In Chapter 5, we introduce the concept of the connected edge monophonic number of a connected graph $G$. For a connected graph $G = (V, E)$, an edge monophonic set $M \subseteq V(G)$ is called a connected edge monophonic set if the subgraph $G[M]$ induced by $M$ is connected. The minimum cardinality of a connected edge monophonic set of $G$ is the connected edge monophonic number of $G$ and is denoted by $m_{1c}(G)$. Connected graphs of order $p$ with connected edge monophonic number 2 or $p$ are characterized. It is shown that for every two integers $a, b$ and $c$ such that $2 \leq a < b < c$, there exists a connected graph $G$ with $m(G) = a$, $m_1(G) = b$ and $m_{1c}(G) = c$, where $m(G)$ is the monophonic number and $m_1(G)$ is the edge monophonic number of $G$. A connected edge monophonic set $M$ in a connected graph $G = (V, E)$ is called a minimal connected edge monophonic set if no proper subset of $M$ is a connected edge monophonic set of $G$. The upper connected edge monophonic number $m_{1c}^+(G)$ is the maximum cardinality of a minimal edge connected monophonic set of $G$. Connected graphs of order $p$ with upper connected edge monophonic number 2 or $p$ are characterized. It is shown that for any positive integers $2 \leq a < b \leq c$, there exists a connected graph $G$ with $m_1(G) = a$, $m_{1c}(G) = b$ and $m_{1c}^+(G) = c$, where $m_1(G)$ is the edge monophonic number and $m_{1c}(G)$ is the connected edge monophonic number of a graph $G$. Let $M$ be a minimum connected edge monophonic set of $G$. A subset $T \subseteq M$ is called a forcing subset for $M$ if $M$ is the unique minimum connected edge monophonic set containing $T$. A forcing subset for $M$ of minimum cardinality is a minimum forcing subset of $M$. The forcing connected edge monophonic number of $M$, denoted by $f_{m_{1c}}(M)$, is the cardinality of a minimum forcing subset of $M$. The forcing connected edge monophonic number of $G$, denoted by $f_{m_{1c}}(G)$, is $f_{m_{1c}}(G) = \min \{f_{m_{1c}}(M)\}$, where the minimum is taken over all minimum connected edge monophonic sets $M$ in $G$. It is shown that for every integers $a$
MONOPHONIC CONCEPTS
IN GRAPHS
Preliminaries

In this chapter we collect the basic definitions and theorems which are needed for the subsequent chapters. For graph theoretic terminology, we refer to [3, 11].

Definition 1.1. A graph $G$ is a pair $(V, E)$, where $V$ is a nonempty set whose elements are called vertices of $G$ and $E$ is a set of 2-element subsets of $V$, whose elements are called edges of $G$. The sets $V$ and $E$ are the vertex set and edge set of $G$, respectively. We write $V(G)$ and $E(G)$ rather than $V$ and $E$ to emphasize that these are the vertex and edge sets of a particular graph $G$. The number of vertices in $G$, denoted by $p = \left| V(G) \right|$, is called the order of $G$ while the number of edges in $G$, denoted by $q = \left| E(G) \right|$, is called the size of $G$. A graph of order $p$ and size $q$ is called a $(p, q)$-graph.

Definition 1.2. If $e = \{u, v\}$ is an edge of a graph $G$, we write $e = uv$ we say that $e$ joins the vertices $u$ and $v$; $u$ and $v$ are adjacent vertices; $u$ and $v$ are incident with $e$. If two vertices are not joined, then we say that they are non-adjacent. If two distinct edges $e$ and $f$ are incident with a common vertex $v$, then $e$ and $f$ are said to be adjacent to each other. A set of vertices in a graph is independent if no two vertices in the set are adjacent. Similarly, a set of edges in a graph is independent if no two edges in the set are adjacent. If two or more edges join the same pair of (distinct) vertices, then these edges are called parallel edges. If an edge $e$ joins a vertex $v$ to itself, then $e$ is called to be a loop. A graph $G$ without loops and parallel edges is called a simple graph.

