same as six-noded solid triangular element but contains additional terms in r & s directions, viz.,

\[ \{ \Phi_3 \}^T = [1 \ r \ s \ t \ rs \ st \ rt \ rst \ r^2 \ s^2 \ rt^2 st^2] \]

This element is compatible with the sixteen noded isoparametric elements.

5.10.4 HEXAHEDRAL ISOPARAMETRIC ELEMENTS-

The eight noded element shown in figure is the simplest hexahedral element. The polynomial function defining the geometry & displacement variation is

\[ \{ \Phi_3 \}^T = [1 \ r \ s \ t \ rs \ rt \ rst \ ] \]
The twenty noded element shown in fig. has eight corner nodes & twelve nodes located at the midpoints of the edges thus capable of accommodating curved boundaries.

The polynomial function defining the geometry & variation of displacements is given by,
\[ \{ \Phi_3 \}^T = [1 \ r \ s \ t \ rs \ rt \ rst \ r_s t \ t_s r \ t_r s \ t_s r \ t r s \ r_t s \ r t_s \ r s t] \]

Figure shows a sixteen noded hexahedral element which is similar to the twenty noded element but with no mid-side nodes along the thickness direction t. The sixteen nodes form a solid bounded by six quadrilateral faces, two of each can be curved.

The polynomial function defining the geometry and displacement variation is the same as for the eight noded hexahedral element but contains additional terms in the r & s direction & is given by
\[ \{ \Phi_3 \}^T = [1 \ r \ s \ t \ rs \ st \ rt \ rst \ r_s t \ t_s r \ t_r s \ t_s r \ t r s \ r_t s \ r s t] \]

Apart from the above mentioned elements, there are certain other isoparametric elements that have different number of mid-surface nodes in the curvilinear directions. These elements may be useful for problems where the variation in displacement function in one direction is of a higher order than in other directions. In the subsequent sections the formulation of two elements, eight noded & twenty noded isoparametric elements are presented.

**5.10.5 EIGHT NODED ISOPARAMETRIC SOLID ELEMENT**

The three-dimensional eight noded isoparametric element shown in figure has eight nodes located at the corners.

Thus, the geometry of the element is described as,
\[
\begin{align*}
X & = \sum_{i=1}^{8} N_i x_i \\
Y & = \sum_{i=1}^{8} N_i y_i \\
Z & = \sum_{i=1}^{8} N_i z_i \\
U & = \sum_{i=1}^{8} N_i u_i \\
V & = \sum_{i=1}^{8} N_i v_i \\
W & = \sum_{i=1}^{8} N_i w_i
\end{align*}
\]
The displacement function given by equation above is incomplete, in that it does not contain all quadratic terms in shape functions given by equation. This can be illustrated by subjecting a regular hexahedral element to pure bending moment acting on faces normal to y-axis & in yz plane as shown in figure. Taking the origin of coordinates at the centroid of the element, the stress components by the elementary bending theory, are

\[ \sigma_y = \frac{Ez}{R}, \quad \sigma_x = \sigma_z = \tau_{xy} = \tau_{yz} = \tau_{zx} = 0 \]

Where E is the modulus of the material & z is the distance of the layer from the neutral axis & R is the radius of curvature.

Using the Hooke’s law, the strain components are computed as follows:

\[ \epsilon_y = \frac{\delta v}{\delta y} = \frac{z}{R} \]
\[ \epsilon_x = \frac{\delta u}{\delta x} = -\mu z/R \]
\[ \epsilon_z = \frac{\delta w}{\delta z} = -\mu z/R \]
\[ \delta = \frac{\delta u}{\delta y} + \frac{\delta v}{\delta x} = 0 \]
\[ \delta = \frac{\delta v}{\delta z} + \frac{\delta w}{\delta y} = 0 \]
\[ \delta = \frac{\delta w}{\delta x} + \frac{\delta u}{\delta z} = 0 \]

Where \( \mu \) is the Poisson’s ratio.

