CHAPTER 3

A CO-ROTATED UPDATED LAGRANGIAN
PROCEDURE FOR THE ELASTIC POSTBUCKLING
ANALYSIS OF SPACE FRAMES

3.1 GENERAL

Structural systems composed of linear steel elements of one kind or another, such as in geodesic domes, reticulated shells, have gained importance in India. As mentioned earlier, their usefulness and economy in covering large space is well established. Thus, accurate evaluation of behaviour, performance and reliability of various truss type structures composed of discrete elements has become important. The theoretical knowledge of the static response of truss structures exhibiting nonlinear elastic behaviour has progressed considerably in the last twenty years. It is important to remember that in large space structures, more often the instability results from a change of physical states and the change of geometry is the cause of failure. The change in physical states can be identified by the ‘postbuckling analysis’ of the truss structural systems and hence the geometric nonlinearity is important. The relevant geometric nonlinear finite element formulations for space truss element have been derived, based on updated Lagrangian technique overlaid on a co-rotational framework.
3.2 Co-rotational procedure in geometric nonlinear analysis of SPACE structures

3.2.1 Concept of co-rotational procedure

The commonly used approaches for the analysis of nonlinear problems are the Total Lagrangian (TL) procedure and the Updated Lagrangian (UL) procedure. Another approach called the co-rotational (CR) formulation has also been used by researchers. The basic idea of the CR formulation is to separate out the rigid body motion from the local deformation of the element. This is accomplished by a local co-ordinate system, which is so selected that some of the nodal displacement components become zero. This co-ordinate system continuously translates and rotates with the deformation of the element. This co-ordinate axes eliminates the rigid body displacements of the element, leaving only the deformational displacements or the natural displacements to be accounted through strain-displacement relations. These strain-displacement relations can be handled locally using either a TL or a UL procedure. Thus we can have a combination of CR-TL and CR-UL procedures for nonlinear analysis of structures. The number of natural degree of freedom (DOF), which represent the minimum number of geometric variables necessary to completely determine the deformations of the element, is equal to the number of global DOF minus the number of rigid body modes of the element. The CR formulation is very efficient when an element has more rigid body modes than the deformational modes. Obviously, space truss elements are the most appropriate for CR formulation, because a space truss element has five rigid body modes (three translations and two rotations) and only one deformational mode.
3.3 DESCRIPTION OF MOTION IN CR FORMULATION

Figure 3.1 shows the space truss element in its original or reference configuration at time \( T=0 \), co-rotated configuration at time \( T=t \) and the neighbouring configuration at time \( T=t+\Delta t \). Figure 3.1 also shows displacements (for clarity this is shown only for node 1) experienced by the element from reference to co-rotated configuration. The primed ('\) quantities are used to represent the co-rotated quantities. If \( x,y,z \) are the coordinates of the material point on the element in the reference configuration, then \( x', y', z' \) are the coordinates of the same material point of the element in the co-rotated or current configuration. The unit vector \( e_x \) is the co-rotating base vector which describe the motion of the truss element in the co-rotated system. Within the co-rotating system, the deformation of the element to an adjacent or neighbouring configuration can be described either by a TL or UL format.

![Figure 3.1 Space truss element in its reference and co-rotated co-ordinates](image)

The conventional 3-D truss element is shown in Figure 3.2(a) with global displacements and their associated nodal forces. The space truss
element in the co-rotated system is shown in Figure 3.2(b). The co-rotated element can be obtained by attaching the origin of the local co-ordinate to node number 1 of the deformed truss element in the co-rotated configuration as shown in Figure 3.1. The co-ordinate of node 2 can be continuously recalculated using the base vector of the element in the original undeformed and current configurations. As noted earlier, for the space truss element, the extension and shortening of the truss element is the only strain producing deformation and all the other five degrees of freedom (three translations and two rotations) are obtained by rigid body rotation.