Definition 1.3. The degree of a vertex $v$ in a graph $G$ is the number of edges of $G$ incident with $v$ and is denoted by $\deg_G v$ or $\deg v$. A vertex of degree 0 in $G$ is called an
isolated vertex and a vertex of degree 1 is called a pendent vertex or an end-vertex of G. A graph is said to be $k$-regular if every vertex of $G$ has degree $k$.

**Theorem 1.4.[11]**

a) The sum of the degrees of the vertices of a $(p, q)$-graph $G$ is $2q$.

b) The number of vertices of odd degree in a graph $G$ is even.

**Definition 1.5.** A graph $H$ is called a subgraph of $G$, written $H \subseteq G$, if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. If $H \subseteq G$ and either $V(H)$ is a proper subset of $V(G)$ or $E(H)$ is a proper subset of $E(G)$, then $H$ is a proper subgraph of $G$. A spanning subgraph of $G$ is a subgraph $H$ with $V(H) = V(G)$. For any set $S$ of vertices of $G$, the induced subgraph $G[S]$ is the maximal subgraph of $G$ with vertex set $S$. Thus two vertices of $S$ are adjacent in $G[S]$ if and only if they are adjacent in $G$. Let $v$ be a vertex of a graph $G$. The induced subgraph $G[V(G) – \{v\}]$ is denoted by $G – v$; it is the subgraph of $G$ obtained by the removal of $v$ and edges incident with $v$. Similarly, if $e$ is an edge of a graph $G$, then $G – e$ is the subgraph of $G$ having the same vertex set as $G$ and whose edge set consists of all edges of $G$ except $e$. For a nonempty set $X$ of edges, the subgraph $[X]$ induced by $X$ has edge set $X$ and consists of all vertices that are incident with at least one edge in $X$.

**Definition 1.6.** Two graphs $G$ and $H$ are equal if $V(G) = V(H)$ and $E(G) = E(H)$. A graph $G$ is said to be isomorphic to a graph $H$, if there exists a one-to-one correspondence $\phi$ from $V(G)$ to $V(H)$ such that $uv \in E(G)$ if and only if $\phi(u) \phi(v) \in E(H)$.
Definition 1.7. A graph $G$ is complete if every two distinct vertices of $G$ are adjacent. A complete graph of order $p$ is denoted by $K_p$.

Definition 1.8. A bipartite graph $G$ is a graph whose vertex set $V(G)$ can be partitioned into two subsets $V_1$ and $V_2$ such that every edge of $G$ joins $V_1$ with $V_2$; $(V_1, V_2)$ is called a bipartition of $G$. If $G$ contains every edge joining $V_1$ and $V_2$, then $G$ is called a complete bipartite graph. The complete bipartite graph with bipartition $(V_1, V_2)$ such that $|V_1| = m$ and $|V_2| = n$ is denoted by $K_{m,n}$. A star is the complete bipartite graph $K_{1,n}$.

Example 1.9. The graph $G$ given in Figure 1.1 is a complete bipartite graph $K_{3,4}$. The graph $G$ given in Figure 1.2 is a star $K_{1,4}$.
Definition 1.10. Let $u$ and $v$ be vertices of a graph $G$. A $u \rightarrow v$ walk of $G$ is a finite, alternating sequence $u = u_0, e_1, u_1, e_2, ..., e_n, u_n = v$ of vertices and edges in $G$ beginning with vertex $u$ and ending with vertex $v$ such that $e_i = u_{i-1}u_i$, $i = 1, 2, ..., n$. The number $n$ is called the length of the walk. The walk is said to be open if $u$ and $v$ are distinct vertices; it is closed otherwise. A walk $u_0, e_1, u_1, e_2, u_2, ..., e_n, u_n$ is determined by the sequence $u_0, u_1, u_2, ..., u_n$ of its vertices and hence we specify this walk by $W: u_0, u_1, u_2, ..., u_n$. A walk in which all the vertices are distinct is called a path. A closed walk $u_0, u_1, u_2, ..., u_n$ in which $u_0, u_1, u_2, ..., u_{n-1}$ are distinct is called a cycle. A path on $p$ vertices is denoted by $P_p$ and a cycle on $p$ vertices is denoted by $C_p$. If a graph $G$ has a spanning cycle $C$, then $G$ is called a hamilton graph and $C$ a hamilton cycle.

