CHAPTER 5

OPTIMAL SERIES - PARALLEL SYSTEMS
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1. INTRODUCION

The study of reliability theory has its origins in the discussion of replacement policies for many simple automobile engineering systems. The earliest work in this direction was initiated by Lotka [2]. An excellent account of the early developments in this direction is given in Barlow and Proschan [1]. In recent times, the interest in optimisation techniques has been steadily increasing largely due to the potentialities of application in a variety of practical problems covering particularly those arising in such complex systems like automobile engineering, computer network, and aero-space reliability models (see for example [3,4,6,7, 8,9,10]).

In this chapter, we develop an optimal procedure leading to the most economical number of parallel units \((N^*)\) in a \((m,N)\) series-parallel system, based on the optimal criterion of minimal per unit costs (including repair costs). The provision of repair facility into the modelling set up may result in some gainful operational advantages.

Based on the optimisation procedure some empirical work is presented, illustrating the applicational value.
2. THE MODEL

Consider a series-parallel reliability system of order \((m,N)\), i.e., a system comprising of \(m\) identical parallel system of order \(N\), connected in series with \(m \geq 2\). The system is provided with repair facility. The system is assumed to be as good as new after each repair, in the sense that its mean time before failure (MTBF) remains the same after each repair. The maintenance and the repair costs (step functions in \(m,N\) and \(n\)) are assumed to be strictly increasing in the arguments \(m,N\) as well as \(n\) (the number of repairs).

**NOTATION**

\[
\begin{align*}
C_1 & \quad \text{acquisition cost of each unit} \\
C_2 & \quad \text{replacement costs} \\
P(t) & \quad \text{the failure time distribution of each of the } Nm \text{ (identical) units} \\
n & \quad \text{number of repairs, } n=0,1,2, \ldots \\
\mu_m(N) & \quad \text{MTBF of the series-parallel system of order } (m,N) \\
C_2e^{-a/mN(i+1)} & \quad \text{the maintenance costs of the system (inclusive of repair costs) during } \mu_m(N) \text{ units of time after } 'i' \text{ repairs, } i=0,1,2, \ldots \, a>0, \text{ is constant characterising these costs as per the physics of the problem and model. For example, if number of repairs increase indefinitely, clearly maintenance costs equal } C_2 \text{ when it is advisable to stop any more repairs. In this context } 'a' \text{ has a role in determining the pace at which } C_2 \text{ is approached with increase in repairs (as also with } N \text{ or } m).}
\end{align*}
\]

For this model, we develop a method to obtain the optimal \(N^*\) for given \(m\) and \(n\).

In the following section, we present the theoretical results.
3. THE OPTIMISATION TECHNIQUE

The assumptions on the model imply:

\[ C_2 e^{-a/m(N+1)(i+1)} > C_2 e^{-a/Nm(i+1)} \]  \hspace{1cm} (1)

The expected cost for the system during \( n \) repairs is given by

\[ NmC_1 + \sum_{i=0}^{n} C_2 e^{-a/Nm(i+1)} \]  \hspace{1cm} (2)

so that, in the steady state case, the expected cost per unit time, say \( C(N,m,n) \) is given by:

\[
C(N,m,n) = \frac{NmC_1 + C_2 \sum_{i=0}^{n} e^{-a/Nm(i+1)}}{(n+1) \mu_m(N)} \hspace{1cm} (3)
\]

We now state and prove the following:

THEOREM I: The \( N^* \) which minimises \( C(N,m,n) \) given in (3), satisfies the pair of inequalities:

\[
D(N,m,n) > \frac{C_2}{C_1} \]  \hspace{1cm} (4)

and \( D(N-1,m,n) < \frac{C_2}{C_1} \)  \hspace{1cm} (5)
where

\[
D(N,m,n) = \frac{(N+1)m\mu_m(N) - Nm\mu_m(N+1)}{\mu_m(N+1) \sum_{i=0}^{n} e^{-a/Nm(i+1)} - \mu_m(N) \sum_{i=0}^{n} e^{-a/m(N+1)(i+1)}}.
\]

Further \( N^* \) is finite and unique.

