Bohr compactification of a topological group $G$ is a compact Hausdorff topological group $H$, that may be canonically associated to $G$. Analogous to this concept, M.T. Philomina [PH] discussed the Bohr-type compactification of frame-groups. Here our attempt is to study this concept in the fuzzy context. We also discuss the ‘product theorem’ for fuzzy frame-groups, which is a very powerful tool in computing the Bohr compactifications.

7.1 Optimal Fuzzy Frame-groups

In this section we discuss the optimality and compactness of fuzzy frame-groups. A frame-group $(X,\cdot,L,\models)$ is an optimal frame-group if $L$ is optimal. It is observed that optimal frame-groups behave nicely with respect to the properties like, connectedness, compactness and regularity. Consequently, we define an optimal fuzzy frame-group.

A fuzzy frame is an optimal fuzzy frame if

$$\mu(a) = 1 \iff a = 0_L \text{ or } a = e_L,$$

by Definition 6.1.2.

But for a frame-group $F = (X,\cdot,L,\models)$, no $\xi \in X$ satisfies $0_L$. So we define an optimal fuzzy frame-group as follows. We denote it as $(F,\mu)$.

Definition 7.1.1. Let $F = (X,.,L,\models)$ be an optimal frame-group. If $\mu = (\mu_\xi,\mu^\alpha)$ where $\mu_\xi$ is an optimal fuzzy frame on $L$ with respect to $\models$ then $(F,\mu)$ is an optimal fuzzy frame-group.

i.e. if $\forall \xi \in X, \mu_\xi(a) = 1 \Leftrightarrow a = e_L$

Example 7.1.1. Consider an optimal frame-group $F = (Q,+,\Omega R,\models)$.

For every $\xi \in Q$, define $\mu_\xi(a) = 1/2$, $\forall a \in \Omega R$, $a \neq Q$

$= 1$, if $a = Q$

Then $\mu = (\mu_\xi,\mu^\alpha)$ is an optimal fuzzy frame-group on $F$.

Let $L$ be an optimal frame. Then $L$ is connected if there exists $a, b \in L$ such that $a \land b = 0$ and $a \lor b = 1$ $\Rightarrow$ either $a = 0$ or $b = 0$.

Definition 7.1.2. A fuzzy frame $\mu$ on $L$ is said to be fuzzy connected, if for some $b, c \in L$, such that

$\mu(b \land c) \geq \mu(a), \forall a \in L$ and $\mu(b \lor c) \geq \mu(a)$, for all $a \in L$ $\Rightarrow$ either $\mu(b) \geq \mu(a), \forall a \in L$ or $\mu(c) \geq \mu(a), \forall a \in L$.

Definition 7.1.3. A fuzzy frame-group $(F,\mu)$, where $F = (X,.,L,\models)$ is said to be fuzzy connected if $(L,\mu)$ is fuzzy connected.

Example 7.1.2. Consider the optimal frame-group $F = (Q,+,\Omega R,\models)$, where $\Omega R$ is the usual metric topology on $R$. The open interval $a = (1, 2) \in \Omega R$ is connected, since there are opens $b = (1, \sqrt{2})$ and $c = (\sqrt{2}, 2)$ in $\Omega R$ such that $b \land c = \emptyset$ and $b \lor c = a$.

Define a fuzzy frame $\mu$ as

$\mu(a) = 1$, for $a = (1, 2)$

$= 1/3$, for all other $a \in \Omega R$. 

101
Now $\mu(b \land c) \geq \min\{\mu(b), \mu(c)\} = 1/3$ and $\mu(b \lor c) \geq \sup\{\mu(b), \mu(c)\} = 1/3$.

i.e. $\mu(b \land c) = \mu(b \lor c) = 1/3 \Rightarrow$ either $\mu(b) \geq \mu(p), \forall p \neq a \in \Omega R$ or $\mu(c) \geq \mu(p), \forall p \neq a \in \Omega R$. Thus $\mu$ is fuzzy connected.

**Definition 7.1.4.** A fuzzy frame-group $(F, \mu)$, where $F = (X,.,L,\models)$ is said to be fuzzy compact if $(L, \mu)$ is fuzzy compact.

**Definition 7.1.5.** A fuzzy frame-group $(F, \mu)$, where $F = (X,.,L,\models)$ is said to be fuzzy Hausdorff if for $x \neq y$ in $X$, $\exists a, b \in L$ such that

$$\mu(x, a) > 0, \mu(y, b) > 0 \text{ and for each } \xi \in X, \mu(\xi (a \land b)) \geq \mu(\xi (c)), \forall c \in L.$$  

**Proposition 7.1.6.** Let $F = (X,.,L,\models)$ be an optimal frame-group. Let $T_L$ denote the spectrum of $L$. Then $F$ is Hausdorff if $(X, T_L)$ is Hausdorff.

**Proof.** Let $(X,.,L,\models)$ be optimal.

$$U_a = \{x \in X \mid x \models a\} \text{ and } T_L = \{U_a \mid a \in L\}. \text{ Then } U_a = \emptyset \iff a = 0_L.$$  

Let $x, y \in X$, $x \neq y$. Let $(X, T_L)$ be Hausdorff. Then find $a, b \in L$ with

$$x \in U_a, \ y \in U_b \text{ and } U_a \cap U_b = \emptyset.$$  

Now $U_a \cap U_b = U_{a \land b} = \emptyset$

$$\Rightarrow a \land b = 0_L$$  

Hence $(X,.,L,\models)$ is Hausdorff.

**Proposition 7.1.7.** Let $F = (X,.,L,\models)$ be an optimal frame-group. Then the optimal fuzzy frame-group $(F, \mu)$, is fuzzy Hausdorff if $(X, T_L)$ is Hausdorff.
Proof. Let \((F, \mu)\) be optimal. Then \(\forall \xi \in X, \mu_\xi(a) = 0 \iff a = 0_L\) and

Let \(x, y \in X, x \neq y\).

If \((X, T_L)\) is Hausdorff, then find \(a, b \in L\) with \(x \in U_a, y \in U_b\) and \(U_a \cap U_b = \emptyset\). Then \(U_{a\land b} = \emptyset\)

i.e., \(a \land b = 0\), since \(F\) is optimal.

\[\therefore \mu_\xi(a \land b) \geq \mu_\xi(c), \forall c \in L, \forall \xi \in X,\]

Thus \(\mu(x, a) > 0, \mu(y, b) > 0\) and \(\forall \xi \in X, \mu_\xi(a \land b) \geq \mu_\xi(c), \forall \xi \in X, \forall c \in L, \mu_\xi(c)\). Hence the result. \(\square\)

In the last chapter we showed that the product and quotient of fuzzy compact fuzzy frames are fuzzy compact fuzzy frames. Here we establish the same for fuzzy frame-groups.

**Proposition 7.1.8.** Product of two fuzzy compact fuzzy frame-groups is again a fuzzy compact fuzzy frame-group.

Proof. A frame-group is compact if its underlying frame is compact and hence the proof follows by Proposition 6.1.5. \(\square\)

**Proposition 7.1.9.** Let \(\mu\) be a fuzzy compact fuzzy frame-group on \((X, ., L, \models)\) and let \((X/Y, ., \bar{L}, \models)\) be the quotient group of the frame-group \((X, ., L, \models)\). Then the fuzzy quotient frame-group \(\Phi(\mu)\) on \((X/Y, ., \bar{L}, \models)\) is also fuzzy compact.

Proof. We prove that \(\Phi(\mu)\) is fuzzy compact on \((X/Y, ., \bar{L}, \models)\).

