CHAPTER 1

PRELIMINARIES

In this chapter, the basic definitions and theorems on graphs which are needed for the subsequent chapters are presented. For graph theoretic terminology, Harary [11] and Chartrand and Ping Zhang [9] are referred to. For graph labeling, Gallian [8] is referred to.

Definition 1.1. A graph $G$ is a finite non-empty set of objects called vertices together with a set of unordered pairs of distinct vertices of $G$ called edges. The vertex set and edge set of $G$ are denoted by $V(G)$ and $E(G)$ respectively. The number of vertices in $G$ is called the order of $G$ and the number of edges in $G$ is called the size of $G$. A graph of order $p$ and size $q$ is called a $(p, q)$-graph. A graph is trivial if its vertex set is a singleton.

If $e = uv$ is an edge of $G$, we say that $u$ and $v$ are adjacent and each is incident with $e$. If two distinct edges are incident with a common vertex, then they are said to be adjacent edges.
Definition 1.2. A graph $H$ is called a subgraph of $G$ if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A subgraph $H$ of a graph $G$ is a proper subgraph of $G$ if either $V(H) \neq V(G)$ or $E(H) \neq E(G)$. A spanning subgraph of $G$ is a subgraph $H$ of $G$ with $V(G) = V(H)$.

For a set $S$ of vertices of $G$, the induced subgraph is the maximal subgraph of $G$ with vertex set $S$ and is denoted by $\langle S \rangle$. Thus, two vertices of $S$ are adjacent in $\langle S \rangle$ if and only if they are adjacent in $G$. The induced subgraph $\langle S \rangle$ is also denoted by $G[S]$. Similarly, for a subset $E'$ of $E(G)$, the edge induced subgraph $\langle E' \rangle$ is the subgraph of $G$ whose vertex set is the set of ends of edges in $E'$ and whose edge set is $E'$.

Let $v$ be a vertex of a graph $G$ and $|V(G)| \geq 2$. Then the induced subgraph $\langle V(G) - \{v\} \rangle$ is denoted by $G - v$ and it is the subgraph of $G$ obtained by the removal of $v$ and the edges incident with $v$. If $e \in E(G)$, the spanning subgraph with edge set $E(G) - \{e\}$ is denoted by $G - e$ and it is the subgraph of $G$ obtained by the removal of the edge $e$. The graph obtained from $G$ by adding an edge $e$ is denoted by $G + e$.

Definition 1.3. The degree of a vertex $v$ in a graph $G$ is defined to be the number of edges incident with $v$ and is denoted by $\deg v$.

A vertex of degree zero is an isolated vertex and a vertex of degree
one is a pendant vertex. An edge $e$ in a graph $G$ is called a pendant edge if it is incident with a pendant vertex.

The minimum of $\{\text{deg } v : v \in V(G)\}$ is denoted by $\delta$ and the maximum of $\{\text{deg } v : v \in V(G)\}$ is denoted by $\Delta$.

**Definition 1.4.** A graph $G$ is called $r$-regular if every vertex of $G$ has degree $r$.

A graph is said to be regular if it is $r$-regular for some non-negative integer $r$. In particular, a 3-regular graph is called a cubic graph.

**Definition 1.5.** A walk of a graph $G$ is an alternating sequence of points and lines $v_0, e_1, v_1, e_2, \ldots, v_{n-1}, e_n, v_n$ beginning with vertex $v_0$ and ending with vertex $v_n$ such that $e_i = v_{i-1}v_i (i=1, 2, \ldots, n)$. It is called a $v_0-v_n$ walk and $n$ is called the length of the walk.

A walk in which all the edges are distinct is called a trial and a walk in which all the vertices are distinct is called a path.

If the two end points $v_0$ and $v_n$ coincide in a path, it is called a cycle.

We henceforth denote by $C_n$ the cycle consisting of $n$ vertices and by $P_n$ a path on $n$ points. The cycle $C_3$ is also called a triangle.

**Definition 1.6.** A closed trial is called a cycle.
Definition 1.7. A graph $G$ is said to be *connected* if any two vertices of $G$ are joined by a path. Otherwise, it is called *disconnected*.

Definition 1.8. A graph is a *acyclic* if it has no cycles.

Definition 1.9. A connected, acyclic graph is called a *tree*.

Definition 1.10. Any graph without cycles is called a *forest*. Thus the components of a forest are trees.