![Diagram](image)

**Figure 3.2 Conventional and natural truss elements**

The geometric nonlinear finite element formulations of a space truss element in the co-rotational context involve two basic steps: (a) relating the reference co-ordinates to the co-rotated ones, and (b) relating the co-rotated co-ordinates to the strain producing (natural) deformations. The former involves the derivation of nonlinear transformation relating the co-rotated variables to the global ones and latter involves the derivation of natural force displacement relations using the natural tangent stiffness matrix. From the polar decomposition theorem in continuum mechanics, it is seen that these steps are completely independent of each other. Combining these two relations the tangent stiffness matrix can be formulated.
3.4 NONLINEAR TRANSFORMATION BETWEEN REFERENCE AND CO-ROTATED VARIABLES

In the present thesis, the Gibbs notation is followed. Bold faced letters in expressions represents vectors or tensors. Ordinary letters represent scalar variables. The origin of the co-rotating coordinate system \((x',y',z')\) coincides with node 1 of the deformed element and \(x'\) axis is defined to pass through the node 2 as shown in Figure 3.1. The \(y'\) and the \(z'\) axes are perpendicular to the \(x'\) axis and their definition is not needed for the CR truss finite element. In the finite element context, let the global displacement and force DOFs be defined by translation displacements at node 1 and node 2 as \(p = \{u_1, v_1, w_1, u_2, v_2, w_2\}\) and \(q = \{U_1, V_1, W_1, U_2, V_2, W_2\}\) respectively. The natural force and displacement are single DOF quantities represented by \(q_n\) and \(p_n\) respectively as in Figure 3.2 (b). The length of the truss element in the reference and co-rotated states are denoted by \(L_0\) and \(L'\) respectively. Correspondingly the area of cross sections are \(A_0\) and \(A'\) respectively, as shown in Figure 3.1. The co-rotating base vector \(e'_x\) referred to the global coordinate system can be written in component form as \(e'_x = 1/L' \begin{bmatrix} x'_21 \\ y'_21 \\ z'_21 \end{bmatrix} \) where \(x'_21 = x'_2 - x'_1\) and so on. The current length is found to be \(L' = (x'_21)^2 + (y'_21)^2 + (z'_21)^2)^{1/2}\). Let \(A\) be a transformation matrix whose first row is equal to the components of the base vector \(e'_x\). The rest of the elements can be written in terms of these three components. The local or co-rotated displacements and their rates can be related to the global displacements and their rates as

\[
\dot{p}' = A \dot{p} \tag{3.1}
\]

where a dot represents the material time derivative. The dotted quantities may be regarded as small changes and referred to as \(\delta'\) (del). By using the symbol \(\delta\) to represent small changes in quantities, Equation (3.1) can be rewritten as
\[
\dot{\mathbf{p}}' = A \dot{\mathbf{p}}
\]  

(3.2)

Assuming that the truss element deforms into an adjacent configuration denoted by quantities with an over bar at \( t + \Delta t \), the increment in natural displacement \( \dot{\mathbf{p}}_n \) is given by \( \bar{x}_2 - x_2' \). The new length \( \bar{x}_2 \) can be obtained as \( \bar{x}_2 = \bar{e}_x^T \bar{X}_{21} \) where \( \bar{e}_x^T \) is the co-rotating base vector at the adjacent configuration and \( \bar{X}_{21} = (\bar{x}_{21}, \bar{y}_{21}, \bar{z}_{21}) \). This again can be written as \( \dot{\mathbf{p}}_n = \bar{L} - L' \). The natural displacement increments and the displacement increments at current co-ordinates can be related as

\[
[\delta \mathbf{p}_n] = \begin{bmatrix} \delta u'_1 \\ \delta v'_1 \\ \delta w'_1 \\ \delta u'_2 \\ \delta v'_2 \\ \delta w'_2 \end{bmatrix} = [-1,0,0,1,0,0]
\]  

(3.3)

which is written in rate form as \( \delta \mathbf{p}_n = \mathbf{H} \dot{\mathbf{p}}' \). In a general co-rotational context \( \mathbf{H} \) is a second-order tensor and hence in the present formulation also it is represented as bold faced capital letter, even though \( \mathbf{H} \) happens to be vector. Therefore, the natural and global variables can be related by combining Equations (3.2) and (3.3) as \( \dot{\mathbf{p}}_n = \mathbf{H} A \dot{\mathbf{p}} \).

### 3.5 NODAL FORCE RELATIONS

The global force vector \( \mathbf{q} \) and co-rotated force vector \( \mathbf{q}' \) are related by the transformation matrix \( A \) as

\[
\mathbf{q} = A^T \mathbf{q}'
\]  

(3.4)
Since the node 1 is attached to the origin of the co-rotated system and in the $x'$ direction of the co-rotating system, the corresponding reaction forces $R_{x1}$ at node 1 is obtained by equilibrium as $R_{x1} = -q_n$. Hence, the local or co-rotated forces and the natural force $q_n$ are related by

$$
\begin{bmatrix}
U_1' \\
V_1' \\
W_1'
\end{bmatrix}
= \begin{bmatrix}
-1 \\
0 \\
0
\end{bmatrix}
\begin{bmatrix}
q_n
\end{bmatrix}
$$