Let \(\Phi(\mu)(\lor S) \geq \Phi(\mu)(\bar{a})\), for \(\bar{S} \subseteq \bar{L}\), where \(\Phi(\mu) = ((\mu/Y^\bar{a}), (\mu/Y)_\bar{x})\)

i.e., \((\mu/Y)_\bar{x}(\lor S) \geq (\mu/Y)(\bar{a})\), for \(\bar{x} \in X/Y\).

i.e., \((\mu/Y)(\bar{x}, \lor S) \geq (\mu/Y)(\bar{x}, \bar{a})\), for \(\bar{x} \in X/Y\).
i.e. \( \mu(x, \lor S) \geq \mu(x, a) \), since \( x \models \bar{a} \iff x \models a \), for \( a \in L, \bar{a} \in \bar{L} \)

\[ \Rightarrow \mu_x(\lor T) \geq \mu(x, a), \text{ for finite } T \subseteq L, \text{ since } \mu \text{ is fuzzy compact.} \]

\[ \Rightarrow (\mu/Y)_x(\lor T) \geq (\mu/Y)(\bar{x}, \bar{a}) \]

Also, \( (\mu/Y)_x(\lor T) = (\mu/Y)(\bar{x}, \lor T) \)

\[ = \mu(x, \lor T) \]

\[ = \mu(x, \lor S) \]

\[ = (\mu/Y)(\bar{x}, \lor S) \]

\[ = (\mu/Y)_x(\lor S) \]

\[ \therefore (\mu/Y) \text{ is fuzzy compact on } \bar{L} \]

That is, \( \Phi(\mu) \text{ is fuzzy compact.} \)

i.e., \( \Phi(\mu) \) is a fuzzy quotient group of fuzzy frame-group \( \mu \) and it is fuzzy compact. \( \Box \)

**Corollary 7.1.10.** The fuzzy homomorphic image of a fuzzy compact fuzzy frame-group is also a fuzzy compact fuzzy frame-group.

**Remark 7.1.11.** If \( R \) is a fuzzy congruence on the fuzzy frame-group \((F, \mu)\) which is fuzzy compact and \( \Phi(\mu) \) is the fuzzy quotient fuzzy frame-group on \((X/Y, , \bar{L}, \models)\) then \( \Phi(\mu) \) is also fuzzy compact.

Next, we discuss the concept of compactifications of fuzzy frame-groups.

### 7.2 Fuzzy Compactification of fuzzy Frame-groups

Bohr-type compactification of frame-groups was discussed in [PH]. Here we discuss the fuzzy compactification of fuzzy frame-groups and we establish some relation between the compactifications of frame-groups in crisp and fuzzy cases. We also prove
the product theorem for the Bohr-type compactifications of frame-groups and fuzzy frame-groups.

**Definition 7.2.1.** Let $F^* = (F, \mu)$ be a fuzzy frame-group on $F = (X, ., L, \models)$. Let $G^* = (G, \lambda)$ be a fuzzy compact fuzzy Hausdorff fuzzy frame-group on $G = (Y, ., M, \models)$ and let $\hat{\Phi} = (f, f^*)$ be a continuous dense onto fuzzy homomorphism from $F$ to $G$ such that $\mu = \lambda \circ \hat{\Phi}$. Then $\hat{\Phi}F^*$ is said to be a fuzzy frame-group compactification of $F^*$.

**Theorem 7.2.2.** Let $F = (X, ., L, \models)$ be an optimal frame-group and $T = (Y, ., M, \models)$ be an optimal, compact Hausdorff frame-group. If $\hat{\Phi} : F \to T$ is a continuous frame-group homomorphism, then $\hat{\Phi}F$ is a frame-group compactification of $F$.

**Proof.** Let $F$ and $T$ be two optimal frame-groups.

A homomorphism $\hat{\Phi}$ is dense if $\hat{\Phi}(F)$ is dense. Let $\hat{\Phi} : F \to T$ be a continuous homomorphism. Then we claim that $\hat{\Phi}$ is onto and dense in $T$.

By Theorem 2.5.13, $\phi : X \to Y$ is dense $\iff \phi^* : M \to L$ is dense.

Let $\phi^*(b) = 0$, for $b \in M$. If $b \neq 0$, then there is a $\phi(x) \models b$, since $T$ is optimal.

$$\Rightarrow x \models \phi^*(b), \text{ since } \hat{\phi} \text{ is a frame-group homomorphism.}$$

But $\phi^*(b) = 0$ and there is no $x \models 0$ and hence a contradiction. Therefore $b = 0$.

i.e., $\phi^*(b) = 0 \Rightarrow b = 0$.

Hence $\phi^*$ is dense and so is $\phi$.

The converse also holds in the same manner. So we conclude that $\hat{\Phi} = (\phi, \phi^*)$ is dense.

Also $\phi^*$ is onto. Hence $\hat{\Phi}F$ is a frame-group compactification of $F$. \qed
Corollary 7.2.3. Let $\tilde{F}^*= (F, \mu)$ be a fuzzy frame-group on optimal frame-group $\tilde{F} = (X, ., L, \vdash)$. Let $\tilde{G}^* = (G, \lambda)$ be a fuzzy compact fuzzy Hausdorff fuzzy frame-group on optimal frame-group $\tilde{G} = (Y, ., M, \vdash)$ and let $\hat{\Phi} = (f, f^*)$ be a continuous dense onto fuzzy homomorphism from $\tilde{F}$ to $\tilde{G}$ such that $\mu = \lambda \circ \hat{\Phi}$. Then $\hat{\Phi} \tilde{F}^*$ is said to be a fuzzy frame-group compactification of $\tilde{F}^*$.

Proof. $(Y, ., M, \vdash)$ is a frame-group compactification of $(X, ., L, \vdash)$ by the above theorem.

$$
\mu : (X, ., L, \vdash) \to [0, 1], \quad \lambda : (Y, ., M, \vdash) \to [0, 1],
\hat{\Phi} : (X, ., L, \vdash) \to (Y, ., M, \vdash), \quad \lambda \circ \hat{\Phi} : (X, ., L, \vdash) \to [0, 1]
$$

i.e., $\lambda \circ \hat{\Phi}$ is a fuzzy frame-group on $(X, ., L, \vdash)$ and thus $\mu = \lambda \circ \hat{\Phi}$.

i.e., $\hat{\Phi}$ is a fuzzy homomorphism between the fuzzy frame-groups $(F, \mu)$ and $(G, \lambda)$.

Also, $\hat{\Phi} F$ is a frame-group compactification of $F$ where $F = (X, ., L, \vdash)$, by the above theorem.

Hence $\hat{\Phi} \tilde{F}^*$ is a fuzzy frame-group compactification of $\tilde{F}^*$.

7.3 Bohr-type Fuzzy Compactifications of Fuzzy Frame-groups.

The study of Bohr compactification of topological semigroups given by J.H. Carruth, J.A. Hildebrant and R.J. Koch [CA; H; K] motivates us to discuss the same for frame-groups and in the fuzzy environment. First we review some results of Bohr-type compactification of frame-groups [PH].

Lemma 7.3.1. [PH] Let $(X, ., L, \vdash)$ be a Hausdorff frame-group. Then there exists a collection $\{(\hat{\phi}_a, X_a, ., L_a, \vdash)\}$, for some index set $A$, where $(X_a, ., L_a, \vdash)$ is a compact
Hausdorff frame-group for each $\alpha \in A$ and

$$\hat{\phi}_\alpha : (X, ., L, \models) \to (X_\alpha, ., L_\alpha, \models)$$

is a dense continuous homomorphism for each $\alpha \in A$ such that if

$$\hat{g} : (X, ., L, \models) \to (Y, ., M, \models)$$

is a dense continuous homomorphism from $(X, ., L, \models)$ into a compact Hausdorff frame-group $(Y, ., M, \models)$, then there exists some $\beta \in A$ and a frame-group isomorphism

$$\hat{f} : (X_\beta, ., L_\beta, \models) \to (Y, ., M, \models)$$

such that the diagram commutes.