Definition 1.11. A graph $G$ is said to be connected if any two distinct vertices of $G$ are joined by a path. A maximal connected subgraph of $G$ is called a component of $G$.

Definition 1.12. A cut-vertex (cut-edge) of a graph $G$ is a vertex (edge) whose removal increases the number of components. A non separable graph is connected, nontrivial and has no cut-vertices. A block of a graph is a maximal non separable subgraph. A graph in which each block is complete is called a block graph. For a cut-vertex $v$ in a
connected graph $G$ and a component $H$ of $G - v$, the subgraph $H$ and the vertex $v$ together with all edges joining $v$ and $V(H)$ is called a branch of $G$ at $v$. An end-block of $G$ is a block containing exactly one cut-vertex of $G$. Thus every end-block is a branch of $G$.

**Theorem 1.13.**[11]  

a) Let $v$ be a cut-vertex of a connected graph $G$, and let $u$ and $w$ be vertices in distinct components of $G - v$. Then $v$ lies on every $u - w$ path in $G$.

b) Let $e$ be a cut-edge of a connected graph $G$, and let $u$ and $w$ be vertices in distinct components of $G - e$. Then $e$ lies on every $u - w$ path in $G$.

**Definition 1.14.** A graph $G$ is called acyclic if it has no cycles. A connected acyclic graph is called a tree. A caterpillar is a tree of order 3 or more, for which the removal of all end-vertices leaves a path. A wounded spider is the graph formed by subdividing at most $t - 1$ of the edges of a star $K_{1,t}$ for $t \geq 0$. A nontrivial path is a tree with exactly two end-vertices.

**Example 1.15.** The graph $G$ given in Figure 1.3 is a caterpillar of order 10.

![Graph G](image.png)
**Definition 1.16** For vertices $u$ and $v$ in a connected graph $G$, the *distance* $d(u, v)$ is the length of a shortest $u - v$ path in $G$. A $u - v$ path of length $d(u, v)$ is called a $u - v$ geodesic. The *eccentricity* $e(v)$ of a vertex $v$ in $G$ is the maximum distance from $v$ and a vertex of $G$. The minimum eccentricity among the vertices of $G$ is the *radius*, $\text{rad } G$ or $r(G)$ and the maximum eccentricity is its *diameter*, $\text{diam } G$ of $G$. Two vertices $u$ and $v$ of $G$ are *antipodal* if $d(u, v) = \text{diam } G$ or $d(G)$. A *double star* is a tree of diameter 3.

**Example 1.17.** For the graph $G$ given in Figure 1.4, $e(v_1) = 3$, $e(v_2) = 2$, $e(v_3) = 2$, $e(v_4) = 2$, $e(v_5) = 3$, $\text{rad } G = 2$, centres of $G$ are $v_3$ and $v_4$ and $\text{diam } G = 3$. Here $d(v_1, v_5) = 3 = \text{diam } G$. Therefore the vertices $v_1$ and $v_5$ are antipodal.

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**Figure 1.4**
Example 1.18. The graph $G$ given in Figure 1.5 is a double star.

![Graph G with vertices $v_1, v_2, v_3, v_4, v_5, v_6$]

Figure 1.5

Theorem 1.19. [14] For every connected graph $G$, $rad G \leq diam G \leq 2 \ rad G$.

Definition 1.20. A graph $G$ is geodetic if each pair of vertices in $G$ is joined by a unique shortest path.

Theorem 1.21. [3] A graph $G$ is geodetic if and only if for every vertex $v$, each vertex $u \in N_i(v)$ is adjacent to a unique vertex in $N_{k;1}(v)$ for $2 \leq k \leq e(v)$, where for any integer $i \geq 1$, let $N_i(v) = \{ u \in V : d(u, v) = i \}$.

Definition 1.22. A vertex $v$ is a simplicial vertex of a graph $G$ if the subgraph induced by its neighbors is complete.

Remark 1.23. Every end vertex is a simplicial vertex, but the converse is not true. For the graph $G$ given in Figure 1.6, $v_1$ is a simplicial vertex of $G$, but it is not an end vertex of $G$. 
Definition 1.24. A geodetic set of $G$ is a set $S \subseteq V(G)$ such that every vertex of $G$ is contained in a geodesic joining some pair of vertices in $S$. The geodetic number $g(G)$ of $G$ is the minimum order of its geodetic sets and any geodetic set of order $g(G)$ is a geodetic basis.