(3.5)

Equation (3.5) can be rewritten as

$$
q' = H^T q_n
$$

(3.6)

This is the pivotal equation which forms the basis of the co-rotational procedure. Taking material time derivative of Equation (3.6), it can be written as

$$
\dot{q}' = H^T \dot{q}_n + \dot{H}^T q_n
$$

(3.7)

Here, the first term of the right hand side represents the change of forces in the co-rotated system due to the change in the natural force $q_n$. This term can easily be obtained by the natural stiffness relation $\dot{q}_n = K' \dot{p}_n$, where $K'$ is the natural stiffness matrix which is a scalar quantity as there is only one natural degree of freedom for the space truss element. The second term on the right hand side of Equation (3.7) can be interpreted as change in the forces due to rotation of the natural force with its magnitude kept invariant. This also follows from the polar decomposition theorem of continuum mechanics which states that change in the force system of a deformable body can be expressed as a 'stretch' (first term on the right hand side of Equation (3.7))
followed by a ‘rotation’ (second term on the right hand side of Equation (3.7)) and vice versa. The most important relation in the co-rotational procedure is the evaluation of the vector $\delta H^T$ and this can be derived in several ways. In the present study, $\delta H^T$ is be derived from geometric principles. The vector $\delta H^T$ has two components and hence $\delta H^T = \delta H_t^T + \delta H_r^T$ where $H_t$ and $H_r$ are the components due to translation and rotation respectively. As illustrated in Figure 3.3, the truss element from the co-rotated configuration undergoes displacement $\{\delta u', \delta v', \delta w'\}$ at the node 1 and $\{\delta u_2', \delta v_2', \delta w_2'\}$ at the node 2 (not shown in the Figure 3.3 for clarity). For the purpose of evaluating the change in the co-rotated forces, let us translate back the adjacent configuration by subtracting the displacements $\{\delta u_1, \delta v_1, \delta w_1\}$ from both node 1 and node 2 of the element. This configuration is called the ‘translated back configuration’.

![Figure 3.3 Change in the co-rotated force vector from geometric principles](image)

At node 2, this translation amounts to a displacement of $\{\delta u_2 - \delta u_1', \delta v_2 - \delta v_1', \delta w_2 - \delta w_1'\}$ which (by our earlier convention) can be written as $\{\delta u_2' - \delta u_1', \delta v_2' - \delta v_1', \delta w_2' - \delta w_1'\}$. This translation does not cause any change in the co-rotated forces, as the natural force does not undergo any rotation. Hence
$\delta(H_1)^T$ will be a null vector and this also implies that a variation on the vector $H$ in Equation (3.3) or Equation (3.5), which are functions of translation, would also be a null vector. Now the change in the co-rotated forces can be found out by a reverse process of rotating the natural force back to the co-rotated configuration from the translated back configuration. This involves rotation of the force about $z$ axis and then about $y$ axis. From geometry it is seen that such a rotation involves a change of force $(q_n \delta w'_{21}/L)$ in the $z$ axis and $(q_n \delta v'_{21}/L)$ about the $y$ axis. $\delta(H_1)^T$ which signifies the change in the co-rotated force due to the rotation of natural force along with the changes due to the reaction can be written as

$$
\delta(H_1)^T = \frac{1}{L} \begin{bmatrix}
\delta U_1' \\
\delta V_1' \\
\delta W_1' \\
\delta U_2' \\
\delta V_2' \\
\delta W_2'
\end{bmatrix} = \begin{bmatrix}
0 \\
-\delta v'_{21} \\
-\delta w_{21} \\
0 \\
\delta v'_{21} \\
\delta w_{21}
\end{bmatrix} [q_n] \tag{3.8}
$$

Since $\delta H^T = (0) + \delta(H_1)^T$, the right hand side of Equation (3.8) can be written as $\delta H^T q_n$. It should be noted that Equation (3.8) coincides with the equation presented by Mattiasson and Samuelsson (1984) using continuum mechanics principles. Now we have derived all relevant terms in Equation (3.7). By taking a variation of Equation (3.4) along with the use of Equation (3.7), we can show that,

$$
\delta q = A^T H^T \delta q_n + A^T \delta H^T q_n \tag{3.9}
$$

The vector $\delta H^T$ can be written in terms of co-rotated quantities by a matrix $B$ as $\delta H^T = B \dot{\phi}$ where the matrix $B$ is given by
Now Equation (3.9) can be rewritten using the relation $\dot{\ddot{q}}_n = K' \ddot{\ddot{p}}_n$ and Equation (3.3) as

$$\dot{\ddot{q}} = A^T H^T K' H \ddot{\ddot{p}} + A^T q_n B \ddot{\ddot{p}}'$$

(3.11)

