The Connected Edge Monophonic Number of a Graph

In this chapter we introduce the connected edge monophonic number $m_{1c}(G)$ of a graph $G$ and some of its general properties are studied. The connected edge monophonic number of certain classes of graphs are determined. Connected graphs of order $p$ with connected edge monophonic number 2 or $p$ are characterized. It is shown that for every two integers $a$, $b$ and $c$ such that $2 \leq a < b < c$, there exists a connected graph $G$ with $m(G) = a$, $m_1(G) = b$ and $m_{1c}(G)$, where $m(G)$ is the monophonic number and $m_1(G)$ is the edge monophonic number of $G$. The upper connected edge monophonic number $m_{1c}^+(G)$ of $G$ is introduced and some of its general properties are studied. Connected graphs of order $p$ with upper connected edge monophonic number 2 or $p$ are characterized. It is shown that for any positive integers $2 \leq a < b \leq c$, there exists a connected graph $G$ with $m_1(G) = a$, $m_{1c}(G) = b$ and $m_{1c}^+(G) = c$, where $m_1(G)$ is the edge monophonic number and $m_{1c}(G)$ is the connected edge monophonic number of a graph $G$. The forcing connected edge monophonic number $f_{m_{1c}}(G)$ of $G$ is introduced. It is shown that for every integers $a$ and $b$ with $a < b$, and $b - 2a - 2 > 0$, there exists a connected graph $G$ such that $f_{m_{1c}}(G) = a$ and $m_{1c} = b$.

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THE CONNECTED EDGE MONOPHONIC NUMBER OF A GRAPH

**Definition 5.1.** A set $M \subseteq V(G)$ is called a connected edge monophonic set if the subgraph $G[M]$ induced by $M$ is connected. The minimum cardinality of a connected edge monophonic set of $G$ is the connected edge monophonic number of $G$ and is denoted by $m_{1e}(G)$. A connected edge monophonic set of size $m_{1e}(G)$ is said to be a $m_{1e}$-set.

**Example 5.2.** For the graph $G$ given in Figure 5.1, $M_1 = \{v_1, v_2, v_3, v_4\}$, $M_2 = \{v_1, v_5, v_6, v_7\}$, $M_3 = \{v_1, v_2, v_3, v_7\}$ and $M_4 = \{v_1, v_2, v_3, v_6\}$ are the only four connected edge monophonic sets of $G$ so that $M_{1e}(G) = 4$.

![Figure 5.1](image)

**Theorem 5.3.** Each semi-simplicial vertex of a graph $G$ belongs to every connected edge monophonic set of $G$.

**Proof.** Let $M$ be a connected edge monophonic set of $G$. Let $v$ be a semi-simplicial vertex of $G$. Suppose that $v \notin M$. Let $u$ be a vertex of $< N(v) >$ such that $\deg_{< N(v) >}(u) = |N(v)| - 1$. Let $u_1, u_2, ..., u_k (k \geq 2)$ be the neighbors of $u$ in $< N(v) >$. Since $M$ is a connected edge monophonic set of $G$, the edge $uv$ lies on the monophonic path $P: x, x_1, ..., u_{i}, u, v, u_j, ..., y$, where $x, y \in M$. Since $v$ is a semi –
simplicial vertex of $G$, $u$ and $u_j$ are adjacent in $G$ and so $P$ is not a monophonic path of $G$, which is a contradiction.

**Corollary 5.4.** Each simplicial vertex of a graph $G$ belongs to every connected edge monophonic set of $G$.

**Proof.** Since every simplicial vertex of $G$ is a semi-simplicial vertex of $G$, the result follows from Theorem 5.3.

**Theorem 5.5.** Let $G$ be a connected graph, $v$ be a cut vertex of $G$ and let $M$ be a connected edge monophonic set of $G$. Then every component of $G - v$ contains an element of $M$.

**Proof.** Let $v$ be a cut vertex of $G$ and $M$ be a connected edge monophonic set of $G$. Suppose there exists a component, say $G_1$ of $G - v$ such that $G_1$ contains no vertex of $M$. By Corollary 5.4, $M$ contains all the simplicial vertices of $G$ and hence it follows that $G_1$ does not contain any simplicial vertex of $G$. Thus $G_1$ contains at least one edge, say $xy$. Since $M$ is a connected edge monophonic set, $xy$ lies on the $u - w$ monophonic path $P: u, u_1, u_2, \ldots, v, \ldots, x, y, \ldots, v_1, \ldots, v, \ldots, w$. Since $v$ is a cut vertex of $G$, the $u - x$ and $y - w$ sub paths of $P$ both contain $v$ and so $P$ is not a monophonic path, which is a contradiction.

**Theorem 5.6.** Each cut vertex of a connected graph $G$ belongs to every minimum connected edge monophonic set of $G$.

**Proof.** Let $v$ be any cut vertex of $G$ and let $G_1, G_2, \ldots, G_r (r \geq 2)$ be the components of $G - v$. Let $M$ be any connected edge monophonic set of $G$. Then by Theorem 5.5, $M$
contains at least one element from each $G_i$; $(1 \leq i \leq r)$. Since $< M >$ is connected, it follows that $v \in M$.

**Corollary 5.7.** For a connected graph $G$ with $k$ semi-simplicial vertices and $l$ cut vertices, $m_{1c}(G) \geq \max\{2, k + l\}

**Proof.** This follows from Corollary 5.4 and Theorems 5.6.

In the following we determine the connected edge monophonic member of some standard graphs.

**Corollary 5.8.**

i) For any non-trivial tree $T$ of order $p$, $m_{1c}(T) = p$.

ii) For the complete graph $K_p(p \geq 2)$, $m_{1c}(K_p) = p$.