Example 1.25. For the graph $G$ given in Figure 1.7, $S = \{v_1, v_3, v_4\}$ is a geodetic basis of $G$ so that $g(G) = 3$.

Definition 1.26. Let $G$ be a connected graph with at least two vertices. A connected geodetic set of a graph $G$ is a geodetic set $S$ such that the subgraph $G[S]$ induced by $S$ is
connected. The minimum cardinality of a connected geodetic set of $G$ is the *connected geodetic number* of $G$ and is denoted by $g_c(G)$. A connected geodetic set of cardinality $g_c(G)$ is called a $g_c$-set of $G$ or a *connected geodetic basis* of $G$.

**Example 1.27.** Consider the graph $G$ of Figure 1.8. For the vertices $u$ and $y$ in $G$, $d(u, y) = 3$ and every vertex of $G$ lies on an $u - y$ geodesic in $G$. Thus $S = \{v_1, v_4\}$ is the unique minimum geodetic set of $G$ and so $g(G) = 2$. Here the induced subgraph $G[S]$ is not connected so that $S$ is not a connected geodetic set of $G$. Now, it is clear that $T = \{v_1, v_2, v_3, v_4\}$ is a minimum connected geodetic set of $G$ and so $g_c(G) = 4$.

![Figure 1.8](image)

**Definition 1.28.** Let $G$ be a connected graph with at least two vertices. An *edge geodetic set* of $G$ is a set $S \subseteq V(G)$ such that every edge of $G$ is contained in a geodesic joining some pair of vertices in $S$. The *edge geodetic number* $g_1(G)$ of $G$ is the minimum order of its edge geodetic sets and any edge geodetic set of order $g_1(G)$ is an *edge geodetic basis* of $G$ or a $g_1$-set of $G$.

**Example 1.29.** For the graph $G$ given in Figure 1.9, $S = \{v_1, v_2, v_4\}$ is an edge geodetic basis for $G$ so that $g_1(G) = 3$. 
**Definition 1.30.** An edge geodetic set $S$ in a connected graph $G$ is called a minimal edge geodetic set if no proper subset of $S$ is an edge geodetic set of $G$. The upper edge geodetic number $g_1^+(G)$ of $G$ is the maximum cardinality of a minimal edge geodetic set of $G$.

**Example 1.31.** For the graph $G$ given in Figure 1.10, $S = \{v_2, v_4, v_5\}$ is an edge geodetic basis of $G$ so that $g_1(G) = 3$. The set $S' = \{v_1, v_3, v_4, v_5\}$ is an edge geodetic set of $G$ and it is clear that no proper subset of $S'$ is an edge geodetic set of $G$ and so $S'$ is a minimal edge geodetic set of $G$. Since $|V(G)| = 5$, it follows that $g_1^+(G) = 4$. 
**Definition 1.32.** Let $G$ be a connected graph and $S$ an edge geodetic basis of $G$. A subset $T \subseteq S$ is called a *forcing subset* for $S$ if $S$ is the unique edge geodetic basis containing $T$. A forcing subset for $S$ of minimum cardinality is a *minimum forcing subset of $S$*. The *forcing edge geodetic number* of $S$, denoted by $f_1(S)$, is the cardinality of a minimum forcing subset of $S$. The *forcing edge geodetic number* of $G$, denoted by $f_1(G)$, is $f_1(G) = \min\{f_1(S)\}$, where the minimum is taken over all edge geodetic bases $S$ in $G$.

**Example 1.33.** For the graph $G$ given in Figure 1.9, $S = \{v_1, v_2, v_4\}$ is the unique edge geodetic basis of $G$ so that $f_1(G) = 0$ and for the graph $G$ given in Figure 1.11, $S_1 = \{v_1, v_5, v_7\}$ and $S_2 = \{v_1, v_5, v_6\}$ are the only two edge geodetic bases of $G$. It is clear that $f_1(S_1) = f_1(S_2) = 1$ so that $f_1(G) = 1$.