Definition 1.11. A connected graph with a unique cycle is called a *unicyclic* graph.

Definition 1.12. A graph $G$ is *complete* if every pair of distinct vertices of $G$ are adjacent in $G$. A complete graph on $p$ vertices is denoted by $K_p$.

The size of $K_p$ is $\frac{p(p-1)}{2}$.

Definition 1.13. A *bipartite* graph is a graph $G$ whose vertex set $V(G)$ can be partitioned into two non-empty subsets $X$ and $Y$ such that each edge of $G$ has one end in $X$ and the other end in $Y$. The pair $(X,Y)$ is called a *bipartition* of $G$. If further, every vertex in $X$ is adjacent to all the vertices of $Y$, then $G$ is called a *complete bipartite* graph. The complete bipartite graph
with bipartition \((X, Y)\) such that \(|X| = m\) and \(|Y| = n\) is denoted by \(K_{m,n}\).

The graph \(K_{1,n}\) is called a star.

**Remark 1.14.** The vertex of \(K_{1,n}\) with degree \(n\) is called the central vertex or apex.

**Definition 1.15.** Two graphs \(G_1\) and \(G_2\) are said to be isomorphic if there exists a bijection \(\phi : V(G_1) \to V(G_2)\) such that \(uv \in E(G_1)\) if and only if \(\phi(u)\phi(v) \in E(G_2)\); such a function \(\phi\) is called an isomorphism from \(G_1\) to \(G_2\). If \(G_1\) and \(G_2\) are isomorphic, it is written as \(G_1 \cong G_2\).

**Definition 1.16.** The complement \(\bar{G}\) of a graph \(G\) is that graph whose vertex set is \(V(G)\) and such that for each pair \(u, v\) of vertices of \(G\), \(uv\) is an edge of \(\bar{G}\) if and only if \(uv\) is not an edge of \(G\).

The graph \(\bar{K}_n\) has \(n\) vertices and no edges. It is called empty graph of order \(n\).

**Definition 1.17.** Let \(G_1 = (V_1, E_1)\) and \(G_2 = (V_2, E_2)\) be two graphs. It is said that \(G_1\) and \(G_2\) are disjoint if they have no vertex in common and edge disjoint if they have no edge in common.

The graph \(G = (V, E)\), where \(V = V_1 \cup V_2\) and \(E = E_1 \cup E_2\) is called the union of \(G_1\) and \(G_2\) and is denoted by \(G_1 \cup G_2\).
If \( V_1 \cap V_2 \neq \emptyset \), then the graph \( G = (V, E) \), where \( V = V_1 \cap V_2 \) and \( E = E_1 \cap E_2 \) is called the intersection of \( G_1 \) and \( G_2 \) and is denoted by \( G_1 \cap G_2 \).

If \( G_1 \) and \( G_2 \) are disjoint graphs, then the join of \( G_1 \) and \( G_2 \) is denoted by \( G_1 + G_2 \) and is defined as \( V(G_1 + G_2) = V_1 \cup V_2 \) and \( E(G_1 + G_2) = E_1 \cup E_2 \cup \{uv : u \in V_1, v \in V_2\} \).

**Definition 1.18.** An \((n,t)\)-kite consists of a cycle of length \( n \) with a \( t \)-edge path (called tail) attached to one vertex of the cycle.

The \((n,1)\)-kite is called a kite with tail length 1.

**Definition 1.19.** For \( p \geq 4 \), the wheel on \( p \) vertices, denoted by \( W_p \), is defined to be the graph \( K_1 + C_{p-1} \). Note that \( q = 2(p-1) \).

**Remark 1.20.** The graph \( F_n = P_n + K_1 \) is called a fan and \( P_n + 2K_1 \) is called a double fan. The graph \( W_n = C_n + K_1 \) \((n \geq 3)\) is called a wheel and the graph \( C_n + 2K_1 \) \((n \geq 3)\) is called a double cone.

**Definition 1.21.** The cartesian product \( G_1 \) and \( G_2 \) of two graphs is defined to be the graph with vertex set \( V_1 \) and \( V_2 \) and two vertices \( u = (u_1, u_2) \) and \( v = (v_1, v_2) \) are adjacent in \( G_1 \) and \( G_2 \) if either \( u_1 = v_1 \) and \( u_2 \) is adjacent to \( v_2 \) or \( u_2 = v_2 \) and \( u_1 \) is adjacent to \( v_1 \). It is denoted by \( G_1 \times G_2 \).
The graph $L_n = P_n \times P_2$ is called a ladder and $P_m \times P_n$ is called a planar grid. The graph $C_n \times P_2$ is called a prism.