**Theorem 5.9.** For the cycle $C_p(p \geq 3)$, $m_{1c}(K_p) = 3$

**Proof.** Let $v_1, v_2, ..., v_p, v_1$ be a cycle of length $p$. Let $x, y \in V(C_p)$ such that $d_m(x, y) = 2$. Then $M = \{x, y\}$ is an edge monophonic set of $C_p$. But $< M >$ is not connected. Let $u$ be a vertex of $C_p$ which is adjacent to both $x$ and $y$. Then $M \cup \{u\}$ is a connected edge monophonic set of $G$ so that $m_{1c}(C_p) = 3$.

**Theorem 5.10.** For the complete bipartite graph $G = K_{m,n}$,

(i) $m_{1c}(G) = 2$ if $m = n = 1$.

(ii) $m_{1c}(G) = n + 1$ if $m = 1, n \geq 2$.

(iii) $m_{1c}(G) = \min\{m,n\} + 1$, if $m, n \geq 2$. 

96
Proof.

i) This follows from Corollary 5.8 (ii)

ii) This follows from Corollary 5.8 (i)

iii) Let $m, n \geq 2$. First assume that $m < n$. Let $U = \{u_1, u_2, \ldots, u_m\}$ and $W = \{w_1, w_2, \ldots, w_n\}$ be a partition of $G$. Let $M = U \cup \{w_1\}$. We prove that $M$ is a minimum connected edge monophonic set of $G$. Any edge $u_i w_j$ (where $1 \leq i \leq m, 1 \leq j \leq n$) lies on the monophonic path $u_i, w_j, u_k$ for any $k \neq i$ so that $M$ is an edge monophonic set of $G$.

Since $G[M]$ is connected, $M$ is a connected edge monophonic set of $G$. Let $T$ be any set of vertices such that $|T| < |M|$. If $T \subset U, G[T]$ is not connected and so $T$ is not a connected edge monophonic set of $G$. If $T \subset W$, again $T$ is not a connected edge monophonic set of $G$ by a similar argument. If $T \supseteq U$, then since $|T| < |S|$, we have $T = U$, which is not a connected edge monophonic set of $G$. Similarly, since $|T| < |S|$, $T$ cannot contain $W$. For if $T \supseteq W$, then $|T| \geq n \geq m + 1 = |M|$, which is a contradiction. Thus $T \nsubseteq U \cup W$ such that $T$ contains at least one vertex from each of $U$ and $W$. Then since $|T| < |M|$, there exists vertices $u_i \in U$ and $w_j \in W$ such that $u_i \notin T$ and $w_j \notin T$. Then clearly $u_i w_j$ does not lie on a monophonic path connecting two vertices of $T$ so that $T$ is not a connected edge monophonic set of $G$. Thus in any cases $T$ is not a connected edge monophonic set of $G$. Hence $M$ is a minimum connected edge monophonic set of $G$ so that $m_{1c}(K_{m,n}) = m + 1$. Now, if $m = n$, we can prove similarly that $M = U \cup \{y\}$, where $y \in W$ is a minimum connected edge monophonic set of $G$. Hence the theorem follows.

\[\Box\]
**Theorem 5.11.** For any connected graph $G$ of order, $2 \leq m_1(G) \leq m_{1c}(G) \leq p$.

**Proof.** Any edge monophonic set needs at least two vertices and so $m_{1c}(G) \geq 2$. Since every connected edge monophonic set is also an edge monophonic set, it follows that $m_1(G) \leq m_{1c}(G)$. Also, since $\langle V(G) \rangle$ induces a connected edge monophonic set of $G$, it is clear that $m_{1c}(G) \leq p$. \hfill \square

**Remark 5.12.** The bounds in Theorem 5.11 are sharp. For any non-trivial path $P, m_1(P) = 2$. For the complete graph $K_p, m_1(K_p) = m_{1c}(K_p) = p$. Also, all the inequalities in the theorem are strict. For the graph $G$ given in Figure 5.2, $m_1(G) = 3, m_{1c}(G) = 5$ and $p = 6$ so that $2 < m_1(G) < m_{1c}(G) < p$.

![Figure 5.2]

The following Theorems 5.13 and 5.14 characterize graphs for which $m_{1c}(G) = 2$ and $m_{1c}(G) = p$, respectively.

**Theorem 5.13.** Let $G$ be a connected graph of order $p \geq 2$. Then $G = K_2$ if and only if $m_{1c}(G) = 2$. 
**Proof.** Let \( G = K_2 \), then \( m_{1c}(G) = 2 \). Conversely, let \( m_{1c}(G) = 2 \). Let \( M = \{u, v\} \) be a minimum connected edge monophonic set of \( G \). Then \( uv \) is an edge. If \( G \neq K_2 \), then there exists an edge \( xy \) different from \( uv \). Since \( uv \) is a chord, the edge \( xy \) cannot lie on any \( u - v \) monophonic path so that \( M \) is not a \( m_{1c} \) - set, which is a contradiction. Thus \( G = K_2 \). \( \blacksquare \)

**Theorem 5.14.** Let \( G \) be a connected graph. Then every vertex of \( G \) is either a cut vertex or a semi-simplicial vertex if and only if \( m_{1c}(G) = p \).

**Proof.** Let \( G \) be a connected graph with every vertex of \( G \) is either a cut vertex or a semi-simplicial vertex. Then the result follows from Theorems 5.3 and 5.6. Conversely, let \( m_{1c}(G) = p \). Suppose that there is a vertex \( x \) in \( G \) which is neither a cut vertex nor a semi-simplicial vertex. Since \( x \) is not a semi-simplicial vertex, \( N(x) \) does not induce a complete sub graph and hence there exists \( u \) and \( v \) in \( N(x) \) such that \( d_m(u, v) = 2 \). Clearly the edge \( xy \), where \( y \in N(x) \) lies on a \( u - v \) monophonic path in \( G \). Also, since \( x \) is not a cut vertex of \( G \), \( G - x \) is connected. Thus \( V(G) - \{x\} \) is a connected edge monophonic set of \( G \) and so \( m_{1c}(G) \leq |V(G) - \{x\}| = p - 1 \), which is a contradiction. \( \blacksquare \)

**Theorem 5.15.** If \( G \) is a non-complete connected graph such that it has a minimum cut set of \( G \) consisting of \( i \) independent vertices, then \( m_{1c}(G) \leq p - i + 1 \).