Theorem 7.3.2. [PH] Let $(X, ., L, \models)$ be a Hausdorff frame-group. Then there exists a Bohr-type compactification of $(X, ., L, \models)$.

Definition 7.3.3. Let $(X, ., L, \models, \mu)$ be a fuzzy frame-group. Let $(Y, ., M, \models, \lambda)$ be a fuzzy compact fuzzy Hausdorff fuzzy frame-group and let $\hat{\Phi} = (f, f^*)$ be a continuous fuzzy homomorphism from $(X, ., L, \models)$ to $(Y, ., M, \models)$ such that $\mu = \lambda \circ \hat{\Phi}$. If $\hat{g} : (X, ., L, \models) \to (Z, ., N, \models)$ is a continuous dense onto fuzzy homomorphism from $(X, ., L, \models)$ to another fuzzy compact fuzzy Hausdorff fuzzy frame-group $(Z, ., N, \models)$, then there exists a unique continuous fuzzy isomorphism $\hat{f} : (Y, ., M, \models) \to (Z, ., N, \models)$ such that $\hat{f} \circ \hat{\Phi} = \hat{g}$, where $\lambda = \eta \circ \hat{f}$. Then $(\hat{\Phi}, Y, ., M, \models, \lambda)$ is said
to be the Bohr-type fuzzy compactification of fuzzy frame-group $(X, L, \models, \mu)$, where

$$\mu = \lambda \circ \hat{\Phi} \text{ and } \mu = \eta \circ \hat{g}.$$ 

**Theorem 7.3.4. (Existence of Bohr-type fuzzy compactification)** Let $(F, \mu)$ be a fuzzy frame-group, where $F = (X, L, \models)$. Then there exists a Bohr-type fuzzy compactification for $(F, \mu)$.

**Proof.** Let $(X, L, \models, \mu)$ be a Hausdorff fuzzy frame-group. Then by Lemma 7.3.1[PH], there exists a collection

$$\Gamma = \{(X_\alpha, L_\alpha, \models, \mu_\alpha)\}_{\alpha \in A}$$

for some index set $A$, where $(X_\alpha, L_\alpha, \models, \mu_\alpha)$ is a compact Hausdorff fuzzy frame-group $\forall \alpha$ and

$$\hat{\Phi}_\alpha : (X, L, \models) \to (X_\alpha, L_\alpha, \models)$$

is a dense continuous homomorphism for each $\alpha \in A$ such that if

$$\hat{g} : (X, L, \models) \to (Y, M, \models)$$

is a dense continuous fuzzy homomorphism from $(X, L, \models)$ into a compact Hausdorff fuzzy frame group $(Y, M, \models)$ then $\exists$ some $\beta \in A$ and fuzzy frame-group isomorphism
\[ \hat{f} : (X_\beta, L_\beta, \models) \rightarrow (Y, M, \models) \]

such that the diagram commutes.

Now, by defining \( \phi : X \rightarrow \prod X_\alpha \) and \( \phi^* : \otimes L_\alpha \rightarrow L \) by

\[
\phi(x) = \langle \phi_\alpha(x) \rangle \quad \text{and} \quad \phi^*(\otimes a_\alpha) = \wedge \{ \phi^*_\alpha(a_\alpha) | \alpha \in A \},
\]

Philomena M. T. [PH] proved that \( \hat{\Phi} = (\phi, \phi^*) \) is a continuous dense frame-group homomorphism from \( (X, L, \models) \rightarrow (\prod X_\alpha, \otimes L_\alpha, \models) \) and \( (\hat{\Phi}, \overline{\phi(x), \otimes L_\alpha, \models}) \) is a Bohr-type compactification for the frame-group \( (X, L, \models) \).

In the fuzzy case,

\[
\hat{\Phi} : (X, L, \models) \rightarrow (\prod X_\alpha, \otimes L_\alpha, \models)
\]

is a fuzzy continuous dense fuzzy homomorphism between the fuzzy frame-groups \( (X, L, \models) \) and \( (\prod X_\alpha, \otimes L_\alpha, \models) \) with \( \mu = \prod \mu_\alpha \circ \hat{\Phi} \).

If \( (Y, M, \models, \lambda) \) is a fuzzy compact Hausdorff fuzzy frame-group, and

\[
\hat{g} : (X, L, \models) \rightarrow (Y, M, \models)
\]

is a fuzzy continuous, dense fuzzy homomorphism, then \( \exists (X_\beta, L_\beta, \models) \in \Gamma \) and a fuzzy frame-group isomorphism

\[
\hat{h} : (X_\beta, L_\beta, \models) \rightarrow (Y, M, \models) \]

such that

\[
\hat{g} = \hat{h} \circ \hat{\Phi}_\beta
\]
where \( \mu_\beta = \lambda \circ \hat{h} \), by Lemma 7.3.1.[PH]

Consider \( \pi_\beta : \prod X_\alpha \to X_\beta \) and \( \pi^*_\beta : L_\beta \to \otimes L_\alpha \).

Then \( \hat{\pi}_\beta = (\pi_\beta, \pi^*_\beta) \) is a fuzzy continuous, dense, fuzzy homomorphism between
\((\prod X_\alpha, ,, \otimes L_\alpha, \models)\) and \((X_\beta, ,, L_\beta, \models)\) such that
\[ \prod \mu_\alpha = \mu_\beta \circ \hat{\pi}_\beta. \]

Define \( \hat{f} : (\overline{\phi}(X), ,, \otimes L_\alpha, \models) \to (Y, ,, M, \models) \) by \( \hat{f} = \hat{h} \circ \hat{\pi}_\beta \), where \( \prod \mu_\alpha = \lambda \circ \hat{f} \).

\[ (X_\beta, L_\beta, \models) \quad \hat{\pi}_\beta \quad (\overline{\phi}(X), ,, \otimes L_\alpha, \models) \]

\[ (X, L, \models) \quad \hat{g} \quad (Y, M, \models). \]

Now, \( \mu = \prod \mu_\alpha \circ \hat{\phi} = \lambda \circ \hat{f} \circ \hat{\phi} \)
i.e. \( \mu = \lambda \circ (\hat{f} \circ \hat{\phi}) \), and hence \( \hat{f} \circ \hat{\phi} \) is a fuzzy frame-group homomorphism from
\((X, ,, L, \models) \to (Y, ,, M, \models). \)

Thus \( \hat{f} \) is a unique (since \( \hat{g} \) is dense) fuzzy continuous dense fuzzy homomorphism and
\( \hat{g} = \hat{f} \circ \hat{\phi}. \)

i.e. \( \hat{g} \) is a fuzzy morphism from \((X, ,, L, \models)\) to \((Y, ,, M, \models)\).

\[ \square \]

**Theorem 7.3.5. (Uniqueness of Bohr-type fuzzy compactifications)** Let \((X, ,, L, \models, \mu)\) be a Hausdorff fuzzy frame-group. Let \((Y, ,, M, \models, \lambda)\) and \((Z, ,, N, \models, \eta)\) be Bohr-type fuzzy frame-group compactifications of \((X, ,, L, \models, \mu)\). Then \( \exists \) a fuzzy frame-group isomorphism
\[ \hat{\Psi} : (Y, ,, M, \models) \to (Z, ,, N, \models) \]
such that the diagram commutes.

Proof. Since \((\hat{f}, Y, M, =, \lambda)\) is a Bohr-type fuzzy compactification of \((X, L, =, \mu)\), \(\exists\) a unique fuzzy continuous, dense fuzzy homomorphism \(\hat{\Psi} : (Y, M, =) \to (Z, N, =)\) such that the diagram commutes, with \(\lambda = \eta \circ \hat{\Psi}\) and \(\hat{\Psi} \circ \hat{f} = \hat{g}\).