By the use of Equation (3.2) the above equation can be written in terms of global co-ordinates as

$$\dot{\ddot{q}} = A^T H^T K' H A \ddot{\ddot{p}} + A^T q_n B A \ddot{\ddot{p}}$$

(3.12)

which can be re-written in terms of the tangent stiffness relations in the global coordinate as

$$\dot{\ddot{q}} = A^T (H^T K' H + q_n B) A \ddot{\ddot{p}}$$

(3.13)

Carrying out the multiplication in Equation (3.13), the global tangent stiffness matrix can be written as

$$K_T = A^T (H^T K' H + q_n B) A = \begin{bmatrix} K_C & -K_C \\ -K_C & K_C \end{bmatrix}$$

(3.14)

Where the component matrices $K_C$ can be worked out to be

$$K_C = (K' - q_n / L) \begin{bmatrix} A_{11}^2 & A_{11} A_{12} & A_{11} A_{13} \\ A_{12} A_{11} & A_{12}^2 & A_{12} A_{13} \\ A_{13} A_{11} & A_{13} A_{12} & A_{13}^2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \frac{q_n}{L}$$

(3.15)
The values of $A_{11}$, $A_{12}$ and $A_{13}$ are the components of the transformation matrix that relate local and global co-ordinates and are well defined in standard text books on finite elements.

### 3.6 Natural Stiffness Matrix Using Updated Lagrangian (UL) Procedure

In Equation (3.15), $K'$ represents the natural stiffness relations or the local tangent stiffness matrix which can be derived using local or co-rotated coordinates. In this study, a UL procedure is used for formulating the local tangent stiffness matrix. The equilibrium of the element is formulated as

$$q_n = [B^L \sigma^l] dV = AL(B^L \sigma^l)$$

(3.16)

where $\sigma^l$ is the true stress $\sigma$ which is nothing but the actual stress which is also called as Cauchy stress in the truss member and $A$ and $L$ are the current area and current length of the truss members and $q_n$ is the natural force in the element. It may be shown that $B^L = I/L$. Hence the natural force in the member at the current geometry can be shown to be

$$q_n = A \sigma$$

(3.17)

The natural stiffness matrix at the UL format can be shown to be

$$K' = K^L + K^\sigma$$

(3.18)

After applying the relation between displacements and the corresponding shape functions $K^L$ and $K^\sigma$ can be shown to be
3.7 INFLUENCE OF INDIVIDUAL MEMBER INSTABILITY

In a structural assembly such as space trusses or in reticulated shells, the individual discrete members buckle locally and globally. Many of the practical struts in space trusses have cross sections, which are symmetric about two perpendicular axes and possess substantial torsion stiffness. Such struts tend to fail locally by buckling about the weaker axis. The bifurcational local buckling of such struts have a significant influence on the subsequent load transfer of the structure and its overall postbuckling response. Hence, it is of importance to include the effect of individual member buckling in elastic postbuckling analysis of truss structures. The postbuckled stiffness of a pin-ended axially loaded strut has been shown to be a constant \( \frac{dQ}{dA} = \left( \frac{Q_E}{2L_0} \right) \) by Britvec (1973, 1995), where \( Q_E \) is the Euler Critical load, \( Q_A \) and \( A \) are the applied load and axial displacement, respectively. This relation was used by Kondoh and Atluri (1985), Tanaka et al (1985), Kondoh and Tanaka (1986) for modelling compression member instability in an UL formulation. Yang and Yang (1996) considered the effect of member instability and yielding using plastification of cross section of the members. In the present study, individual struts of space frames are assumed to be hinged at both the ends and are axially loaded. Based on these assumptions the stiffness

\[
K'_{L} = \frac{AE}{L} 
\]

\[
K'_{\sigma} = \frac{\sigma A}{L} 
\]

where \( E \) is the Young's modulus. And hence the local tangent stiffness matrix in the UL format can be written as

\[
K' = \frac{A}{L} (E + \sigma) 
\]

\[
(3.19) 
\]

\[
(3.20) 
\]

\[
(3.21) 
\]
characteristics of individual struts in the postbuckling stage can be obtained as follows. The initial imperfection of the strut of length $L$ can be written as (Figure 3.4)

$$w_0 = \delta_0 \sin(\pi x/L)$$

(3.25)

in which $\delta_0$ is the amplitude of initial imperfection. The total deflection of the strut can be written as

$$w = \delta \sin(\pi x/L) = \left[ \frac{\delta_0}{(1 - Q_A/Q_E)} \right] \sin(\pi x/L)$$

(3.26)

where $Q_A$ is the axial load and $Q_E$ is the Euler critical load of the strut.

