Chapter 4

Design of Wavelet for MST Radar clear air echo

As mentioned earlier, a wavelet suitable for the clear-air echoes of MST radar is designed in this work. Details of this design methodology and the wavelet thus obtained and its properties are described in this chapter. The MRA filter coefficients computed for this wavelet are also presented in this chapter. The wavelet designed in this work is compared with standard wavelets in its efficacy in providing sparse representation of clear air echoes. This methodology is on the same lines of one which is developed by Soon-Huat [2002] described in the Chapter 3, but instead of Genetic algorithm, Levenberg-Marquardt’s algorithm is used as an optimization technique.

4.1 Selection of mother wavelet to MST radar data

As the MST radar clear-air Doppler echo is of have predominantly Gaussian distribution in off-vertical direction [Anandan et al., 2005], it can be assumed that the wavelet basis function is a shifted Gaussian in frequency domain, which can be written as

\[
X(f) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(\frac{(f - f_0)^2}{2\sigma^2}\right)
\]  

(4.1)

Then chosen mother wavelet in time domain is given by the following equation.

\[
x(t) = \phi(t) = \exp\left(-2\pi^2\sigma^2 t^2\right)\exp\left(j\omega_0 t\right)
\]  

(4.2)

For typical values of \(\sigma = 0.12\) and \(f_0 = 0.4\) Hz (which are typical values of line width and Doppler shifts of MST radar clear-air echoes) shape of chosen mother wavelet centered on 3.5 sec is shown in Fig (4.1). The corresponding frequency domain spectrum is shown in Fig (4.2).
Figure 4.1: Proposed wavelet basis suitable to MST Radar clear air echo in time domain.

Figure 4.2: The frequency domain representation of the proposed mother wavelet.
4.2 Computing MRA filter coefficients

A wavelet function should exhibit normalization, orthogonally properties and it should have one zero at $\pi$ in the frequency response. It should be ensured that the chosen wavelet basic function and the corresponding MRA filter coefficients satisfy these properties. Then according to Soon-Huat the corresponding MRA filter coefficients can be calculated by representing the scaling filter coefficients in terms of trigonometric functions of Sine and Cosine represented by parametric angles, such that the above conditions are satisfied. As explained in following result section it was found that eight filter coefficients can represent chosen mother wavelet.

Using trigonometric encoding method [Soon and Huat, 2002] an 8-tap filter is represented by 3 parametric angles $a$, $b$, and $c$ where

$$c = \frac{\pi}{4} - a - b$$  \hspace{1cm} (4.3)

The coefficients of 8-tap low pass filter are described in terms of the above angles, by the following equations (Soon and Huat, 2002)

$$h(0)= \cos(a) \cdot \cos(b) \cdot \cos(c)$$
$$h(1)= \sin(a) \cdot \cos(b) \cdot \cos(c)$$
$$h(2)= - \cos(a) \cdot \sin(b) \cdot \sin(c)$$
$$h(3)= - \sin(a) \cdot \sin(b) \cdot \sin(c)$$
$$h(4)= - \sin(a) \cdot \sin(b) \cdot \cos(c)$$
$$h(5)= \cos(a) \cdot \sin(b) \cdot \cos(c)$$
$$h(6)= - \sin(a) \cdot \cos(b) \cdot \sin(c)$$
$$h(7)= \cos(a) \cdot \cos(b) \cdot \sin(c)$$  \hspace{1cm} (4.4)

The parameters $a$, $b$ and $c$ can be obtained by solving the recursive relation between scaling function and low pass filter/scaling filter coefficients discussed in Chapter 1, Eq.(1.25) repeated below

$$\phi(t) = \sqrt{2} \sum_{n=0}^{N} h(n)\phi(2t - n)$$  \hspace{1cm} (4.5)
Substituting 8-tap filter coefficients Eq. (4.4) in recursion relation Eq. (4.5), and replacing \( c \) with Eq. (4.3) we get