Integrating Equations we get

\[ V = zy/R + f_1(x,z) \]
\[ U = -\mu zx/R + g_1(y,z) \]
\[ W = -\frac{\mu z^2}{2R} + h_1(x,y) \]

Where \( f_1, \ g_1 \) & \( h_1 \) are constants of integration.

\[ V = zy/R \]
\[ U = -\mu zx/R \]

From equations &, we get

\[ \frac{\delta w}{\delta y} = -\frac{\delta v}{\delta z} = -\frac{y}{R} \]

Or

\[ W = -\frac{y^2}{2R} + h_2(x,z) \]
Where $h_2$ is a constant of integration
From equation we get
\[ \frac{\delta w}{\delta x} = -\frac{\delta u}{\delta z} = \mu x/R \]
\[ W = \mu x^2/2R + h_3 (y,z) \]
From equations we get
\[ W = -y^2/2R - \mu z^2/2R + \mu x^2/2R + c_1 \]
The constants $c_1$ in Equation is evaluated by using the condition w displacements at the eight corners of the element are zero.
\[ 0 = -b^2/2R - \mu c^2/2R + \mu a^2/2R + c_1 \]
\[ C1 = b^2/2R + \mu c^2/2R - \mu a^2/2R \]
Substituting the above value of $c_1$ into Equation
\[ W = b^2/2R (1 - y^2/b^2) + \mu c^2/2R (1 - z^2/c^2) - \mu a^2/2R (1-x^2/a^2) \]
Generalising, Equations & can be expressed as
\[ U = \alpha_1 xz \]
\[ V = \alpha_2 yz \]
\[ W = \alpha_3 (1 - y^2/b^2) + \alpha_4 (1 - z^2/c^2) + \alpha_5 (1-x^2/a^2) \]
Where $\alpha_1, \ldots, \alpha_5$ are constants which are functions of $\mu$, $R$, $a$, $b$ & $c$.
Similarly, if the element is subject to pure bending moment on faces normal to the x axis, and in the xy plane, the displacement variation can be expressed as,
\[ U = \beta_1 xy \]
\[ V = \beta_2 (1 - y^2/b^2) + \beta_3 (1 - z^2/c^2) + \beta_4 (1-x^2/a^2) \]
\[ W = \beta_5 yz \]
Where $\beta_1, \ldots, \beta_5$ are constants.

When the element is subjected to pure bending moment on faces normal to z- axis and in xz plane, the displacement variation can be expressed as
\[ U = \delta_1 (1 - y^2/b^2) + \delta_2 (1 - z^2/c^2) + \delta_3 (1-x^2/a^2) \]
\[ V = \delta_4 xy \]
\[ W = \delta_5 xz \]
Where $\delta_1, \ldots, \delta_5$ are constants.
When we refer equations these displacement variations are not represented by the shape functions used for the eight noded solid element. This element is able to represent only displacements given by Equations.

Absence of quadratic terms,

\[(1-x^2/a^2), (1- y^2/b^2), (1- z^2/c^2)\] which can be expressed as \[(1- r^2), (1- s^2), (1- t^2)\] by putting \(r = x/a, s = y/b, t = z/c\) for the element shown in figure. is the primary source of error in the solution when the element is under flexural action.

To simulate adequately the flexural response, Wilson et. al. introduced additional displacement modes to the general eight noded solid element & these additional modes have the same form & order as the terms in . These additional modes are represented by functions of the type,

\[\begin{align*}
    P_1 &= (1- r^2), \\
    P_2 &= (1- s^2), \\
    P_3 &= (1- t^2)
\end{align*}\]

These additional modes are called incompatible modes & they are not activated at the nodes of the element.

\[
\begin{pmatrix}
    u \\
    v \\
    w
\end{pmatrix} = \sum_{i=1}^{8} N_i \begin{pmatrix}
    u_i \\
    v_i \\
    w_i
\end{pmatrix} + [P] \{\alpha\}
\]

Where

\[
[P] = \begin{pmatrix}
    P_1 & P_2 & P_3 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & P_1 & P_2 & P_3 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & P_1 & P_2 & P_3
\end{pmatrix}
\]

\[
\{\alpha\}^T = [\alpha_1 \alpha_2 \alpha_3 \alpha_4 \alpha_5 \alpha_6 \alpha_7 \alpha_8]
\]
5.10.6 STRAIN – DISPLACEMENT MATRIX \([B]\)

As the formulation is done in terms of natural coordinates, we have to establish the relationship between the derivatives of function in the natural coordinates & the derivates in Cartesian coordinates. This is derived by evaluating the Jacobian given by Equation as,

\[
\begin{bmatrix}
\frac{\delta}{\delta r} \\
\frac{\delta}{\delta s} \\
\frac{\delta}{\delta t}
\end{bmatrix}
= [J]
\begin{bmatrix}
\frac{\delta}{\delta x} \\
\frac{\delta}{\delta y} \\
\frac{\delta}{\delta z}
\end{bmatrix}
\]