![Figure 1.11](image)

**Definition 1.34.** A connected geodetic set $S$ in a connected graph $G$ is called a *minimal connected geodetic set* if no proper subset of $S$ is a connected geodetic set of $G$. The *upper connected geodetic number* $g_c^+(G)$ is the maximum cardinality of a minimal connected geodetic set of $G$. 

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**Example 1.35.** For the graph $G$ given in Figure 1.12, $S_1 = \{v_2, v_4, v_5, v_6\}$, $S_2 = \{v_1, v_2, v_4, v_5\}$, $S_3 = \{v_1, v_2, v_4, v_6\}$, $S_4 = \{v_2, v_3, v_4, v_5\}$ and $S_5 = \{v_2, v_3, v_4, v_6\}$ are the minimum connected geodetic sets of $G$ so that $g_c(G) = 4$. The set $S' = \{v_1, v_3, v_4, v_5, v_6\}$ is also a connected geodetic set of $G$ and it is clear that no proper subset of $S'$ is a connected geodetic set so that $S'$ is a minimal connected geodetic set of $G$. Hence $g_c^+(G) = 5$.

![Figure 1.12](image)

**Definition 1.36.** Let $G$ be a connected graph and $S$ a connected geodetic basis of $G$. A subset $T \subseteq S$ is called a *forcing subset* for $S$ if $S$ is the unique connected geodetic basis containing $T$. A forcing subset for $S$ of minimum cardinality is a *minimum forcing subset* of $S$. The *forcing connected geodetic number* of $S$, denoted by $f_c(S)$, is the cardinality of a minimum forcing subset of $S$. The *forcing connected geodetic number* of $G$, denoted by $f_c(G)$, is $f_c(G) = \min\{f_c(S)\}$, where the minimum is taken over all connected geodetic bases $S$ in $G$.

**Example 1.37.** For the graph $G$ given in Figure 1.13, $S_1 = \{v_1, v_2, v_4\}$, $S_2 = \{v_1, v_3, v_5\}$, $S_3 = \{v_2, v_3, v_4\}$, $S_4 = \{v_2, v_4, v_5\}$, $S_5 = \{v_2, v_3, v_5\}$ and $S_6 = \{v_3, v_4, v_5\}$ are the only connected geodetic bases of $G$ such that $f_c(S_1) = f_c(S_2) = 2$ and $f_c(S_3) = f_c(S_4) = f_c(S_5) = f_c(S_6) = 3$. Thus $f_c(G) = 2$. 

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**Definition 1.38.** Let $G$ be a connected graph with at least two vertices. A *connected edge geodetic set* of $G$ is an edge geodetic set $S$ such that the subgraph $G[S]$ induced by $S$ is connected. The minimum cardinality of a connected edge geodetic set of $G$ is the *connected edge geodetic number* of $G$ and is denoted by $g_{1c}(G)$. A connected edge geodetic set of cardinality $g_{1c}(G)$ is called a $g_{1c}$-set of $G$ or a *connected edge geodetic basis* of $G$.

**Example 1.39.** For the graph $G$ given in Figure 1.14, $S_1 = \{v_1, v_2, v_3, v_4\}$ is a $g_{1c}$-set so that $g_{1c}(G) = 4$. Also, $S_2 = \{v_1, v_2, v_3, v_5\}$ is another $g_{1c}$-set of $G$.

**Definition 1.40.** A connected edge geodetic set $S$ in a connected graph $G$ is called a *minimal connected edge geodetic set* if no proper subset of $S$ is a connected edge
geodetic set of $G$. The upper connected edge geodetic number $g_{1c}^+(G)$ is the maximum cardinality of a minimal connected edge geodetic set of $G$.

**Example 1.41.** For the graph $G$ given in Figure 1.15, $S = \{v_1, v_3, v_4\}$ is a connected edge geodetic basis of $G$ so that $g_{1c}(G) = 3$ and $S' = \{v_2, v_3, v_4, v_5\}$ is a connected edge geodetic set of $G$ and it is clear that no proper subset of $S'$ is a connected edge geodetic set of $G$. Since $|V| = 5$, it follows that $g_{1c}^+(G) = 4$.