Similarly, since \((Z, N, =, \eta)\) is a Bohr-type fuzzy compactification, \(\exists\) a unique fuzzy continuous dense fuzzy homomorphism

\[ \hat{\Phi} : (Z, N, =) \to (Y, M, =) \]

such that the diagram commutes and \(\hat{f} = \hat{\Phi} \circ \hat{g}\) with \(\eta = \lambda \circ \hat{\Phi}\). Now, by uniqueness of \(\hat{\Psi}\) and \(\hat{\Phi}\)

\[ \hat{\Phi} \circ \hat{\Psi} : (Y, M, =) \to (Y, M, =) \text{ and } \hat{\Psi} \circ \hat{\Phi} : (Z, N, =) \to (Z, N, =) \]

are identity fuzzy morphisms.

So \(\hat{\Psi} = (\psi, \psi^*)\) is a fuzzy frame-group isomorphism.
7.4 Fuzzy congruence on a fuzzy frame-group

Here we discuss fuzzy congruence on fuzzy frame-groups and then the quotient of fuzzy frame-groups. Finally, we establish that any fuzzy frame-group compactification is a quotient of Bohr-type fuzzy frame-group compactification.

Definition 7.4.1. Let $F = (X, ., L, \models) = \{ (x, a) \in X \times L \mid x \in X, a \in L \text{ and } x \models a \}$ be a frame-group. Let $R$ be a relation defined on $F$. Then $R$ is said to be an equivalence relation if

i. $((x, a), (x, a)) \in R$, $\forall (x, a) \in F$ (Reflexivity)

ii. $((x, a), (y, b)) \in R \Rightarrow ((y, b), (x, a)) \in R$, for $(x, a), (y, b) \in F$ (Symmetry)

iii. $((x, a), (y, b)) \in R$ and $((y, b), (z, c)) \in R \Rightarrow ((x, a), (z, c)) \in R$, for $(x, a), (y, b), (z, c) \in F$ (Transitivity).

Definition 7.4.2. An equivalence relation $R$ on $F$ is said to be a congruence if the following conditions are satisfied.

1. If $((x, a_{\alpha}), (y, b_{\alpha})) \in R$, $\forall \alpha \in \Lambda$ then $((x, \forall a_{\alpha}), (y, \forall b_{\alpha})) \in R$.

2. If $(x, a_i) \in F$ and $(y, b_i) \in F, i = 1, 2$, then $((x, a_1), (y, b_1)) \in R$ and $((x, a_2), (y, b_2)) \in R \Rightarrow ((x, a_1 \land a_2), (y, b_1 \land b_2)) \in R$.

Definition 7.4.3. Let $F$ be a frame-group. A fuzzy binary relation $\mathcal{R}$ on $F$ is an equivalence relation if the following conditions are satisfied.

i. $\mathcal{R}((x, a), (x, a)) = 1$

ii. $\mathcal{R}((x, a), (y, b)) = \mathcal{R}((y, b), (x, a))$, for $(x, a), (y, b) \in F$
iii. \( \mathcal{R}((x, a), (z, c)) \geq \min\{\mathcal{R}((x, a), (y, b)), \mathcal{R}((y, b), (z, c))\} \),
for \((x, a), (y, b), (z, c) \in F\).

**Definition 7.4.4.** \( \mathcal{R} \) is said to be a fuzzy congruence on frame-group \( F \), if the following conditions are satisfied.

1. For \((x, a_\alpha), (y, b_\alpha) \in F, \alpha \in \Lambda, \)
\[ \mathcal{R}((x, \vee a_\alpha), (y, \vee b_\alpha)) \geq \sup_\alpha \{\mathcal{R}((x, a_\alpha), (y, b_\alpha))\} \]

2. For \((x, a_i), (y, b_i) \in F, i = 1, 2, \)
\[ \mathcal{R}((x, a_1 \land a_2), (y, b_1 \land b_2)) \geq \min\{\mathcal{R}((x, a_1), (y, b_1)), \mathcal{R}((x, a_2), (y, b_2))\} \]

**Definition 7.4.5.** Let \( \mathcal{R} \) be a fuzzy binary relation on a frame-group \( F \). Then the fuzzy binary relation \( \mathcal{R}_F \) on fuzzy frame-group \((F, \mu)\) is defined as
\[ \mathcal{R}_F((x, a), (y, b)) = \mathcal{R}((x, a), (y, b)) \mu(x, a) \mu(y, b). \]

**Definition 7.4.6.** \( \mathcal{R}_F \) is said to be a fuzzy congruence on a fuzzy frame-group \((F, \mu)\), if

1. For \((x, a_\alpha), (y, b_\alpha) \in F, \alpha \in \Lambda, \)
\[ \mathcal{R}_F((x, \vee a_\alpha), (y, \vee b_\alpha)) \geq \sup_\alpha \{\mathcal{R}_F((x, a_\alpha), (y, b_\alpha))\} \]

2. For \((x, a_i), (y, b_i) \in F, i = 1, 2, \)
\[ \mathcal{R}_F((x, a_1 \land a_2), (y, b_1 \land b_2)) \geq \min\{\mathcal{R}_F((x, a_1), (y, b_1)), \mathcal{R}_F((x, a_2), (y, b_2))\} \]

**Proposition 7.4.7.** Let \( F, G \) be frame-groups and let \( \hat{\theta} : F \to G \) be a frame-group homomorphism, which defines a binary relation
\[ \mathcal{R}(\hat{\theta}) = \{(x, a), (y, b) \in F \times F | \hat{\theta}(x, a) = \hat{\theta}(y, b)\} \text{ on } F. \]

Then \( \mathcal{R}(\hat{\theta}) = (\mathcal{R}_\theta, \mathcal{R}_{\theta^*}) \) is a congruence on \( F. \)
Proof. Let $F = (X, \cdot, L, \models)$, $\hat{\theta} : F \to G$, $\hat{\theta} = (\theta, \theta^*)$, where

$\theta : X \to Y$ is a group homomorphism and

$\theta^* : M \to L$ is a frame homomorphism

such that

$x \models \theta^*(b) \iff \theta(x) \models b$, $x \in X$, $b \in M$.

1. Let $((x, a_\alpha), (y, b_\alpha)) \in \mathcal{R}(\hat{\theta})$, $\alpha \in \Lambda$.

\[ \Rightarrow \hat{\theta}(x, a_\alpha) = \hat{\theta}(y, b_\alpha), \ \forall \alpha \]

\[ \Rightarrow \theta(x) = \theta(y) \quad \text{and} \quad \theta^*(a_\alpha) = \theta^*(b_\alpha) \]

\[ \Rightarrow \theta(x) = \theta(y) \quad \text{and} \quad \theta^*(\vee a_\alpha) = \theta^*(\vee b_\alpha) \]

\[ \therefore (x, a_\alpha), (y, b_\alpha)) \in \mathcal{R}(\hat{\theta}), \ \forall \alpha \in \Lambda \Rightarrow ((x, \vee a_\alpha), (y, \vee b_\alpha)) \in \mathcal{R}(\hat{\theta}), \]

Similarly, $((x, a_1), (y, b_1)) \in \mathcal{R}(\hat{\theta}), ((x, a_2), (y, b_2)) \in \mathcal{R}(\hat{\theta})$

\[ \Rightarrow ((x, a_1 \land a_2), (y, b_1 \land b_2)) \in \mathcal{R}(\hat{\theta}). \]

\[ \therefore \mathcal{R}(\hat{\theta}) \text{ is a congruence on } F. \]

As we have seen in frames, the congruence $\mathcal{R}(\hat{\theta})$ on frame-group $F$ partitions $F$ into disjoint equivalence classes and the set of all equivalence classes form a quotient group of the frame-group, denoted by $F/\mathcal{R}(\hat{\theta})$. Then the frame-group homomorphism $\hat{\alpha} = (\alpha, \alpha^*)$ from $F$ to $F/\mathcal{R}(\hat{\theta})$, where

$\alpha : X \to X/\mathcal{R}_\theta$ is a surjective group homomorphism and

$\alpha^* : L/\mathcal{R}_{\theta^*} \to L$ is a dense onto frame homomorphism,

is called the quotient of $\hat{\theta}$.