\[
\phi(t) = \sqrt{2} \left[ \left( \cos(a) \cos(b) \cos(\frac{\pi}{4} - a - b) \right) \phi(2t) \\
+ \left( \sin(a) \cos(b) \cos(\frac{\pi}{4} - a - b) \right) \phi(2t - 1) \\
+ \left( - \cos(a) \sin(b) \sin(\frac{\pi}{4} - a - b) \right) \phi(2t - 2) \\
+ \left( \sin(a) \sin(b) \sin(\frac{\pi}{4} - a - b) \right) \phi(2t - 3) \\
+ \left( - \sin(a) \cos(b) \cos(\frac{\pi}{4} - a - b) \right) \phi(2t - 4) \\
+ \left( \cos(a) \sin(b) \cos(\frac{\pi}{4} - a - b) \right) \phi(2t - 5) \\
+ \left( - \sin(a) \cos(b) \sin(\frac{\pi}{4} - a - b) \right) \phi(2t - 6) \\
+ \left( \cos(a) \sin(b) \sin(\frac{\pi}{4} - a - b) \right) \phi(2t - 7) \right]
\]

(4.6)

Levenberg-Marquardt’s (LM) algorithm, details of which are given in following section, finds the values of these parameters by optimizing (minimizing) mean square error between wavelet computed by recursion equation Eq. (4.6) and mother wavelet (Eq. (4.2)). LM optimizing algorithm needs some guess values as initial values to these parameters, then this algorithm tries to optimize mean square error of optimizing function Eq. (4.6) with respect to mother wavelet i.e. Eq. (4.2) by adjusting parameter values iteratively. This procedure is terminated when the mean square error satisfies minimum value in two successive iteration steps.

As described in Chapter 3, for the relations between low pass and high pass filter coefficients and synthesis and analysis filter coefficients for an orthogonal wavelet are given as

\[
\begin{align*}
    h_{14}(k) &= (-1)^k h_{04}(N - k - 1) \\
    h_{05}(k) &= h_{04}(N - 1 - k) \\
    h_{15}(k) &= h_{14}(N - 1 - k)
\end{align*}
\]

(4.7)

Where, \( h_{04}(k) \) and \( h_{14}(k) \) are analysis scaling/low pass filter and wavelet/high pass filter coefficients respectively, where as \( h_{05}(k) \) and \( h_{15}(k) \) are synthesis scaling/low pass filter and wavelet/high pass filter coefficients respectively.
Therefore, as orthogonality condition is imposed while encoding filter coefficients, Eq. (4.7) can be used to calculate remaining MRA filter coefficients from analysis scaling filter coefficients.

4.3 Levenberg-Marquardt’s (LM) algorithm

The LM algorithm is an iterative technique that locates a local minimum of a multivariate function that is expressed as the sum of squares of several non-linear, real-valued functions. It became a standard technique for nonlinear least-squares problems, widely adopted in various disciplines for dealing with data-fitting applications. LM can be thought of as a combination of steepest descent and the Gauss-Newton method. When the current solution is far from a local minimum, the algorithm behaves like a steepest descent method: slow, but guaranteed to converge. When the current solution is close to local minimum, it becomes Gauss-Newton method and exhibits fast convergence. Detailed analysis of LM algorithm is available in [Madsen et al., 2004, Nocedal and Wright, 1999 and Kelley, 1999]. A complete LM algorithm is shown in pseudo code below [Manolis, 2005]

\[
\text{Input: A vector function } f: \mathbb{R}^m \rightarrow \mathbb{R}^n \text{ with } n \geq m, \\
\text{A measurement vector } x \in \mathbb{R}^n \text{ and initial parameters} \\
\text{Estimate } P_0 \in \mathbb{R}^m \\
\text{Output: A vector } P^+ \in \mathbb{R}^m \text{ minimizing } ||x-f(P)||^2. \\
\text{Algorithm:}
\]