**Definition 1.22.** The corona $G_1 \odot G_2$ of two graphs $G_1$ and $G_2$ is defined as the graph obtained by taking one copy of $G_1$ (with $p_1$ vertices) and $p_1$ copies of $G_2$ and then joining the $i^{th}$ vertex of $G_1$ to all the vertices in the $i^{th}$ copy of $G_2$.

The graph $P_n \odot K_1$ is called a comb and the graph $C_n \odot K_1$ is called a crown.

**Definition 1.23.** The Helm $H_n$ is the graph obtained from a wheel $W_n$ by attaching a pendant edge at each vertex of the $n$-cycle of the wheel.

**Definition 1.24.** A Dragon is a graph obtained by joining an end point of a path $P_m$ to a vertex of the cycle $C_n$. It is denoted by $C_n \odot P_m$.

**Notation 1.25.** The one point union of $t$ cycles, each of length $n$ is denoted by $C_n^{(t)}$ is called the friendship graph.

**Definition 1.26.** The Bistar $B(m,n)$ is the graph obtained by making adjacent the two central vertices of $K_{l,m}$ and $K_{l,n}$.

**Definition 1.27.** A caterpillar is a tree with the property that the removal of its end vertices results in a path.
Definition 1.28. A lobstar is a tree with the property that the removal of its end vertices results in a caterpillar.

Definition 1.29. A vertex labeling of a graph $G$ is an arrangement $f$ of labels to the vertices of $G$ which induces for each edge $uv$ a label depending on the vertex labels $f(u)$ and $f(v)$. Similarly an edge labeling of a graph $G$ is an assignment of labels to the edges of $G$ that induces for each vertex a label depending on the labels of the edges incident to it.

Definition 1.30. Let $G = (V, E)$ be a graph. A mapping $f : V(G) \rightarrow \{0,1\}$ is called binary vertex labeling of $G$ and $f(v)$ is called the label of the vertex $v$ of $G$ under $f$.

Next, we give some concepts in Number Theory [6], which will be used to label the vertices of a graph.

Definition 1.31. Let $a$ and $b$ be two integers. If $a$ divides $b$ it means that there is a positive integer $k$ such that $b = ka$. It is denoted by $a \mid b$.

If $a$ does not divide $b$, then it is denoted by $a \nmid b$.

Now we give the definition of divisor function.

Definition 1.32. The divisor function of integer $d(n)$ is defined
by $d(n)=\Sigma 1$. That is, $d(n)$ denotes the number of all divisors of an integer $n$.

**Example 1.33.** Consider the integer 6. The divisors of 6 are 1, 2, 3, 6. Then $d(6) = 4$.

Next, we define the divisor summability function.

**Definition 1.34.** Let $n$ be an integer and $x$ be a real number. The divisor summability function is defined as $D(x) = \Sigma d(n)$. That is, $D(x)$ is the sum of the number of all divisors of $n$ for $n \leq x$.

**Example 1.35.** If $x = 6.7$, then

$$D(6.7) = d(1) + d(2) + d(3) + d(4) + d(5) + d(6)$$

$$= 1 + 2 + 2 + 3 + 2 + 4$$

$$= 14$$

The big $O$ notation is defined as follows.

**Definition 1.36.** Let $f(x)$ and $g(x)$ be two functions defined on some subset of the real numbers. $f(x) = O(g(x))$ as $x \to \infty$ if and only if there is a positive real number $M$ and a real number $x_0$ such that $|f(x)| \leq M |g(x)|$ for all $x > x_0$.

Next, we state Dirichlets divisor problem as follows.
Result 1.37. \( D(x) = x \log x + x(2\gamma - 1) + \Delta(x) \) where \( \gamma \) is the Euler-Mascheroni Constant given by \( \gamma \approx 0.577 \) approximately and \( \Delta(x) = O(\sqrt{x}) \).

Theorem 1.38. (Tchebychef):

For any positive integer \( n > 1 \), there is at least one prime \( p \) such that \( n < p < 2n \).