Proposition 7.4.8. Let $F$, $G$ be frame-groups and let $\hat{\Phi} : F \to G$ be a continuous frame-group homomorphism. Let $\hat{\Phi}$ define a fuzzy congruence $\mathcal{R}$ on $F$ as
$$\mathcal{R} = \{( (x, a), (y, b) ) \in F \times F \mid \hat{\Phi}(x, a) = \hat{\Phi}(y, b) \}.$$ 

Then $\mathcal{R}_F$, defined by 

$$\mathcal{R}_F((x, a), (y, b)) = \mathcal{R}((x, a), (y, b)) \mu(x, a) \mu(y, b)$$

is a fuzzy congruence on the fuzzy frame-group $(F, \mu)$. 

**Proof.** 

Since $\mathcal{R} = \{( (x, a), (y, b) ) \in F \times F \mid (x, a) = (y, b) \}$, we have, 

1. For $(x, a_\alpha), (y, b_\alpha) \in F$, \( \forall \alpha \in \Lambda \),

$$\mathcal{R}(x, \lor a_\alpha), (y, \lor b_\alpha) \geq \sup \alpha \{ \mathcal{R}((x, a_\alpha), (y, b_\alpha)) \}$$

2. $\mathcal{R}((x_1 \land a_2), (y_1 \land b_2)) \geq \min \{ \mathcal{R}((x_1, a_1), (y_1, b_1)), \mathcal{R}((x_2, a_2), (y_2, b_2)) \}$

Now, $\mathcal{R}_F((x, a), (y, b)) = \mathcal{R}((x, a), (y, b)) \mu(x, a) \mu(y, b)$

$$\mathcal{R}_F((x, \lor a_\alpha), (y, \lor b_\alpha)) = \mathcal{R}((x, \lor a_\alpha), (y, \lor b_\alpha)) \mu(x, \lor a_\alpha) \mu(y, \lor b_\alpha)$$

$$\geq \sup \alpha \{ \mathcal{R}((x, a_\alpha), (y, b_\alpha)) \mu(x, a_\alpha) \mu(y, b_\alpha) \}$$

$$= \sup \alpha \{ \mathcal{R}_F((x, a_\alpha), (y, b_\alpha)) \}$$

$$\mathcal{R}_F((x_1 \land a_2), (y_1 \land b_2)) = \mathcal{R}((x_1 \land a_2), (y_1 \land b_2)) \mu(x_1 \land a_2) \mu(y_1 \land b_2)$$

$$\geq \min \{ \mathcal{R}((x_1, a_1), (y_1, b_1)) \mu(x_1, a_1) \mu(y_1, b_1), \mathcal{R}((x_2, a_2), (y_2, b_2)) \mu(x_2, a_2) \mu(y_2, b_2) \}$$

$$= \min \{ \mathcal{R}_F((x_1, a_1), (y_1, b_1)), \mathcal{R}_F((x_2, a_2), (y_2, b_2)) \}$$

Therefore, $\mathcal{R}_F$ is a fuzzy congruence on $(F, \mu)$. \( \square \)
Theorem 7.4.9. Let $F^*$ be fuzzy frame-group with Bohr-type fuzzy compactification $(\hat{\beta}, B^*)$. If $\hat{\alpha}F^*$ is any fuzzy frame-group compactification, then $\hat{\alpha}F^*$ is the quotient of $(\hat{\beta}, B^*)$.

Proof. If $\hat{\alpha}F^*$ is a fuzzy compactification of $F^*$, then $\hat{\alpha}$ is a dense onto fuzzy homomorphism from $F$ to $A$. From the definition of Bohr-type fuzzy compactification, there exists a continuous fuzzy isomorphism $\hat{\theta} : B \to A$ such that $\hat{\theta}\hat{\beta} = \hat{\alpha}$.

Since $\hat{\alpha}F^*$ is a fuzzy compactification of $F^*$, $\hat{\theta}$ is a dense onto fuzzy homomorphism.

Let $\mathcal{R}_F$ be a fuzzy relation defined by $\hat{\theta}$.

i.e., $\mathcal{R}_F = \mathcal{R}\{(x, a), (y, b)) \in B \times B \mid \hat{\theta}(x, a) = \hat{\theta}(y, b)$  
$\mu(x, a) = \mu(y, b))\}$

By Proposition 7.4.2, $\mathcal{R}_F$ is a fuzzy congruence determined by $\hat{\theta}$ on $B$.

Therefore, $\hat{\theta} : B \to A$ is a quotient map, since it is a fuzzy continuous surmorphism.

Therefore, $A$ is a quotient of $B$.

i.e. $\hat{\alpha}F^*$ is the quotient of $(\hat{\beta}, B^*)$. \hfill \qed

Remark 7.4.10. Any fuzzy frame-group compactification is a quotient of Bohr-type fuzzy compactification of fuzzy frame-groups.

Theorem 7.4.11. Let $F$ be a fuzzy frame-group with Bohr-type fuzzy compactification $(\hat{\beta}, B)$. If $\mathcal{R}_F$ is a fuzzy congruence on $B$, then $\exists$ a fuzzy frame-group compactification $\hat{\alpha}F$ of $F$ so that the fuzzy congruence defined by this compactification is $\mathcal{R}_F$.

Proof. If $\mathcal{R}_F$ is a fuzzy congruence on $B$, then $\mathcal{R} : B \to B/\mathcal{R}$ is a fuzzy quotient.

The projection map $\pi : B \to B/\mathcal{R}$ is a fuzzy homomorphism and fuzzy homomorphic image of a fuzzy compact fuzzy Hausdorff fuzzy frame-group is a fuzzy compact fuzzy.
Hausdorff fuzzy frame-group.

i.e. $B/\mathcal{R}$ is fuzzy compact fuzzy Hausdorff fuzzy frame-group. Let $A = B/\mathcal{R}$. Define $\hat{\alpha} : F \to A$ as $\hat{\alpha} = \pi \hat{\beta}$. Then $\hat{\alpha}$ is a continuous fuzzy homomorphism and we show that $\hat{\alpha}(F)$ is dense in $A$.

Let $\hat{\alpha}(a) = 0$

$\Rightarrow \pi \hat{\beta}(a) = 0$

$\Rightarrow \hat{\beta}(a) = 0$, since $\pi$ is dense.

$\Rightarrow a = 0$, since $\hat{\beta}$ is dense.

Therefore, $\hat{\alpha}F$ is a fuzzy frame-group compactification of $F$.

Also since $(\hat{\beta}, B)$ is a Bohr type fuzzy compactification of $F$, $\exists$ unique continuous fuzzy isomorphism $\hat{\Phi} : B \to A$ such that $\hat{\Phi}\hat{\beta} = \hat{\alpha}$.

Therefore, $F$ is a fuzzy frame-group compactification and the congruence determined by $\hat{\alpha}F$ is that determined by $\mathcal{R}_F$ itself.

In the next section, we discuss the product of Bohr-type compactifications of frame-groups, in view of the product theorem for topological semigroups.
7.5 Product Theorem for fuzzy frame-groups

In 1961, K. Deleeuw and I. Glicksberg [DE; G] showed that the product of Bohr compactifications of a collection of abelian topological monoids is the Bohr compactification of their product. They showed by an example that the identity is not necessary. Using this fact, K.S. Kripalini(1990)[KR], proved that the product theorem holds for compactifications of topological semigroups. She has also proved that any semigroup compactification is a quotient of Bohr compactification. In 2008, Philomina M.T.[PH] discussed the concept of Bohr compactification of frame-groups and called it as Bohr-type compactification. Here we try to establish the product theorem for the Bohr-type compactifications of a family of frame-groups and we discuss the same for the family of fuzzy frame-groups.