Where the Jacobian \([J]\) is given by,

\[
[J] = \frac{\partial x}{\partial r} \quad \frac{\partial y}{\partial r} \quad \frac{\partial z}{\partial r} \\
\frac{\partial x}{\partial s} \quad \frac{\partial y}{\partial s} \quad \frac{\partial z}{\partial s} \\
\frac{\partial x}{\partial t} \quad \frac{\partial y}{\partial t} \quad \frac{\partial z}{\partial t}
\]

Substituting Eq. for \(x, y & z\) into Eq. we get,

\[
[J] = \sum_{i=1}^{8} \begin{bmatrix}
x_i \frac{\delta N_i}{\delta r} \quad y_i \frac{\delta N_i}{\delta r} \quad z_i \frac{\delta N_i}{\delta r} \\
x_i \frac{\delta N_i}{\delta s} \quad y_i \frac{\delta N_i}{\delta s} \quad z_i \frac{\delta N_i}{\delta s} \\
x_i \frac{\delta N_i}{\delta t} \quad y_i \frac{\delta N_i}{\delta t} \quad z_i \frac{\delta N_i}{\delta t}
\end{bmatrix}
\]

\[
[B]^{-1} = \begin{bmatrix}
J_{11} & J_{12} & J_{13} \\
J_{21} & J_{22} & J_{23} \\
J_{31} & J_{32} & J_{33}
\end{bmatrix}
\]

\[
\epsilon = [B] \{d\}
\]

\[
\{d\}^T = [u_1, v_1, w_1, \ldots, u_8, v_8, w_8, \alpha_1, \alpha_4, \alpha_7, \alpha_2, \alpha_5, \alpha_8, \alpha_3, \alpha_6, \alpha_9]
\]

Equation can be written as

\[
\epsilon = \sum_{i=1}^{8} [B_i] \{d_i\} + [P'] \{\alpha^*\}
\]
\[ \sum_{i=1}^{8} \begin{pmatrix} u_i \\ v_i \\ w_i \end{pmatrix} = \begin{pmatrix} u_1 & \delta N_i/\delta r \\ v_1 & \delta N_i/\delta s \\ w_1 & \delta N_i/\delta t \end{pmatrix} \begin{pmatrix} \alpha_1 & -2r \alpha_1 & -2r \alpha_7 \\ \alpha_4 & -2r \alpha_4 & -2r \alpha_7 \\ \alpha_7 & -2r \alpha_7 & -2r \alpha_7 \end{pmatrix} \]

\[ \begin{pmatrix} \delta u/\delta x & \delta v/\delta x & \delta w/\delta x \\ \delta u/\delta y & \delta v/\delta y & \delta w/\delta y \\ \delta u/\delta z & \delta v/\delta z & \delta w/\delta z \end{pmatrix} = \begin{pmatrix} \delta u/\delta r & \delta v/\delta r & \delta w/\delta r \\ \delta u/\delta s & \delta v/\delta s & \delta w/\delta s \\ \delta u/\delta t & \delta v/\delta t & \delta w/\delta t \end{pmatrix} \begin{pmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{pmatrix} \begin{pmatrix} \delta u/\delta r & \delta v/\delta r & \delta w/\delta r \\ \delta u/\delta s & \delta v/\delta s & \delta w/\delta s \\ \delta u/\delta t & \delta v/\delta t & \delta w/\delta t \end{pmatrix} \]

In order to find the derivatives of the displacements in global coordinates we have to first obtain the derivatives of the displacements in Natural coordinates. Differentiating equation with respect to \( r, s, t \) we get
Expanding the above equation, we get

\[ \delta u / \delta z = J_{11} \delta u / \delta r + J_{12} \delta u / \delta s + J_{13} \delta u / \delta t \]

Substituting from equation for \( u \) in the above equation-

\[ \delta u / \delta z = \sum_{i=1}^{8} (J_{11} \delta N_i / \delta r + J_{12} \delta N_i / \delta s + J_{13} \delta N_i / \delta t)u_i - 2r J_{11} + 2s J_{12} + 2t J_{13} \alpha_3 \]

Simplifying equation we get,

\[ \delta u / \delta x = \sum_{i=1}^{8} \delta N_i / \delta x u_i - 2r J_{11} + 2s J_{12} + 2t J_{13} \alpha_3 \]