**Proof.** Since \( G \) is non-complete, it is clear that \( 1 \leq i \leq p - 2 \). Let \( U = \{v_1, v_2, \ldots, v_i\} \) be a minimum independent cut set of vertices of \( G \). Let \( G_1, G_2, \ldots, G_m (m \geq 2) \) be the components of \( G - U \) and let \( M = V(G) - U \). Then every vertex \( v_j (1 \leq j \leq i - 1) \) is adjacent to at least one vertex of \( G_t \) for any \( t(1 \leq t \leq m) \). Let \( uv \) be an edge of \( G \). If \( uv \)
lies in one of $G_t$ for any $t(1 \leq t \leq m)$, then clearly $uv$ lies on the monophonic path ($uv$ itself) joining two vertices $u$ and $v$ of $M$. Otherwise $uv$ is of the form $v_ju$ ($1 \leq j \leq i$), where $u \in G_t$ for some $t$ such that $1 \leq t \leq m$. As $m \geq 2$, $v_j$ is adjacent to some $w$ in $G_s$ for some $s \neq t$ such that $1 \leq s \leq m$. Thus $v_ju$ lies on the monophonic path $u, v_j, w$. Thus $M$ is an edge monophonic set of $G$, such that $< M >$ is not connected. However $M \cup \{x\}, x \notin U$ is a connected edge monophonic set of $G$ so that $m_{1c}(G) \leq |V(G) - (U \cup \{x\})| = p - i + 1$.

**Realisation Results**

**Theorem 5.16.** For positive integers $r_m, d_m$ and $l > d_m - r_m + 3$ with $r_m < d_m \leq 2r_m$, there exists a connected graph $G$ with $rad_m(G) = r_m$, $diam_m(G) = d_m$ and $m_{1c}(G) = l$.

**Proof.** When $r_m = 1$, we let $G = k_{1, l-1}$. Then the result follows from Corollary 4.7(i).

Let $r_m \geq 2$. let $C_{r_m+2}: v_1, v_2, ..., v_{r_m+2}$ be a cycle of length $r_m + 2$ and let $P_{d_m-r_m+1}: u_0, u_1, u_2, ..., u_{d_m-r_m}$ be a path of length $d_m - r_m + 1$. Let $H$ be a graph obtained from $C_{r_m+2}$ and $P_{d_m-r_m+1}$ by identifying $v_1$ in $C_{r_m+2}$ and $u_0$ in $P_{d_m-r_m+1}$.

Now add $l - d_m + r_m - 3$ new vertices $w_1, w_2, ..., w_{l-d_m+r_m-3}$ to $H$ and join each $w_i$ ($1 \leq i < l - d_m + r_m - 3$) to the vertex $u_{d_m-r_m-1}$ and obtain the graph $G$ as shown in Figure 4.3. Then $rad_m(G) = r_m$ and $diam_m(G) = d_m$. Let $M = \{u_0, u_1, u_2, ..., u_{d_m-r_m}, w_1, w_2, ..., w_{l-d_m+r_m-3}\}$ be the set of all cut-vertices and end-vertices of $G$.

By Corollary 5.4 and Theorem 5.6, $M$ is a subset of every connected edge monophonic set of $G$. It is clear that $M$ is not a connected edge monophonic set of $G$. Also $M \cup \{x\}$, where $x \notin M$ is not a connected edge monophonic set of $G$ and so $m_{1c}(G) \geq d_m -
\[ r_m + 1 + l - d_m + r_m - 3 + 1 = l - 1. \] However, \( M \cup \{v_2, v_3\} \) is a connected edge monophonic set of \( G \) so that \( m_{1c}(G) = l \).

**Theorem 5.17.** For every pair \( k, p \) of integers with \( 3 \leq k \leq p \), there exists a connected graph \( G \) of order \( p \) such that \( m_{1c}(G) = k \).

**Proof.** Let \( P_k: u_1, u_2, ..., u_k \) be a path on \( k \) vertices. Add new vertices \( v_1, v_2, ..., v_{p-k} \) and join each \( v_i \) (\( 1 \leq i \leq p - k \)) with \( u_1 \) and \( u_3 \), there by obtaining the graph \( G \) in Figure 5.3. Then \( G \) has order \( p \) and \( M = \{u_3, u_4, ..., u_k\} \) is the set of all cut vertices and simplicial vertices of \( G \). By Corollary 5.4 and Theorem 5.6, \( m_{1c}(G) \geq k - 2 \). Clearly \( M \) is not a connected edge monophonic set of \( G \) and so \( m_{1c}(G) > k - 2 \). Now, either \( M \cup \{v_i\}(1 \leq i \leq p - K) \) nor \( M \cup \{u_2\} \) is an edge monophonic set of \( G \). However \( T = M \cup \{u_1\} \) is an edge monophonic set of \( G \) such that \( G[T] \) is disconnected. It is clear that \( T \cup \{u_2\} \) is a connected edge monophonic set of \( G \) and hence it follows that \( m_{1c}(G) = k \).

![Figure 5.3](image-url)
In view of Theorem 5.11, we have the following realisation theorem.

**Theorem 5.18.** For any positive integers $2 \leq a < b < c$, there exists a connected graph $G$ such that $m(G) = a, \ m_1(G) = b$ and $m_{1c}(G) = c$.