\[
k:=0; \quad v:=2; \quad P:=P_0 \\
A:=J^TJ; \quad \mathbf{e}_p:=x-f(P); \quad g:=J^T\mathbf{e}_p; \\
\text{Stop: } = \left( ||g||_\infty \leq \varepsilon_1 \right); \quad \mu:=\mu^* \max_{i=1,...,m}(A_{ii}); \\
\text{While (notstop) and } (k<k_{\text{max}}) \\
\quad k:=k+1; \\
\quad \text{Repeat} \\
\quad \quad \text{Solve } (A+\mu I) \mathbf{\delta}P = g; \\
\quad \quad \text{If } (||\mathbf{\delta}P|| \leq ||P||) \\
\quad \quad \quad \text{Stop: =true; } \\
\quad \quad \text{Else} \\
\quad \quad \quad P_{\text{new}}:=P+\mathbf{\delta}P; \\
\quad \quad \quad \rho:=\left( ||\mathbf{e}_p||^2 - ||x-f(P_{\text{new}})||^2 \right) / \left( \mathbf{\delta}^T P (\mu \mathbf{\delta}P + g) \right); \\
\quad \quad \quad \text{If } \rho>0 \\
\quad \quad \quad \quad P:=P_{\text{new}}; \\
\quad \quad \quad A:=J^TJ; \quad \mathbf{e}_p:=x-f(P); \quad g:=J^T\mathbf{e}_p; \\
\quad \quad \quad \text{Stop: } = \left( ||g||_\infty \leq \varepsilon_1 \right); \\
\quad \quad \quad \mu:=\mu^* \max \left( 1/3, 1-(2\rho-1)^3 \right); \quad v:=2* v; \\
\]
Else
  \( \mu = \mu^* \), \( v = 2^* v \);
End if
End if

Until \( \rho > 0 \) or (stop)

End while

\( P^+ = P \);

Figure 4.3: LM Algorithm

In above algorithm vectors and arrays appear in bold face and \( ^T \) is used to denote transposition. Also, \( \| \cdot \| \) and \( \| \cdot \|_\infty \) respectively denote the 2 and infinite norms. Let \( f \) be an assumed functional relation which maps a parametric vector \( P \in \mathbb{R}^m \) to an estimated measurement vector \( \hat{X} = f(P) \), \( \hat{X} \in \mathbb{R}^m \). An initial parametric estimate \( P_0 \) and a measured vector \( X \) are provided and it is desired to find the best vector \( P^+ \) that best satisfies the functional relation \( f \) locally, i.e. minimizes the squared distance \( \epsilon^T \epsilon = (X - \hat{X}) \) for all \( P \). The LM algorithm terminates when at least one of the following conditions met [Manolis, 2005]:

- The gradient’s magnitude drops below a threshold \( \epsilon_1 \)
- The relative change in the magnitude of \( \delta P \) drops below a threshold \( \epsilon_2 \)
- A maximum number of iterations \( k_{\text{max}} \) is reached

4.4 Results

LM algorithm optimizes the measurement vector by adjusting the optimizing parameters. In this work recursion equation Eq. (4.5) is considered as a measurement vector (\( \hat{X} \) in standard LM algorithm) because this equation can be represented in terms of parameter vector Eq. (4.6) and it is equivalent of mother wavelet Eq. (4.2), so it can be compared with mother wavelet for optimization. Appendix A shows Matlab code used in this work for LM algorithm and necessary functions for it. The main program \texttt{waveletfitfn2para.m} takes X and Y data (X vector in standard LM algorithm), input parameter vector (\( P_0 \) in standard LM algorithm) with some initial guess values and it calls the \texttt{lmfit.m} program which is the actual optimizing program. It can allow maximum of three functions for optimization. In this work Y data is the amplitude of mother wavelet with respect to time X data and two functions, real and imaginary parts of recursion relation (\texttt{RecWaveletreal.m} and \texttt{ImagWaveletimag.m}) are used as optimizing functions.
These two functions are represented in terms of parameter vector (or angles). The LM algorithm computes real and imaginary parts of a wavelet using recursion relation for initial guess values of given parameter vector, and then calculates mean square error between this wavelet and mother wavelet (X and Y data) i.e. $X - \hat{X}$ in standard LM algorithm. If this mean square error doesn’t satisfy minimum allowed value i.e. 0.0001, it adjusts parameter vector ($a, b, c, \ldots$) and this process is continued. LM algorithm terminates when the mean square error satisfies minimum value in two successive iteration steps. In this work various filters, 4- tap filter with 2 parameter vector, 8-tap with 3 parameter vector, 16- tap filter with 4 parameter vector and 32- tap filter with 5 parameter vector and with different sets of initial values were studied. The filter type and initial guess parameter vector, which result desired frequency response of the filter coefficients derived using these values is selected as a best filter type and initial guess parameter vector. After extensive study it was found that 8-tap filter represented with 3 parameter vector Eq. (4.4) and initial guessed parameter vector $P = [\pi/4, \pi /4]$, corresponding parameter vector values ($a=1.0404$, and $b=1.6215$) resulted as desired frequency response as shown is Fig (4.4).