**Theorem 7.5.1. (Product Theorem for frame-groups)**

Let \( \{(X_\alpha, ., L_\alpha, |.)\}_{\alpha \in \Lambda} \) be a collection of frame-groups and \((\beta_\alpha, B_\alpha)\) be the Bohr-type compactification of \( F_\alpha = (X_\alpha, ., L_\alpha, |.)\), for each \( \alpha \in \Lambda \). Let \( \hat{\beta} : \prod F_\alpha \rightarrow \prod B_\alpha \) be the function defined by \( \hat{\beta}(x)(\delta) = \beta_\delta \pi_\delta(x) \), where \( \pi_\delta : \prod F_\alpha \rightarrow F_\delta \), is the projection for each \( \delta \in \Lambda \), \( x \in \prod F_\alpha \). Then \( (\hat{\beta}, \prod B_\alpha) \) is the Bohr-type compactification of \( \prod F_\alpha \).

**Proof.** Let \( F = \prod F_\alpha = (\prod X_\alpha, ., \otimes L_\alpha, |.) \) and \( B = \prod B_\alpha \), where

\[ B_\alpha = (\beta_\alpha(X_\alpha), ., \otimes L_\alpha, |.) \].

Let \( \hat{g} : F \rightarrow T \) be a continuous homomorphism from \( F \) to \( T \), a compact Hausdorff frame-group. Without loss of generality assume that \( \hat{g} \) is dense, onto. Now we show that \( \exists \) a continuous dense, onto homomorphism \( \hat{\Phi} : B \rightarrow T \) such that the diagram commutes.
Let $1$ denote the identity of $F$ and $1_\alpha$ denote the identity of $F_\alpha$. Define

$$\hat{\psi}(\alpha) : F_\alpha \to F,$$

so that

$$\pi_\delta(\psi_\alpha(z)) = z, \text{ if } \alpha = \delta$$

$$= 1_\delta, \text{ if } \alpha \neq \delta, \text{ where } z \in F_\alpha.$$ 

Then $\hat{\psi}_\alpha$ is the natural map and there exists a unique continuous homomorphism $\hat{g}_\alpha : B_\alpha \to T$ such that the diagram commutes, since $(\beta_\alpha, B_\alpha)$ is a Bohr-type compactification of $F_\alpha$.

We claim that $\hat{\Phi} = (\phi, \phi^*)$ is a continuous frame-group homomorphism from $B$ to $T$. i.e., we prove that

$$\phi : \prod B_\alpha \to Y \text{ is a group homomorphism and }$$

$$\phi^* : M \to L_\alpha \text{ is a homomorphism,}$$

in such a way that

$$\phi(x) \models b \iff x \models \phi^*(b), \text{ where } x \in \prod B_\alpha, b \in M.$$ 

With reference to the product theorem of Bohr compactification of product of
topological semigroups (DeLeuuw and Glicksberg, 1961), define \( \phi(x) = \land \phi_\alpha(x) \), where \( \phi_\alpha(x) = (g_\alpha \circ \beta_\alpha)(x) \).

i.e. \( \phi(x) \) is a continuous group homomorphism and \( \phi^*(b) = \otimes a_\alpha \), where \( \phi_\alpha^*(b) = a_\alpha \).

Now, \( \phi(x) \models b \iff \phi^*(x) \models b, \forall \alpha \).

\[
\iff x \models \phi^*(b) = a_\alpha, \forall \alpha \\
\iff x \models \otimes a_\alpha, \forall \alpha \\
\iff x \models \phi^*(b)
\]

Therefore, \( \hat{\Phi} = (\phi, \phi^*) \) is a continuous frame-group homomorphism.

In addition, \( \hat{\Phi} \) is dense. For, we know that \( \hat{\Phi} \) is dense if both \( \phi \) and \( \phi^* \) are dense, by Theorem 2.5.12.

Let \( \phi^*(b) = 0 \iff \otimes a_\alpha = 0, \forall \alpha \)

\[
\iff \otimes a_\alpha = 0, \forall \alpha \\
\iff \phi_\alpha^*(b) = 0, \forall \alpha \\
\iff b = 0, \text{since each } \phi_\alpha^* \text{ is dense frame homomorphism.}
\]

Hence \( \phi^* \) is dense.

Now \( \phi(x) = \land \phi_\alpha(x) \), each \( \phi_\alpha \) is dense and hence \( \phi \) is dense. Thus \( \hat{\Phi} \) is dense. \( \Box \)

**Corollary 7.5.2. (Product Theorem for fuzzy frame-groups)**

Let \( \{(X_\alpha, \land L_\alpha, |, \mu_\alpha)\}_{\alpha \in \Lambda} \) be a collection of fuzzy frame-groups and \((\beta_\alpha^*, B_\alpha)\) be the Bohr-type fuzzy compactification of \( F_\alpha = (X_\alpha, \land L_\alpha, |, \mu_\alpha) \) for each \( \alpha \).

Let \( \hat{\beta}^* : \prod F_\alpha \to \prod B_\alpha \), the function defined by

\[
\hat{\beta}(x)(\delta) = \beta_\delta \pi_\delta(x),
\]

where \( \pi_\delta : \prod F_\alpha \to F_\delta \), is the projection for each \( \delta \in \Lambda \). Then \((\hat{\beta}^*, \prod B_\alpha)\) is the Bohr-type compactification of \( \prod F_\alpha \), where \( \prod F_\alpha = \{(\prod X_\alpha, \land \otimes L_\alpha, |, \prod \mu_\alpha)\} \) and
\( \hat{\Phi}^* : B \to T \) is a unique continuous dense onto fuzzy homomorphism such that
\[
\prod \mu_\alpha = \hat{\beta}^* \circ \hat{\Phi}^*.
\]

Now we establish the frame analogues of Induced homomorphism theorem and First isomorphism theorem for topological semigroups.

**Theorem 7.5.3. (Induced homomorphism Theorem for frame-groups)** Let \( F, G, H \) be three frame-groups and \( \hat{\alpha} : F \to G \) be a (quotient) surmorphism and \( \hat{\beta} : F \to H \) be a continuous homomorphism such that \( K(\hat{\alpha}) \subset K(\hat{\beta}) \).

Then \( \exists \) a unique continuous frame-group homomorphism \( \hat{\gamma} : G \to H \) such that the diagram commutes.

![Diagram](image)

**Proof.** Let \( F = (X, L, \models) \), \( G = (Y, M, \models) \), \( H = (Z, N, \models) \) be three frame-groups. If \( \hat{\phi} : F \to G \) is a frame-group homomorphism, then let \( K(\hat{\phi}) \) be the relation defined by
\[
\{( (x, a), (y, b) ) \in F \times F \mid \hat{\phi}(x, a) = \hat{\phi}(y, b), \text{ for } (x, a), (y, b) \in F \}
\]

\( \hat{\alpha} : F \to G \) is a (quotient) surmorphism and \( \hat{\beta} : F \to G \) is a continuous frame-group homomorphism.

\( \forall \ (y, b) \in G \), define \( \hat{\gamma}(y, b) = (\hat{\beta}\hat{\alpha}^{-1})(x, a) \), for \( (x, a) \in F \).

Now we observe that
(i) $\hat{\gamma}$ is well defined,

(ii) $\hat{\gamma}$ is a continuous frame-group homomorphism,

(iii) $\hat{\gamma}$ is unique.

(i). Let $(y_1, b_1), (y_2, b_2) \in G$.

$(y_1, b_1) = (y_2, b_2) \Rightarrow \hat{\alpha}(x_1, a_1) = \hat{\alpha}(x_2, a_2)$, since $\hat{\alpha}$ is onto.