**Proof.** Let $G$ be the graph given in Figure 5.4 obtained from the path on $c - b + 2$ vertices $P_{c-b+2}: u_1, u_2, ..., u_{c-b+2}$ by adding $b - 2$ new vertices $v_1, v_2, ..., v_{b-a}, w_1, w_2, ..., w_{a-2}$ to $P_{c-b+2}$ and joining each $v_i (1 \leq i \leq b - a)$ with $u_1, u_2, u_3$ and joining each $w_i (1 \leq i \leq a - 2)$ with $v_2$. Let $M = \{w_1, w_2, ..., w_{a-2}, u_{c-b+2}\}$ be the set of all simplicial vertices of $G$. By Theorem 1.51, $M$ is a subset of every monophonic set of $G$. It is clear that $M$ is not a monophonic set of $G$ and so $m(G) \geq a$. However $M_1 = M \cup \{u_1\}$ is a monophonic set of $G$ so that $m(G) = a$. By Corollary 2.8, $M$ is a subset of every edge monophonic set of $G$. It is easily observed that every edge monophonic set of $G$ contains each $v_i (1 \leq i \leq b - a)$. Let $M_2 = M_1 \cup \{v_1, v_2, ..., v_{b-a}\}$. It is clear that $M_2$ is an edge monophonic set of $G$ so that $m_1(G) = a + b - a = b$. It can be easily varied that $M_2$ is not a connected edge monophonic set of $G$. Let $M' = M_1 \cup \{u_2, u_3, ..., u_{c-b+1}\}$ be the set of simplicial vertices and cut vertices of $G$. By Corollary 5.4 and Theorem 5.6, $M'$ is a subset of every connected edge monophonic set of $G$. It is clear that $M'$ is not a connected edge monophonic set of $G$. It is easily observed that every connected edge monophonic set of $G$ contains each $v_i (1 \leq i \leq b - a)$ so that $m_{1c}(G) \geq c$. However $M_3 = M' \cup \{v_1, v_2, ..., v_{b-a}\}$ is a connected edge monophonic set of $G$ so that $m_{1c}(G) = b + c - b = c$. ■
Definition 5.19. A connected edge monophonic set $M$ in a connected graph $G$ is called a minimal connected edge monophonic set if no proper sub set of $M$ is a connected edge monophonic set of $G$. The upper connected edge monophonic number $m_{1c}^+(G)$ is the maximum cardinality of a minimal connected edge monophonic set of $G$.

Example 5.20. For the graph $G$ given in Figure 5.5, $M_1 = \{v_1, v_2, v_3, v_4\}$, $M_2 = \{v_1, v_2, v_3, v_5\}$, $M_3 = \{v_1, v_2, v_3, v_6\}$ and $M_4 = \{v_1, v_2, v_3, v_7\}$ are minimum connected edge monophonic sets of $G$ so that $M_{1c}(G) = 4$. The sets $M' = \{v_1, v_4, v_5, v_6, v_7\}$, $M'' = \{v_2, v_4, v_5, v_6, v_7\}$, and $M''' = \{v_3, v_4, v_5, v_6, v_7\}$ are also connected edge monophonic sets of $G$ and it is clear that no proper subsets of $M', M''$ and $M'''$ are connected edge monophonic set so that $M', M''$ and $M'''$ are minimal edge monophonic.
sets of $G$. It is easily verified that there is no minimal connected edge monophonic set $M$ with $|M| \geq 5$. Hence it follows that $m_{1c}^+(G) = 4.$

![Figure 5.5](image)

**Remark 5.21.** Every minimum connected edge monophonic set of $G$ is a minimal connected edge monophonic set of $G$. The converse is not true. For the graph $G$ given in Figure 4.5, $M' = \{v_1, v_4, v_5, v_6, v_7\}$ is a minimal connected edge monophonic set and is not a minimum connected edge monophonic set of $G$.

**Theorem 5.22.** For any connected graph $G$, $2 \leq m_{1c}(G) \leq m_{1c}^+(G) \leq p$.

**Proof.** Any connected edge monophonic set need at least two vertices and so $m_c(G) \geq 2$. Since every minimum connected edge monophonic set is a minimal connected edge monophonic set, $m_{1c}(G) \leq m_{1c}^+(G)$. Also, since $V(G)$ induces a connected edge monophonic set of $G$, it is clear that $m_{1c} \leq p$. Thus $2 \leq m_c(G) \leq m_{1c}^+(G) \leq p$. □

**Remark 5.23.** For the graph $K_2$, $m_{1c}(K_2) = 2$. For any non-trivial tree $T$ of order $p$, $m_{1c}^+(T) = p$. Also, all the inequalities in Theorem 5.22, are strict. For the graph $G$ given in Figure 4.5, $m_{1c}(G) = 3, m_{1c}^+(G) = 4$, $p = 6$ so that $2 < m_{1c}(G) < m_{1c}^+(G) < p$.  

104
Theorem 5.24. For any connected graph $G$, $m_1^+(G) = p$ if and only if $m_1^-(G) = p$

Proof. Let $m_1^+(G) = p$. Then $M = V(G)$ is the unique minimal edge monophonic set of $G$. Since no proper subset of $M$ is a connected edge monophonic set, it is clear that $M$ is the unique minimum connected edge monophonic set of $G$ and so $m_1^+(G) = p$. The converse follows from Theorem 5.22.

Theorem 5.25. Every simplicial vertex of a connected graph $G$ belongs to every minimal connected edge monophonic set of $G$.

Proof. Since every minimal connected edge monophonic set is an edge monophonic set, the result follows from Corollary 2.8.

Theorem 5.26. Let $G$ be a connected graph containing a cut-vertex $v$. Let $M$ be a minimal connected edge monophonic set of $G$, then every component of $G - v$ contains an element of $M$.