![Graph G](image)

**Figure 1.15**

**Definition 1.42.** Let $G$ be a connected graph and $S$ a connected edge geodetic basis of $G$. A subset $T \subseteq S$ is called a forcing subset for $S$ if $S$ is the unique connected edge geodetic basis containing $T$. A forcing subset for $S$ of minimum cardinality is a minimum forcing subset of $S$. The forcing connected edge geodetic number of $S$, denoted by $f_{1c}(S)$, is the cardinality of a minimum forcing subset of $S$. The forcing connected edge geodetic number of $G$, denoted by $f_{1c}(G)$, is $f_{1c}(G) = \min\{f_{1c}(S)\}$, where the minimum is taken over all connected edge geodetic bases $S$ in $G$.

**Example 1.43.** For the graph $G$ given in Figure 1.16, $S_1 = \{v_1, v_2, v_3, v_4, v_5\}$, $S_2 = \{v_1, v_2, v_4, v_5, v_6\}$, $S_3 = \{v_1, v_2, v_3, v_4, v_6\}$ and $S_4 = \{v_1, v_3, v_4, v_5, v_6\}$ are the only
connected edge geodetic bases of $G$ so that $f_{1e}(S_1) = f_{1e}(S_2) = f_{1e}(S_3) = f_{1e}(S_4) = 3$.

Thus $f_{1e}(G) = 3$.

\[ \begin{array}{c}
\text{Theorem 1.44.}[6] \quad \text{Each simplicial vertex of a connected graph } G \text{ belongs to every geodetic set of } G.
\\
\text{Theorem 1.45.}[17, 18] \quad \text{Every cut-vertex of a connected graph } G \text{ belongs to every connected geodetic set of } G.
\\
\text{Theorem 1.46.}[17, 18] \quad \text{For every non-trivial tree } T \text{ of order } p, g_e(T) = p.
\\
\text{Definition 1.47.} \quad \text{A chord of a path } u_0, u_1, u_2, \ldots, u_h \text{ is an edge } u_iu_j, \text{ with } j \geq i + 2. \text{ An } u-v \text{ path is called a monophonic path if it is a chordless path.}
\\
\text{Example 1.48.} \quad \text{For the graph } G \text{ given in Figure 1.17, } v_2v_4 \text{ is a chord of the } v_1 - v_5 \text{ path } v_1, v_2, v_3, v_4, v_5. \text{ The monophonic } v_1 - v_5 \text{ path is } v_1, v_2, v_4, v_5.\end{array} \]
**Definition 1.49.** A monophonic set of \( G \) is a set \( M \subseteq V(G) \) such that every vertex of \( G \) is contained in a monophonic path joining some pair of vertices in \( M \). The monophonic number \( m(G) \) of \( G \) is the minimum order of its monophonic sets and any monophonic set of order \( m(G) \) is a minimum monophonic set of \( G \).

**Example 1.50.** For the graph \( G \) given in Figure 1.18, the \( v_1 - v_3 \) monophonic paths are \( P_1: v_1, v_2, v_3; P_2: v_1, v_7, v_9, v_3 \) and \( P_3: v_3, v_4, v_5, v_6, v_1 \). The \( v_1 - v_8 \) monophonic paths are \( P_4: v_1, v_7, v_8; P_2: v_1, v_2, v_3, v_9, v_8 \) and \( P_3: v_1, v_6, v_5, v_4, v_3, v_9, v_8 \). All the vertices of \( G \) lies on \( v_1 - v_8 \) monophonic path. Then \( \{v_1, v_8\} \) is a minimum monophonic set so that \( m(G) = 2 \).
Theorem 1.51.[12] Each simplicial vertex of $G$ belongs to every monophonic set of $G$.

Theorem 1.52.[12] The monophonic number of a tree $T$ is the number of end vertices in $T$.

Definition 1.53. Let $G$ be a connected graph and $M$ be a minimum monophonic set of $G$. A subset $T \subseteq M$ is called a forcing subset for $M$ if $M$ is the unique minimum monophonic set containing $T$. A forcing subset for $M$ of minimum cardinality is a minimum forcing subset of $M$. The forcing monophonic number of $M$, denoted by $f_m(M)$, is the cardinality of a minimum forcing subset of $M$. The forcing monophonic number of $G$, denoted by $f_m(G)$, is $f_m(G) = \min\{f_m(M)\}$, where the minimum is taken over all minimum monophonic sets $M$ in $G$. 