The curved length in the deformed state can be obtained as

$$L_0 = \int_0^L \sqrt{1 + (dw/dx)^2} \, dx$$

(3.27)

![Figure 3.4 Deformed shape of an initially imperfect strut](image)

The axial shortening due to bowing effect can be written as $\Delta_0 = L - L_0$. Since Equation (3.27) cannot be solved explicitly, by assuming a quadratic relation between the curved length and chord length, the integral in Equation (3.27) can be approximated as

$$L = \frac{L_0}{1 + 2/3 (2\delta / L_0)^2}$$

(3.28)
The total end shortening \( \Delta \) due to both the axial and bowing effect can be written as \( \Delta = \Delta_a + \Delta_b \) which is simply

\[
\Delta = \frac{Q_A L_0}{EA} + L_0 - \frac{L_0}{1 + 2/3(2\delta / \ell)^2}
\]

(3.29)

From this relation of \( Q \) and \( \Delta \) the postbuckling stiffness of an imperfect column can be obtained as

\[
\frac{dQ_A}{d\Delta} = \frac{1}{\left[\frac{L_0}{EA} + \frac{L_0}{Q_A (1 - \frac{1}{1 + 2/3(2\delta / \ell)^2})}\right]}
\]

(3.30)

in which \( \delta \) is given by

\[
\delta = \frac{\delta_0}{1 - \frac{Q_A}{Q_E}}
\]

(3.31)

The initial imperfection in the present case has been assumed as 0.001\( L_0 \). When Equation (3.30) is used as the postbuckled stiffness for the strut, the automatic load incrementation procedure along with arc-length methods become easier. Nevertheless, when \( Q_A \) approaches the value of \( Q_E \) the postbuckling stiffness of the strut becomes negligible and hence small load increments near the Euler critical load are unavoidable. Hence a procedure of scaling down the loading increments near the Euler critical load of individual members has been implemented in the present study.

3.8 NUMERICAL STUDIES

The formulation described in the earlier section has been incorporated in a computer program developed for this purpose. Solution procedures such as the arc-length procedure and minimum residual
displacement (MRD) method have been incorporated in the computer program. The load incrementation was done automatically (Crisfield 1991).

For this, the concept of ‘current stiffness parameter (CSP)’ proposed by Bergan and Soreide (1973) and Bergan et al (1978) has been used. Two problems of space truss structures are chosen to calibrate the derived formulation and computer program. The first problem considered is the two bar toggle problem (Figure 3.5), which exhibits a highly nonlinear behaviour. The results of the present study using a CR-UL procedure is compared with a pure TL procedure and are presented in Table 3.1. The results indicate the advantage of the CR formulation for truss structures in terms of number of iterations.

![Figure 3.5 Two bar truss](image)

The second example problem is chosen to verify the postbuckling analysis of trusses with member instability. The 24 bar truss dome (Figure 3.6) in which the buckling of struts in the inner circle and the overall postbuckling of the dome was considered. The results are shown in Figure 3.7 for the case of postbuckling analysis of 24 bar dome, Cichon (1984) presented results only in the limited loading range. The results of the present study compare very well with the results of Cichon (1984) as evident from Figure 3.7.
Table 3.1 Comparison of CR-UL and TL procedures for two bar truss

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Figure 3.6  Geometry of 24 bar truss dome

Figure 3.7  Post buckling response of 24 bar truss dome
3.9 SUMMARY

A local updated Lagrangian (UL) formulation has been implemented on co-rotated finite element framework for space trusses. The relation between the increments in co-rotated forces and the rotation of the natural force has been derived using geometric principles. A computer program that incorporated the derived formulations has been tested against two benchmark problems published in the literature. From the study it was observed that for truss structures exhibiting postbuckling limit point behaviour, inclusion of the second order strain relation and hence Green’s strain is very essential. The importance of inclusion of member instabilities in the overall system behaviour and the relevant equations for implementing in the numerical procedure are presented.