Proof. Let $v$ be a cut-vertex of $G$ and $M$ be a minimal connected edge monophonic set of $G$. Suppose there exists a component say $G_1$ of $G - v$ such that $G_1$ contains no vertex of $M$. By Theorem 5.25, $M$ contains all simplicial vertices of $G$ and hence it follows that $G_1$ does not contain any simplicial vertex of $G$. Thus $G_1$ contains at least one edge say $xy$. Since $M$ is the minimal connected edge monophonic set, $xy$ lies on the $u - w$ monophonic path $P: u, u_1, u_2, ..., v, ..., x, y, ..., v_1, ..., v, ..., w$. Since $v$ is a cut-vertex of $G$, the $u - x$ and $y - w$ sub path of $P$ both contains $v$ and so $P$ is not a path, which is a contradiction.
**Theorem 5.27.** Every cut-vertex of a connected graph $G$ belongs to every minimal connected edge monophonic set of $G$.

**Proof.** Let $v$ be any cut-vertex of $G$ and let $G_1, G_2, \ldots, G_r (r > 2)$ be the components of $G - \{u\}$. Let $M$ be any connected edge monophonic set of $G$. Then $M$ contains at least one element from each $G_i (1 \leq i \leq r)$. Since $G[M]$ is connected, it follows that $u \notin M$. 

**Corollary 5.28.** For a connected graph $G$ with $k$ simplicial vertices and $l$ cut-vertices, $m_{1c}^+(G) \geq \max \{2, k + l\}$.

**Proof.** This follows from Theorem 5.25 and 5.27.

**Corollary 5.29.** For the complete graph $G = K_p, m_{1c}^+(G) = p$.

**Proof.** This is follows from Theorem 5.25.

**Corollary 5.30.** For any tree $T$, $m_{1c}^+(T) = p$.

**Proof.** This follows from Corollary 5.29.

**Realisation Results**

**Theorem 5.31.** For positive integers $r_m, d_m$ and $l > d_m - r_m + 3$ with $r_m < d_m \leq 2r_m$, there exists a connected graph $G$ with $rad_m(G) = r_m, diam_m(G) = d_m$ and $m_{1c}^+(G) = l$.

**Proof.** When $r_m = 1$, we let $G = K_{1, l-1}$. Then the result follows from Corollary 5.30. Let $r_m \geq 2$, let $C_{r_m+2}: v_1, v_2, \ldots, v_{r_m+2}, v_1$ be a cycle of length $r_m+2$ and let $P_{d_m-r_m+1}: u_0, u_1, u_2, \ldots, u_{d_m-r_m}$ be a path of length $d_m-r_m+1$. Let $H$ be a graph obtained from $C_{r_m+2}$ and $P_{d_m-r_m+1}$ by identifying $v_1$ in $C_{r_m+2}$ and $u_0$ in $P_{d_m-r_m+1}$.
Now add \( l - d_m + r_m - 3 \) new vertices \( w_1, w_2, \ldots, w_{l-d_m+r_m-3} \) to \( H \) and join each \( w_i \) \((1 \leq i < l - d_m + r_m - 3)\) to the vertex \( u_{d_m-r_m-1} \) and obtain the graph \( G \) as shown in Figure 4.3. Then \( \text{rad}_m(G) = r_m \) and \( \text{diam}_m(G) = d_m \). Let \( M = \{u_0, u_1, u_2, \ldots, u_{d_m-r_m}, w_1, w_2, \ldots, w_{l-d_m+r_m-3}\} \) be the set of cut-vertices and end-vertices of \( G \). By Corollary 5.4 and Theorem 5.6, \( M \) is a subset of every connected edge monophonic set of \( G \). It is clear that \( M \) is not a connected edge monophonic set of \( G \). Also \( M \cup \{x\} \), where \( x \notin M \) is not a connected edge monophonic set of \( G \). However \( M_1 = M \cup \{v_2, v_3\} \) is a connected edge monophonic set of \( G \). Now, we show that \( M_1 \) is a minimal connected edge monophonic set of \( G \). Assume, to the contrary, that \( M_1 \) is not a minimal connected edge monophonic set of \( G \). Then there is a proper subset \( T \) of \( M_1 \) such that \( T \) is connected edge monophonic set of \( G \). Let \( y \in M_1 \) and \( y \notin T \). By Theorem 5.3, \( y \neq w_i (1 \leq i \leq l - d_m + r_m - 3) \). Also by Theorem 5.6, \( y \neq u_i (1 \leq i \leq d_m - r_m) \). Then \( T \) is not a connected edge monophonic set of \( G \), which is a contradiction. Thus, \( M_1 \) is a minimal connected edge monophonic set of \( G \) and so \( m_{1c}^+(G) \geq l \). Let \( M' \) be a minimal connected edge monophonic set of \( G \) such that \( |M'| > l \). By Theorems 5.3 and 5.5, \( M' \) contains \( M \). Since, \( M_1 = M \cup \{v_2, v_3\} \) or \( M_2 = M \cup \{v_2, v_{r_m+2}\} \) or \( M_3 = M \cup \{v_{r_m+1}, v_{r_m+2}\} \) is also a connected edge monophonic set of \( G \) and \( <M'> \) is connected, it follows that \( M' \) contains either \( M_1 \) or \( M_2 \) or \( M_3 \), which is a contradiction to \( M' \) is a minimal connected edge monophonic set of \( G \). Therefore \( m_{1c}^+(G) = l \). 

In view of Theorem 5.22, we have the following realisation result.