Figure 1.18

\[ G \]
Definition 1.54. A vertex \( v \) is said to be a monophonic vertex of \( G \) if \( v \) belongs to every minimum monophonic set of \( G \).

Theorem 1.55. Let \( G \) be a connected graph. Then

(a) \( f_m(G) = 0 \) if and only if \( G \) has a unique minimum monophonic set.

(b) \( f_m(G) = 1 \) if and only if \( G \) has at least two minimum monophonic sets, one of which is a unique minimum monophonic set containing one of its elements.

(c) \( f_m(G) = m(G) \) if and only if no minimum monophonic set of \( G \) is the unique minimum monophonic set containing any of its proper subsets and

(d) \( f_m(G) \leq m(G) - |W| \), where \( W \) is the set of all monophonic vertices of \( G \).

Definition 1.56. For two vertices \( u \) and \( v \) in a connected graph \( G \), the monophonic distance \( d_m(u, v) \) is the length of the longest \( u - v \) monophonic path in \( G \). A \( u - v \) monophonic path of length \( d_m(u, v) \) is called a \( u - v \) monophonic. For a vertex \( v \) of \( G \), the monophonic eccentricity \( e_m(v) \) is the monophonic distance between \( v \) and a vertex farthest from \( v \). The minimum monophonic eccentricity among the vertices is the monophonic radius, \( rad_m(G) \) and the maximum monophonic eccentricity is the monophonic diameter \( diam_m(G) \) of \( G \).

Example 1.57. For the graph \( G \) given in Figure 1.19, \( e_m(v_1) = 4, e_m(v_2) = 2, e_m(v_3) = 4, e_m(v_4) = 3, e_m(v_5) = 2 \) and \( e_m(v_6) = 3 \). Then \( rad_m(G) = 2 \), \( diam_m(G) = 4 \) and monophonic centres are \( v_2 \) and \( v_5 \).
Definition 1.58. The maximum degree of $G$, denoted by $\Delta(G)$, is given by $\Delta(G) = \max \{\deg_G(v) : v \in V(G)\}$. $N(v) = \{u \in V(G) : uv \in E(G)\}$ is called the neighborhood of the vertex $v$ in $G$.

Definition 1.59. If $e = \{u, v\}$ is an edge of a graph $G$ with $d(u) = 1$ and $d(v) > 1$, then we call $e$ a pendent edge, $u$ a leaf and $v$ a support vertex. $L(G)$ denote the set of all leaves of a graph $G$.

Definition 1.60. A vertex $v$ is an universal vertex of a graph $G$, if it is a full degree vertex of $G$.

Definition 1.61. A set of vertices $D$ in a graph $G$ is a dominating set if each vertex of $G$ is dominated by some vertex of $D$. The domination number of $G$ is the minimum
cardinality of a dominating set of $G$ and is denoted by $\gamma(G)$. A dominating set of size $\gamma(G)$ is said to be a $\gamma$-set.

**Example 1.62.** For the graph $G$ given in Figure 1.20, $D = \{v_1, v_3, v_5\}$ is a minimum dominating set so that $\gamma(G) = 3$.

![Figure 1.20](image)

**Definition 1.63.** A set of vertices $M$ in $G$ is called a geodetic dominating set if $M$ is both a geodetic set and a dominating set. The minimum cardinality of a geodetic dominating set of $G$ is its geodetic domination number and is denoted by $\gamma_g(G)$. A geodetic dominating set of size $\gamma_g(G)$ is said to be a $\gamma_g$-set.

**Example 1.64.** For the graph $G$ given in Figure 1.21, $S = \{v_3, v_3, v_4\}$ is a minimum geodetic dominating set so that $\gamma_g(G) = 3$. 

![Figure 1.21](image)
Definition 1.65. A simplex of a graph $G$ is a subgraph of $G$ which is a complete graph.

Example 1.66. For the graph $G$ given in Figure 1.21, the subgraph $H$ with $V(H) = \{v_1, v_2, v_6\}$ is a simplex of $G$. 