$\Rightarrow \hat{\beta}(x_1, a_1) = \hat{\beta}(x_2, a_2)$, since $K(\hat{\alpha}) \subset K(\hat{\beta})$.

$\Rightarrow \hat{\beta}(\hat{\alpha}^{-1}(y_1, b_1)) = \hat{\beta}(\hat{\alpha}^{-1}(y_2, b_2))$

$\Rightarrow \hat{\gamma}(y_1, b_1) = \hat{\gamma}(y_2, b_2)$

$\therefore \hat{\gamma}$ is well defined.

(ii) $\hat{\gamma} = \hat{\beta}\hat{\alpha}^{-1}$. Clearly $\hat{\gamma}$ is continuous.

$\hat{\beta} = (\beta, \beta^*)$, where $\beta : X \to Z$ is a group homomorphism

$\beta^* : N \to L$ is a frame homomorphism,

such that $x \models \beta^*(n) \iff \beta(x) \models n$.

We have to prove that $\hat{\gamma} = (\gamma, \gamma^*)$ where

$\gamma : Y \to Z$ is a group homomorphism

$\gamma^* : N \to M$ is a frame homomorphism,

such that $y \models \gamma^*(n) \iff \gamma(y) \models n, \forall \ n \in N, y \in Y$.

Clearly, $\gamma$ is group homomorphism and $\gamma^*$ is a frame homomorphism.

Let $y \models \gamma^*(n)$. For $y \in Y, \exists x \in X$ such that $\alpha(x) = y$.

Hence $\alpha(x) \models \gamma^*(n) \iff x \models \alpha^*(\gamma^*(n))$

$\iff x \models \beta^*(n)$

$\iff \beta(x) \models n$

$\iff \gamma \alpha(x) \models n$
Therefore $\hat{\gamma} = (\gamma, \gamma^*)$ is a frame-group homomorphism. 

(iii) We prove the uniqueness.

Let if possible, $\hat{\gamma}_1 : G \to H$ be a continuous frame-group homomorphism such that the diagram commutes. Then $\hat{\beta} = \hat{\gamma}_1 \hat{\alpha} = \hat{\gamma} \hat{\alpha}$.

Now, $\hat{\beta}(0_F) = 0_H$.

\[ \text{i.e. } \hat{\gamma} \hat{\alpha}(0_F) = 0_H \text{ and } \hat{\gamma}_1 \hat{\alpha}(0_F) = 0_H. \]

\[ \Rightarrow \hat{\gamma}(0_G) = 0_H \text{ and } \hat{\gamma}_1(0_G) = 0_H. \]

i.e., both $\hat{\gamma}$ and $\hat{\gamma}_1$ maps bottom element of $G$ to bottom element of $H$.

Similarly, $\hat{\gamma}(e_G) = e_H$ and $\hat{\gamma}_1(e_G) = e_H$.

i.e., both $\hat{\gamma}$ and $\hat{\gamma}_1$ maps top element of $G$ to top element of $H$.

Hence $\hat{\gamma} = \hat{\gamma}_1$. \qed

Now, we prove the following lemma, which we will use to prove the First Isomorphism theorem.

**Lemma 7.5.4.** Let $F, G, H$ be three frame-groups and $\hat{f} : F \to G$, $\hat{h} : F \to H$ and $\hat{g} : G \to H$ be the respective morphisms such that the diagram commutes. Then

a) If $\hat{f}$ is a quotient and $\hat{h}$ is a continuous frame-group homomorphism, then $\hat{g}$ is a continuous frame-group homomorphism.

b) If both $\hat{f}$ and $\hat{h}$ are quotients, then $\hat{g}$ is a quotient.

**Proof.** $F = (X, \cdot, L, |=)$, $G = (Y, \cdot, M, |=)$, $H = (Z, \cdot, N, |=)$. Let $\hat{f} : F \to G$ be quotient. We claim that $\hat{g} : G \to H$ is a continuous frame-group homomorphism.

For $y_1, y_2 \in Y, g(y_1 y_2) = g(f(x_1) f(x_2))$, since $f$ is surjective.
\[ g(f(x_1 x_2)) = h(x_1 x_2) = h(x_1) h(x_2) = g f(x_1) g f(x_2) = g(y_1) g(y_2), \text{ since } \hat{h} = \hat{g} \hat{f} \text{ and } \hat{h} \text{ is a continuous frame-group homomorphism.} \]

Hence, \( g \) is a group homomorphism.

\( g^* : N \to M \). For \( n_1, n_2 \in N \),

\[
\begin{align*}
    h^*(n_1 \land n_2) &= f^* g^*(n_1 \land n_2) \\
    h^*(n_1) \land h^*(n_2) &= f^* g^*(n_1 \land n_2) \\
    f^* g^*(n_1) \land f^* g^*(n_2) &= f^* g^*(n_1 \land n_2) \\
    f^*(g^*(n_1) \land g^*(n_2)) &= f^*(g^*(n_1 \land n_2)) \\
    g^*(n_1) \land g^*(n_2) &= g^*(n_1 \land n_2), \text{ since } \hat{f} \text{ is a quotient.}
\end{align*}
\]

Similarly, we can see that \( g^*(\lor S) = \lor g^*(a), \ a \in S, \ S \subset N. \)

Therefore, \( g^* \) is a frame homomorphism.

Let \( y \models g^*(n) \)

\[
\begin{align*}
    \Rightarrow f(x) &\models g^*(n) \\
    \Rightarrow x &\models f^*(g^*(n)) \\
    \Rightarrow x &\models h^*(n)
\end{align*}
\]
\[ \Rightarrow h(x) \models n \]
\[ \Rightarrow gf(x) \models n \]
\[ \Rightarrow g(y) \models n \]

Converse also holds in the same manner and so \( g^* \) is a frame homomorphism.

Hence \( \hat{g} \) is a continuous frame-group homomorphism.

b) Let \( \hat{f}, \hat{g} \) be quotients. Let \( g^*(n) = 0 \)

\[ \Rightarrow f^*(g^*(n)) = 0 \]
\[ \Rightarrow h^*(n) = 0 \]
\[ \Rightarrow n = 0, \text{ since } f^*, h^* \text{ are dense frame homomorphism.} \]

Therefore, \( g^* \) is a dense frame homomorphism.

Also, \( g : Y \rightarrow Z \). Let \( z \in Z \).

\[ \Rightarrow \exists x \text{ such that } h(x) = z. \]
\[ \Rightarrow gf(x) = z \]
\[ \Rightarrow g(y) = z \]

Therefore, \( g^* \) is a surjective group homomorphism and thus \( \hat{g} \) is a quotient.

**Theorem 7.5.5. (First Isomorphism Theorem)** Let \( F = (X,.,L,\models) \) and \( G = (Y,.,M,\models) \) be two frame-groups and \( \hat{\Phi} : F \rightarrow G \) be a continuous surmorphism. \( K(\hat{\Phi}) \) is the relation defined by

\[ K(\hat{\Phi}) = \{ ((x,a),(y,b)) \in F \times F \mid \hat{\Phi}(x,a) = \hat{\Phi}(y,b) \}. \]

Then \( K(\hat{\Phi}) \) is a congruence on \( F \) and there exists a unique algebraic isomorphism \( \psi : X/K_\phi \rightarrow Y \) such that the diagram commutes and the following are equivalent.

i. \( \hat{\Psi}^{-1} \) is a continuous frame-group homomorphism.

ii. \( \hat{\Psi} \) is a frame-group isomorphism.

125
iii. \( \hat{\Phi} \) is quotient.