**Theorem 5.32.** For any positive integers \( 2 \leq a < b \leq c \), there exists a connected graph \( G \) such that \( m_1(G) = a \), \( m_{1c}(G) = b \) and \( m_{1c}^+(G) = c \).
Proof. If $2 \leq a < b = c$, let $G$ be any tree of order $b$ with $a$ end-vertices. Then by Corollary 2.11, $m_1(G) = a$, by Corollary 5.8(i), $m_{1e}(G) = b$ and by Corollary 5.30, $m_{1e}^+(G) = b$. Let $2 \leq a < b < c$. Now, we consider four cases.

Case 1. Let $b > a$ and $b - a \geq 2$. Then $b - a + 2 \geq 4$, let $P_{b-a+2} : v_1, v_2, ..., v_{b-a+2}$ be a path of length $b - a + 1$. Add $c - b + a - 1$ new vertices $w_1, w_2, ..., w_{c-b}, u_1, u_2, ..., u_{a-1}$ to $P_{b-a+2}$ and join $w_1, w_2, ..., w_{c-b}$ to both $v_1$ and $v_3$ and also join $u_1, u_2, ..., u_{a-1}$ to both $v_1$ and $v_2$, there by producing the graph $G$ of Figure 4.6. Let $M = \{u_1, u_2, ..., u_{a-1}, v_{b-a+2}\}$ be the set of all simplicial vertices of $G$. By Corollary 2.8, every edge monophonic set of $G$ contains $M$. It is clear that $M$ is an edge monophonic set of $G$ so that $m_1(G) = a$. Let $M_1 = M \cup \{v_2, v_3, ..., v_{b-a+1}\}$. By Corollary 5.4 and Theorem 5.6 each connected edge monophonic set contains $M_1$. It is clear that $M_1$ is a connected edge monophonic set of $G$ so that $M_{1e}(G) = b$. Let $M_2 = M_1 \cup \{w_1, w_2, ..., w_{c-b}\}$. It is clear that $M_2$ is a connected edge monophonic set of $G$. Now, we show that $M_2$ is a minimal connected edge monophonic set of $G$. Assume, to the contrary, that $M_2$ is not a minimal connected edge monophonic set. Then there is a proper subset $T$ of $M_2$ such that $T$ is a connected edge monophonic set of $G$. Let $v \in M_2$ and $v \notin T$. By Corollary 5.4 and Theorem 5.6 it is clear that $v = w_i$, for some $i = 1, 2, ..., c - b$. Clearly, this $w_i$ does not lie on a monophonic path joining any pair of vertices of $T$ and so $T$ is not a connected edge monophonic set of $G$, which is a contradiction. Thus $M_2$ is a minimal connected edge monophonic set of $G$ and so $m_{1e}^+(G) \geq c$. Since the order of the graph is $c + 1$, it follows that $m_{1e}^+(G) = c$.

Case 2. Let $a > 2$ and $b - a = 1$. Since $c > b$, we have $c - b + 1 \geq 2$. Consider the graph $G$ given in Figure 4.7. Then as in Case 1, $M = \{u_1, u_2, ..., u_{a-1}, u_3\}$ is a minimum.
edge monophonic set, \( M_1 = M \cup \{v_2\} \) is a minimum connected edge monophonic set and \( M_2 = V(G) - \{v_1\} \) is a minimal connected edge monophonic set of \( G \) so that \( m_1(G) = a, m_{1c}(G) = b \) and \( m_{1c}^+(G) = c \).

**Case 3.** Let \( a = 2 \) and \( b - a = 1 \). Then \( b = 3 \). Consider the graph \( G \) given in Figure 4.8. Then as in Case 1, \( M = \{v_1, v_3\} \) is a minimum edge monophonic set, \( M_1 = \{v_1, v_2, v_3\} \) is a minimum connected edge monophonic set and \( M_2 = V(G) - \{v_1\} \) is a minimal connected edge monophonic set of \( G \) so that \( m_1(G) = a, m_{1c}(G) = b \) and \( m_{1c}^+(G) = c \).

**Case 4.** Let \( a = 2 \) and \( b - a \geq 2 \). Then \( b \geq 4 \). Consider the graph \( G \) given in Figure 4.9. Then as in Case 1, \( M = \{v_1, v_b\} \) is a minimum edge monophonic set, \( M_1 = \{v_1, v_2, ..., v_b\} \) is a minimum connected edge monophonic set and \( M_2 = V(G) - \{v_1\} \) is a minimal connected edge monophonic set of \( G \) so that \( m_1(G) = a, m_{1c}(G) = b \) and \( m_{1c}^+(G) = c \).

**THE FORCING CONNECTED EDGE MONOPHONIC NUMBER OF A GRAPH**

**Definition 5.33.** Let \( G \) be a connected graph and \( M \) a minimum connected edge monophonic set of \( G \). A subset \( T \subseteq M \) is called a forcing subset for \( M \) if \( M \) is the unique minimum connected edge monophonic set containing \( T \). A forcing subset for \( M \) of minimum cardinality is a minimum forcing subset of \( M \). The forcing connected edge monophonic number of \( M \), denoted by \( f_{m_{1c}}(M) \), is the cardinality of a minimum forcing subset of \( M \). The forcing connected edge monophonic number of \( G \), denoted by \( f_{m_{1c}}(G) \), is \( f_{m_{1c}}(G) = \min\{f_{m_{1c}}(M)\} \), where the minimum is taken over all minimum connected edge monophonic sets \( M \) in \( G \).
Example 5.34. For the graph $G$ given in Figure 5.6, $M_1 = \{v_2, v_3, v_4, v_5\}$, $M_2 = \{v_1, v_4, v_5, v_6\}$, $M_3 = \{v_1, v_2, v_3, v_4\}$ and $M_4 = \{v_3, v_4, v_5, v_6\}$ are the only four minimum connected edge monophonic sets of $G$ so that $f_{m1c}(M_1) = f_{m1c}(M_2) = f_{m1c}(M_3) = f_{m1c}(M_4) = 2$. Thus $f_{m1c}(G) = 2$.