Similarly we can drive expression for the remaining derivatives of displacement with respect to the Cartesian system and they are as follows:

\[ \delta v / \delta x = \sum_{i=1}^{8} \delta N_i / \delta x v_i - 2r J_{21} + 2s J_{22} + 2t J_{23} \alpha_6 \]

\[ \delta w / \delta x = \sum_{i=1}^{8} \delta N_i / \delta x w_i - 2r J_{31} + 2s J_{32} + 2t J_{33} \alpha_3 \]

\[ \delta u / \delta y = \sum_{i=1}^{8} \delta N_i / \delta y u_i - 2r J^*_{21} + 2s J^*_{22} + 2t J^*_{23} \alpha_3 \]

\[ \delta v / \delta y = \sum_{i=1}^{8} \delta N_i / \delta y v_i - 2r J^*_{21} + 2s J^*_{22} + 2t J^*_{23} \alpha_6 \]

\[ \delta w / \delta y = \sum_{i=1}^{8} \delta N_i / \delta y w_i - 2r J^*_{31} + 2s J^*_{32} + 2t J^*_{33} \alpha_9 \]

\[ \delta u / \delta z = \sum_{i=1}^{8} \delta N_i / \delta z u_i - 2r J^*_{31} + 2s J^*_{32} + 2t J^*_{33} \alpha_3 \]

\[ \delta v / \delta z = \sum_{i=1}^{8} \delta N_i / \delta z v_i - 2r J^*_{31} + 2s J^*_{32} + 2t J^*_{33} \alpha_6 \]

\[ \delta w / \delta z = \sum_{i=1}^{8} \delta N_i / \delta z w_i - 2r J^*_{31} + 2s J^*_{32} + 2t J^*_{33} \alpha_9 \]

Using the equations, we can obtain the values of the matrices \([B_i']\) and \([P_i']\) of Equation as,

\[ [B_i] = \begin{bmatrix} \delta N_i / \delta x & 0 & 0 \\ \end{bmatrix} \]
\begin{align*}
\begin{bmatrix}
0 & \frac{\delta N_i}{\delta y} & 0 \\
0 & 0 & \frac{\delta N_i}{\delta z} \\
\frac{\delta N_i}{\delta y} & \frac{\delta N_i}{\delta x} & 0 \\
0 & \frac{\delta N_i}{\delta z} & \frac{\delta N_i}{\delta y}
\end{bmatrix}
\end{align*}
\begin{align*}
\text{And } [P'] = \begin{bmatrix}
-2r J_{11} & 0 & 0 & -2sJ_{12} & 0 & 0 \\
0 & -2r J_{21} & 0 & 0 & -2sJ_{22} & 0 \\
0 & 0 & -2r J_{31} & 0 & 0 & -2sJ_{32} \\
-2r J_{21} & -2r J_{31} & 0 & -2sJ_{22} & -2sJ_{12} & 0 \\
0 & -2r J_{31} & -2r J_{21} & 0 & -2sJ_{32} & -2sJ_{22} \\
-2r J_{31} & 0 & -2r J_{11} & -2sJ_{32} & 0 & -2sJ_{12} \\
\end{bmatrix}
\end{align*}
\begin{align*}
=- [B_9] [B_{10}] [B_{11}] \\
\text{Where } [B_9] [B_{10}] & \text{ & } [B_{11}] \text{ are submatrices of size 6*3 shown in partition in Equation} \\
\text{.Hence we can write the } [B] \text{ matrix of equation as-}
\end{align*}

\[ 6*3 \quad 6*3 \quad 6*3 \quad 6*3 \quad 6*3 \quad 6*3 \quad 6*3 \quad 6*3 \quad 6*3 \]

### 5.11 CONVERGENCE CRITERIA

The eight noded solid element with incompatible modes was an improvement over the one without incompatible modes was an improvement over the one without incompatible modes in representing the bending behavior of the element. Also under constant deformation state even the regular hexahedron elements give erroneous results. The inconsistencies have been investigated by Taylor et.al[6] & the condition to be satisfied for constant strain state has been derived. The procedure is exactly similar to the derivation presented in chapter for isoparametric four noded quadrilateral element with incompatible modes.

\[ \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} [P'] [J] dr \, ds \, dt = [0] \]

Where J is the determinant of the jacobian matrix [J]. For eight noded solid element with incompatible modes \([P']\) is given by equation.

In essence the Jacobian matrix involved in the formation of \([B_n]\) in equation is evaluated at integration points, while the Jacobian involved in \([P']\) in equation is evaluated at \(r = s = t = 0\).