Substituting $a$ and $b$ in equation (4.3), we get third parameter $c=-1.8766$. Then scaling filter coefficients $h_{0,A}(k)$ are computed with these parametric angles using Eq. (4.4) and remaining MRA filter coefficients were computed from Eq. (4.6). Table (4.1) shows MRA filter coefficients of a chosen wavelet to MST radar clear air echo. It can be noticed that the sum of low pass coefficients is $\sqrt{2}$ and high pass coefficients is zero as required by Eq. (1.55).
CHAPTER 4. Design of Wavelet for MST Radar clear air echo

Table 4. 1: MRA filter coefficients of the designed wavelet

<table>
<thead>
<tr>
<th>Analysis Filters</th>
<th>Synthesis Filters</th>
</tr>
</thead>
<tbody>
<tr>
<td>Low-pass filter coefficients $h_{0a}(k)$</td>
<td>Low-pass filter coefficients $h_{0s}(k)$</td>
</tr>
<tr>
<td>High-pass filter coefficients $h_{1a}(k)$</td>
<td>High-pass filter coefficients $h_{1s}(k)$</td>
</tr>
<tr>
<td>0.0077</td>
<td>0.0245</td>
</tr>
<tr>
<td>-0.0245</td>
<td>-0.0417</td>
</tr>
<tr>
<td>0.0132</td>
<td>-0.0417</td>
</tr>
<tr>
<td>0.4818</td>
<td>-0.1521</td>
</tr>
<tr>
<td>0.8216</td>
<td>-0.1521</td>
</tr>
<tr>
<td>0.2593</td>
<td>0.2593</td>
</tr>
<tr>
<td>-0.8216</td>
<td>0.2593</td>
</tr>
<tr>
<td>-0.1521</td>
<td>0.4818</td>
</tr>
<tr>
<td>-0.0417</td>
<td>0.4818</td>
</tr>
<tr>
<td>0.0245</td>
<td>0.0077</td>
</tr>
<tr>
<td>0.0077</td>
<td>-0.0245</td>
</tr>
</tbody>
</table>

4.5 Frequency response of MRA filter coefficients

One immediate factor to be examined is whether the filters corresponding to the designed wavelet possess the right frequency response. The frequency response of computed MRA filter coefficients derived for the designed wavelet as explained above is shown in Fig (4.4). Here solid line indicates low pass filter where as dot dash line indicates high pass filter. The frequency response of a standard wavelet (Daubechies wavelet db10) is shown in Fig. (4.5). As can be seen from these plots the frequency response of these filter coefficients were found to be as desired i.e. say, compares with that of db10 quite well, though the roll off of the designed wavelets are not as steep as in the case of db10. However this comparatively extended roll of the designed wavelets did not pose any difficulty in the envisaged application, viz., the denoising of the MSR radar signals. In any case, it should be possible to make the cutoff much sharper by increasing the length of the filter coefficients, which is not attempted in this work.
CHAPTER 4. Design of Wavelet for MST Radar clear air echo

4.6 Reconstruction with MRA filters coefficients

The next check on the designed wavelet is to verify its ability to decompose and reconstruct MST radar signal perfectly with minimal artifacts. This is verified in the following way. First MST radar signal is decomposed using newly designed wavelet and then reconstructed back by performing inverse wavelet transform using same wavelet. Fig (4.6) illustrates the perfect reconstruction of designed wavelet. The signal in Fig (4.6a) is
the experimental MST radar signal before applying wavelet analysis; whereas signals in figures (4.6b and 4.6c) are after wavelet analysis (wavelet decomposition and reconstruction) using designed wavelet and standard wavelet Daubechies db10 respectively. It can be observed that the signal is reconstructed without any observable artifacts similar to standard wavelet.

Figure 4.6: Experimental MST radar data (a) Original data (before wavelet analysis) (b) and (c) After wavelet decomposition and reconstruction using newly designed wavelet and standard Daubechies db10 wavelet respectively.
4.7 Conclusions

The wavelet suitable for MST radar clear-air Doppler echo is designed. The corresponding MRA filter coefficients were computed. The frequency response of computed filter coefficients satisfies the desired frequency response as in the case of standard Daubechies wavelet (db10).