![Figure 5.6](image)

The following result follows immediately from the definitions of the connected edge monophonic number and the forcing connected edge monophonic number of a connected graph $G$.

Theorem 5.35. For any connected graph $G$, $0 \leq f_{m1c}(G) \leq m_{1c}(G) \leq p$.

Remark 5.36. For any non-trivial tree $T$, by Corollary 5.8(i), the set of all vertices is the unique $m_{1c}$-set of $G$. It follows that $m_{1c}(T) = p$ and $f_{m1c}(T) = 0$. For the cycle $C_4: u_1, u_2, u_3, u_4, u_1$ of order 4, $M_1 = \{u_1, u_2, u_3\}$, $M_2 = \{u_2, u_3, u_4\}$, $M_3 = \{u_3, u_4, u_1\}$ and $M_4 = \{u_4, u_1, u_2\}$ are the only $m_{1c}$-sets of $C_4$ so that $m_{1c}(C_4) = 3$. Also, it is easily seen that $f_{m1c}(C_4) = 3$. Thus $m_{1c}(C_4) = f_{m1c}(C_4)$. Also, the inequalities in the theorem can be
strict. For the graph $G$ given in Figure 5.6, $f_{m1c}(G) = 2$, $m_{1c}(G) = 4$ and $p = 6$ as in Example 5.34. Thus $0 < f_{m1c}(G) < m_{1c}(G) < p$.

**Definition 5.37.** A vertex $v$ of a connected graph $G$ is said to be a connected edge monophonic vertex of $G$ if $v$ belongs to every minimum connected edge monophonic set of $G$.

**Example 5.38.** For the graph $G$ given in Figure 5.7, $M_1 = \{v_1, v_2, v_3\}$, $M_2 = \{v_1, v_2, v_4\}$ and $M_3 = \{v_1, v_2, v_5\}$ are the only three minimum connected edge monophonic sets of $G$ so that $v_1$ and $v_2$ are the connected edge monophonic vertices of $G$.

![Figure 5.7](image)

**Theorem 5.39.** Let $G$ be a connected graph. Then

(a) $f_{m1c}(G) = 0$ if and only if $G$ has a unique minimum connected edge monophonic set.

(b) $f_{m1c}(G) = 1$ if and only if $G$ has at least two minimum connected edge monophonic sets, one of which is a unique minimum connected edge monophonic set containing one of its elements.
(c) \( f_{m_1c}(G) = m_1c(G) \) if and only if no minimum connected edge monophonic set of \( G \) is the unique minimum connected edge monophonic set containing any of its proper subsets.

**Theorem 5.40.** Let \( G \) be a connected graph and let \( \mathcal{I} \) be the set of relative complements of the minimum forcing subsets in their respective minimum connected edge monophonic sets in \( G \). Then \( \bigcap_{F \in \mathcal{I}} F \) is the set of connected edge monophonic vertices of \( G \).

**Corollary 5.41.** Let \( G \) be a connected graph and \( M \) a minimum connected edge monophonic set of \( G \). Then no connected edge monophonic vertex of \( G \) belongs to any minimum forcing set of \( M \).

**Theorem 5.42.** Let \( G \) be a connected graph and \( W \) be the set of all connected edge monophonic vertices of \( G \). Then \( f_{m_1c}(G) \leq m_1c(G) - |W| \).

**Corollary 5.43.** If \( G \) is a connected graph with \( k \) simplicial vertices and \( l \) cut-vertices, then \( f_{m_1c}(G) \leq m_1c(G) - (k + l) \).

**Remark 5.44.** The bound in Corollary 5.43 is sharp. For the graph \( G \) given in Figure 5.8, \( M_1 = \{v_1, v_2, v_3, v_4, v_5\} \), \( M_2 = \{v_1, v_2, v_3, v_5, v_6\} \) and \( M_3 = \{v_1, v_2, v_3, v_4, v_6\} \) are the only \( m_1c \)-sets of \( G \) so that \( m_1c(G) = 5 \). Also, it is easily seen that \( f_{m_1c}(G) = 2 \) and \( W = \{v_1, v_2, v_3\} \) is the set of connected edge monophonic vertices of \( G \) with \( |W| = 3 \). Thus \( f_{m_1c}(G) = m_1c(G) - |W| \).
Theorem 5.45. For any complete graph $G = K_p$ ($p \geq 2$) or any non-trivial tree $G = T$, $f_{m1c}(G) = 0$.

Proof. For $G = K_p$, it follows from Corollary 5.4 and Theorem 5.6 that the set of all vertices of $G$ is the unique minimum connected edge monophonic set. Now, it follows from Theorem 5.39(a) that $f_{m1c}(G) = 0$.

If $G$ is a non-trivial tree, then by Corollary 5.8(i), the set of all vertices of $G$ is the unique minimum connected edge monophonic set of $G$ and so $f_{m1c}(G) = 0$ by Theorem 5.39(a).

Realisation Result

In view of Theorem 5.35, we have the following realization result.

Theorem 5.46. For every pair $a, b$ of integers with $b - 2a \geq 3$ and $b \geq 3$, there exists a connected graph $G$ such that $f_{m1c}(G) = a$ and $m_{1c}(G) = b$.

Proof. Let $F_i : r_i, s_i, u_i, t_i, r_i$ ($1 \leq i \leq a$) be a copy of $C_4$. Let $G$ be the graph obtained from $F_i$’s ($1 \leq i \leq a$) by first identifying the vertices $t_{i-1}$ of $F_{i-1}$ and $r_i$ of
$F_i \ (2 \leq i \leq a)$ and then adding $b-2a-1$ new vertices $x, z_1, z_2, \ldots, z_{b-2a-2}$ and joining the $b-2a-1$ edges $xr_1, t az_1, \ldots, t az_{b-2a-2}$. The graph $G$ is given in Figure 4.11. Let $Z = \{x, z_1, z_2, \ldots, z_{b-2a-2}, t a, r_1, r_2, \ldots, r_a, r_a\}$ be the set of end-vertices and cut-vertices of $G$. Let $H_i = \{u_i, s_j\} (1 \leq i \leq a)$.