**Proof.** Clearly \( K(\hat{\Phi}) \) is a congruence on \( F \). Then \( F/K(\hat{\Phi}) \) is a quotient group of \( F \), where \( F/K(\hat{\Phi}) = (X/K_\phi, L/K_\phi^*) \). Then \( \hat{\pi} \) is a quotient corresponding to the congruence \( K(\hat{\Phi}) \). Then by induced homomorphism theorem, \( \exists \) a unique continuous frame-group homomorphism \( \hat{\Psi} : F/K(\hat{\Phi}) \to G \) such that the diagram commutes.

Clearly, \( \psi : X/K_\phi \to Y \) is an algebraic isomorphism. Now we prove that conditions (i), (ii) and (iii) are equivalent.

(i) \( \Rightarrow \) (ii)

\( \hat{\Psi}^{-1} \) is a continuous frame-group homomorphism. \( \hat{\Psi} = (\psi, \psi^*) \) is a frame-group isomorphism if \( \psi \) is a group isomorphism and \( \psi^* \) is a continuous and open frame homomorphism. Clearly, \( \hat{\Psi} \) is a continuous frame-group homomorphism. We claim that \( \psi^* : M \to L/K_{\Phi^*} \) is open.

Let \( [a] \land \psi^*(x) = [a] \land \psi^*(y) \), for each \( [a] \in L/K_{\Phi^*} \)

\[ \Leftrightarrow \hat{\Psi}^{-1}[a] \land x = \hat{\Psi}^{-1}[a] \land y \]

\[ \Leftrightarrow b \land x = b \land y \), where \( b = \hat{\Psi}^{-1}[a] \).

Therefore, \( \psi^* \) is an open frame homomorphism and hence \( \hat{\Psi} \) is a frame-group isomorphism if \( \psi \) is a group isomorphism

(ii) \( \Rightarrow \) (iii)

Given, \( \hat{\Phi} \) is a continuous surmorphism. Now \( \hat{\Phi} = \hat{\Psi} \hat{\pi} = (\psi \pi, \pi^* \psi^*) \)
We prove that $\phi^* = \pi^* \psi^*$ is dense, onto.

Let $\phi^*(b) = 0$.

\[ \Rightarrow \pi^* \psi^*(b) = 0 \]
\[ \Rightarrow \psi^*(b) = 0 \]
\[ \Rightarrow b = 0 \]

Therefore $\phi^*$ is dense onto frame homomorphism and hence (iii).

(iii) $\Rightarrow$ (i)

$\hat{\Phi}$ is a quotient and $\hat{\pi}$ is continuous. Therefore, $\hat{\Psi}^{-1}$ is a continuous frame-group homomorphism by Lemma 7.5.4.

Now we prove the following theorem.

**Theorem 7.5.6.** Let $\{F_\alpha\}$ be a collection of frame-groups, each with Bohr-type compactification $(\beta_\alpha, B_\alpha)$. Let $R_\alpha$ be the congruence on $B_\alpha$. Then $(\hat{\gamma}, \prod A_\alpha)$ is a frame-group compactification of $\prod F_\alpha$, where $A_\alpha = B_\alpha / R_\alpha$ for each $\alpha$ and $\prod B_\alpha / R_\alpha$ is isomorphic to $\prod A_\alpha$ where

\[ R = \{(x, a_\alpha) \otimes (y, b_\alpha) \in B \times B \mid ((x, a_\alpha), (y, b_\alpha)) \in R_\alpha, \alpha \in \Lambda\}. \]

**Proof.** Let $\{F_\alpha\}$ be a collection of frame-groups where $F_\alpha = (X_\alpha, \ldots, L_\alpha, \vdash)$. Let $\prod F_\alpha = F$ and $\prod B_\alpha = B$.

Let $(\gamma_\alpha, A_\alpha)$ be any frame-group compactification of $F_\alpha$. $(\beta_\alpha, B_\alpha)$ is the Bohr-type compactification of $F_\alpha$. Let $\hat{\Phi}_\alpha : B_\alpha \to A_\alpha$ be any frame-group compactification.

Then

\[ R_\alpha = \{((x, a_\alpha), (y, b_\alpha)) \in B_\alpha \times B_\alpha \mid \hat{\Phi}_\alpha(x, a_\alpha) = \hat{\Phi}_\alpha(y, b_\alpha), \forall \alpha \in \Lambda, x, y \in X_\alpha\} \]

is a congruence on $B_\alpha \times B_\alpha$ and $B_\alpha / R_\alpha$ is a quotient of $(\beta_\alpha, B_\alpha)$. Let $A_\alpha = B_\alpha / R_\alpha$. 


By product theorem $(\hat{\beta}, B)$ is the Bohr-type compactification of $F$, where $\hat{\beta}$ is a continuous frame-group homomorphism from $F$ to $B$. Define $\hat{\gamma} : F \to \prod A_\alpha$, by

$$\gamma(x)_\delta = \gamma_\delta \pi_\delta(x),$$

where $\pi_\delta : F \to F_\delta$, $\gamma_\delta = \phi_\delta \beta_\delta$, $\phi_\delta : B_\delta \to A_\delta$, since each $B_\delta$ is the Bohr compactification of $F_\delta$, $\exists$ a continuous homomorphism $\phi_\delta : B_\delta \to A_\delta$.

Then $(\hat{\gamma}, \prod A_\alpha)$ is a frame-group compactification of $F$.

Also, $(\hat{\beta}, B)$ is Bohr-type compactification of $F$, by product theorem. Therefore $\exists$ a continuous frame-group homomorphism $\hat{\Phi} : B \to \prod A_\alpha$ such that $\hat{\Phi} \hat{\beta} = \hat{\gamma}$ defined by

$$\hat{\Phi}((x, \otimes a_\alpha)) = (\hat{\Phi}_\alpha(x, a_\alpha))$$

If $R$ is the congruence defined by $\hat{\Phi}$, then $B/R$ is the quotient of $B$.

$$R = \{(x, \otimes a_\alpha), (y, \otimes b_\alpha) \in B \times B \mid \hat{\Phi}(x, \otimes a_\alpha) = \hat{\Phi}(y, \otimes b_\alpha), \alpha \in \Lambda\},$$

where $x = (x_1, x_2, ...)$ and $(x, \otimes a_\alpha), (y, \otimes b_\alpha) \in B$

$$R = \{(x, \otimes a_\alpha), (y, \otimes b_\alpha) \in B \times B \mid \hat{\Phi}_\alpha(x, a_\alpha) = \hat{\Phi}_\alpha(y, b_\alpha), \forall \alpha \in \Lambda\}$$

$$=\{(x, \otimes a_\alpha), (y, \otimes b_\alpha) \in B \times B \mid ((x, a_\alpha), (y, b_\alpha)) \in R_\alpha, \forall \alpha \in \Lambda\}$$

Thus, $(\hat{\gamma}, \prod A_\alpha)$ is a frame-group compactification of $F$, where $A_\alpha = B_\alpha / R_\alpha$.

Let $\hat{\pi} : B \to B/R$ be the natural map. Then $(\hat{\pi} \hat{\beta}, B/R)$ is a frame-group compactification of $F$. Then by Theorem 7.5.3, $\exists$ a unique continuous homomorphism $\hat{\Psi} : B/R \to \prod A_\alpha$ such that the diagram commutes.
Again \( \hat{\Phi} : B \to \prod A_\alpha \) defines a congruence \( R \) on \( B \). Therefore \( B/R \) is a quotient frame-group. Then by Theorem 7.5.5, \( \hat{\Psi} \) is a unique isomorphism such that the diagram commutes.

**Remark 7.5.7.** Any frame-group compactification is a quotient of \( B = \prod B_\alpha \).

Thus we see that the compactification of frame-groups, namely, Bohr-type compactification of frame-group can be extended to fuzzy context. Besides, we conclude that Bohr-type compactification of product of fuzzy frame-groups is the product of Bohr-type compactifications of fuzzy frame-groups.