### 5.12 ELEMENT STIFFNESS MATRIX

The constitutive matrix \([C]\) for isotropic three – dimensional stress analysis is given in equation The element stiffness matrix is evaluated using the equation as,

\[ K = \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} [B]^T [C] [B] [J] \, dr \, ds \, dt \]

\[ 33*33 \quad 33*6 \quad 6*6 \quad 6*33 \]

Numerical integration procedure will be used for evaluating the stiffness matrix. Using Gauss quadrature, the 2*2*2 scheme has been found to be adequate.

### 5.13 TWENTY NODED ISOPARAMETRIC SOLID ELEMENTS

Isoparametric Element shown in fig. has eight corner nodes and other nodes at mid points of the lines joining the successive corner nodes. The shape functions are given below:

\[ N_i = 1/8 \ (1+ r_i \ r) \ (1+ s_i \ s) \ (1+ t_i \ t) \ (r_i \ r + s_i \ s + t_i \ t- 2) \]
For nodes $i = 1$ to $8$

$$M_i = \frac{1}{8} (1 - r_i^2) (1 + s_i s) (1 + t_i t)$$

For nodes $i = 10, 12, 14, 16$

$$M_i = \frac{1}{8} (1 - s_i^2) (1 + r_i r) (1 + t_i t)$$

For nodes $i = 9, 11, 13, 15$

$$M_i = \frac{1}{8} (1 - t_i^2) (1 + r_i r) (1 + s_i s)$$

For nodes $i = 17, 18, 19, 20$

The geometry of the element is described as,

$$\begin{align*}
X & = \sum_{i=1}^{20} N_i \delta x_i \\
Y & = \sum_{i=1}^{20} N_i \delta y_i \\
Z & = \sum_{i=1}^{20} N_i \delta z_i
\end{align*}$$

The variation of displacement over the element is expressed by the same shape function as,

$$\begin{align*}
u & = \sum_{i=1}^{20} N_i \delta u_i \\
v & = \sum_{i=1}^{20} N_i \delta v_i \\
w & = \sum_{i=1}^{20} N_i \delta w_i
\end{align*}$$

The element properties can be derived similar to the eight noded element except that no incompatible modes are assumed in this case. The jacobian matrix $[J]$ and strain – displacement matrix $[B]$ are given below.

$$[J] = \sum_{i=1}^{20} \begin{bmatrix}
 x_i \delta N_i/\delta r & y_i \delta N_i/\delta r & z_i \delta N_i/\delta r \\
 x_i \delta N_i/\delta s & y_i \delta N_i/\delta s & z_i \delta N_i/\delta s \\
 x_i \delta N_i/\delta t & y_i \delta N_i/\delta t & z_i \delta N_i/\delta t
\end{bmatrix}$$
\[ [B] = [[[B_1], [B_2]], \ldots, [B_{20}]] \]

Where \([B_i], i = 1, \ldots, 20\).

The element stiffness matrix is given by the following expression,

\[
K = \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} [B]^T [C] [B] [J] \, dr \, ds \, dt
\]

The stiffness matrix is evaluated using 3*3*3 Gauss points.

Instead of using 3*3*3 integration scheme, a reduced 14 point integration rule \([7, 8, 9]\) can also be used, since this rule gives the same accuracy with less computational effort. Of these 14 points, six correspond to three pairs of points situated symmetrically along each axis of symmetry and the remaining eight correspond to those situated symmetrically about each plane of symmetry just similar to that 2*2*2 Gauss rule in the case of eight noded solid element. The values of weight functions and the coordinates of the sampling points are given below in equation. If \((r,s,t)\) is the function to be integrated numerically, its value is given by,

\[
\int \int \int f(r,s,t) \, dr \, ds \, dt = B_6 \left[ f(-b,0,0) + f(b,0,0) + f(0,-b,0) + \ldots + \text{6 terms} \right] + \\
C_8 \left[ f(-c,-c,-c) + f(-c,-c,+c) + f(-c,+c,-c) + \ldots + \text{8 terms} \right]
\]

Where \(B_6\) & \(C_8\) are the weights given by

\(B_6 = 0.8864265927977839\) \hspace{1cm} \(C_8 = 0.3351800554016621\)

Where \(b\) & \(c\) are sampling point locations from centre of element given by,

\(b = 0.798224257542215\) \hspace{1cm} \(c = 0.7587869106393281\)