First we show that $m_{1c}(G) = b$. By Corollary 5.4 and Theorem 5.6, $m_{1c}(G) \geq b - a$. Since $G[Z]$ is not connected, $Z$ is not a connected edge monophonic set of $G$. We observe that every connected edge monophonic set of $G$ must contain at least one vertex from $H_i (1 \leq i \leq a)$. Thus $m_{1c}(G) \geq b - a + a = b$. On the other hand, since the set $M_1 = Z \cup \{s_1, s_2, \ldots, s_a\}$ is a connected edge monophonic set of $G$, it follows that $m_{1c}(G) \leq |M_1| = b$. Thus $m_{1c}(G) = b$.

Next we show that $f_{m_{1c}}(G) = a$. Since every $m_{1c}$-set of $G$ contains $Z$, it follows from Theorem 5.42 that $f_{m_{1c}}(G) \leq m_{1c}(G) - |Z| = b - (b - a) = a$. Now, since $m_{1c}(G) = b$ and every minimum connected edge monophonic set of $G$ contains $Z$, it is easily seen that every minimum connected edge monophonic set $M$ is of the form $Z \cup \{c_1, c_2, \ldots, c_a\}$ where $c_i \in H_i (1 \leq i \leq a)$. Let $T$ be any proper subset of $M$ with $|T| < a$. Then there is a vertex $c_j (1 \leq j \leq a)$ such that $c_j \notin T$. Let $d_j$ be a vertex of $H_j$ distinct from $c_j$. Then $M_2 = (M - \{c_j\}) \cup \{d_j\}$ is a $m_{1c}$-set properly containing $T$. Thus $M$ is not the unique $m_{1c}$-set containing $T$. Thus $T$ is not a forcing subset of $M$. This is true for all minimum connected edge monophonic sets of $G$ and so it follows that $f_{m_{1c}}(G) = a$. 

**Example 5.47.** For the graph $G$ given in Figure 5.9, $M_1 = \{v_2, v_4, v_3\}, M_2 = \{v_2, v_3, v_4\}$, $M_3 = \{v_1, v_2, v_4\}, M_4 = \{v_1, v_3, v_4\}, M_5 = \{v_1, v_2, v_3\}$ and $M_6 = \{v_1, v_2, v_3\}$ are the only six $m_c$-sets so that $m_c(G) = 3$. Also, $f_{mc}(M_1) = f_{mc}(M_2) = f_{mc}(M_3) = f_{mc}(M_4) = f_{mc}(M_5) = f_{mc}(M_6) =$
$f_{mc}(M_6) = 2$ so that $f_{mc}(G) = 2$. It is clear that $M' = \{v_1, v_2, v_3, v_4\}$ is the unique $m_{1c}$-set of $G$ so that $f_{m_{1c}}(G) = 0$. So we leave the following as open problems.

**Problem 5.48.** For every integers $a$, $b$, $c$ and $d$ with $0 \leq c \leq d$, $a \leq b \leq d$, does there exists a connected graph $G$ such that $f_{m_{1c}}(G) = a$, $f_{mc}(G) = b$, $m_{c}(G) = c$ and $m_{1c}(G) = d$?

**Problem 5.49.** For every integers $a$, $b$, $c$ and $d$ with $a \leq b \leq c \leq d$, does there exists a connected graph $G$ such that $f_{mc}(G) = a$, $f_{m_{1c}}(G) = b$, $m_{c}(G) = c$ and $m_{1c}(G) = d$?

The forcing connected edge monophonic number of a graph is introduced in this chapter. By the similar manner the upper forcing connected edge monophonic number of a graph is defined in the following definition.

**Definition 5.50.** Let $G$ be a connected graph and $M$ a minimum connected edge monophonic set of $G$. A subset $T \subseteq M$ is called a forcing subset for $M$ if $M$ is the unique minimum connected edge monophonic set containing $T$. A forcing subset for $M$ of minimum cardinality is a minimum forcing subset of $M$. The forcing connected edge monophonic number of $M$, denoted by $f_{m_{1c}}(M)$, is the cardinality of a minimum forcing subset of $M$. The upper forcing connected edge monophonic number of $G$, denoted by
$f_{m1c}^+(G)$, is $f_{m1c}^+ (G) = \max \{ f_{m1c}(M) \}$, where the maximum is taken over all minimum connected edge monophonic sets $M$ in $G$.

For the graph $G$ given in Figure 5.10, $M_1 = \{ v_1, v_5, v_6, v_7 \}$, $M_2 = \{ v_2, v_3, v_4, v_7 \}$, $M_3 = \{ v_3, v_4, v_5, v_7 \}$ and $M_4 = \{ v_3, v_4, v_6, v_7 \}$ are the only four $m_{1c}$-sets of $G$ so that $m_{1c}(G) = 4$. Also, $f_{m1c}(M_1) = f_{m1c}(M_2) = 1$, $f_{m1c}(M_3) = f_{m1c}(M_4) = 2$ so that $f_{m1c}(G) = 1$ and $f_{m1c}^+(G) = 2$. So we leave the following problem as open question.

**Problem 5.51.** For every integers $a$, $b$ and $c$ with $0 \leq a \leq b \leq c$, $c \geq 2$, does there exists a connected graph $G$ such that $f_{m1c}(G) = a$, $f_{m1c}^+(G) = b$ and $m_{1c}(G) = c$?