**Proof:** As \( C(N,m,n) \) is discrete in \( N \), we obtain \( N^* \) which minimises \( C(N,m,n) \) through the pair of inequalities:

\[
C(N+1, m, n) > C(N, m, n) \tag{7}
\]

and \( C(N, m, n) < C(N-1, m, n) \). \tag{8}

Using (3) in (7) we obtain:

\[
\frac{(N+1)mC_1 + C_2 \sum_{i=0}^{n} e^{-a/(N+1)m(i+1)}}{(n+1)\mu_m(N+1)} > \frac{NmC_1 + C_2 \sum_{i=0}^{n} e^{-a/Nm(i+1)}}{(n+1)\mu_m(N)}
\]
which after some calculations yields:

\[ \text{i.e., } C_1 \left[ \frac{(N+1)m}{\mu_m(N+1)} - \frac{Nm}{\mu_m(N)} \right] > \]

\[ C_2 \left[ \frac{\sum_{i=0}^{n} e^{-a/Nm(i+1)}}{\mu_m(N)} - \frac{\sum_{i=0}^{n} e^{-a/(N+1)m(i+1)}}{\mu_m(N+1)} \right]. \tag{9} \]

which after some calculations yields:

\[
\frac{(N+1)m}{\mu_m(N+1)} - \frac{Nm}{\mu_m(N)} \geq \frac{C_2}{C_1}. 
\]

\[ \mu_m(N+1) \sum_{i=0}^{n} e^{-a/mN(i+1)} - \mu_m(N) \sum_{i=0}^{n} e^{-a/(N+1)m(i+1)} \]

\[ i.e., D(N, m, n) > \frac{C_2}{C_1} > 0. \tag{10} \]

A similar manipulation using (3) in (8) leads to:

\[ D(N-1, m, n) < \frac{C_2}{C_1}. \tag{11} \]

Thus the first part in the theorem is established through (10) and (11).

To show that N* is finite and unique, it suffices to show that D(N, m, n) is strictly increasing in N and further tends to + as N → ∞. This is done as follows:
First we observe that,

\[
[ \mu_m(N+1) - \mu_m(N) ] = \int_0^1 \left\{ [1 - F(t)^{N+1}]^m - [1 - F(t)^N]^m \right\} dt > 0
\]

and → 0 as \( N \to \infty \), since \( 0 < F(t) < 1 \), \( \psi(t) \), so that \( \{ \mu_m(N+1) - \mu_m(N) \} \) is a decreasing in \( N \). (12)

Recasting \( D(N, m, n) \) as:

\[
D(N,m,n) = \frac{m \left\{ \frac{N[ \mu_m(N) - \mu_m(N+1)] + \mu_m(N) }{\mu_m(N+1) \sum_{i=0}^{n} e^{-a/mN(i+1)} - \mu_m(N) \sum_{i=0}^{n} e^{-a/m(N+1)(i+1)}} \right\}}{i=0}
\]

we obtain:

\[
D(N,m,n) > -m \mu_m(N) - N \quad (13)
\]

Noticing that: (i) \( D(N,m,n) > 0 \), by (10),

(ii) \( \sum_{i=0}^{n} e^{-a/Nm(i+1)} \) is increasing \( N \) but \( +n \) (finite and fixed)

as \( N \to \infty \),
(iii) $\mu_m(N)$ clearly $\to \infty$ as $N \to + \infty$ and

(iv) using (12) in (13), we obtain that:

$D(N,m,n)$ is strictly increasing in $N$ and further tends to $+ \infty$ as $N \to + \infty$.

Hence $D(N,m,n)$ crosses the finite $0 < \frac{C_2}{C_1}$ value just once, so that $N^*$ is unique.

The proof is complete.

4. EMPIRICAL WORK

In this section, we present illustrative numerical work, specialising to the negative exponential law governing the stochastic failure times.

(i) EXPONENTIAL FAILURE TIMES

Let $F(t) = 1 - e^{-\lambda t}$, $t \geq 0$, $\lambda > 0$,  \hspace{1cm} (14)

so that, $\mu_m(N) = \frac{1}{N\lambda} \sum_{j=0}^{N-1} \beta(\frac{j+1}{N}, m)$, \hspace{1cm} (15)

a known result (see Rau [5]).

We now compute the optimal number of units $N^*$ and the resulting cost for some typical values of $\frac{C_2}{C_1}$ and present these in the following tables:
### TABLE 1: $C_2/C_1, N^*, C(N^*, m, n)/C_1, m=2$

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<tr>
<th>$C_2/C_1$</th>
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<th>$n=2$</th>
<th>$n=3$</th>
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<td>$\frac{N \times C(N^*, m, n)}{C_1}$</td>
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### TABLE 2: $C_2/C_1, N^*, C(N^*, m, n)/C_1, m=3$

<table>
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DISCUSSION

The following observations emanate from tables 1, 2 and 3.

(i) the resulting cost increases with the ratio values $C_2/C_1$ as well as $m$. For example there is an increase of about two and half times in the resulting cost when $m=4$ compared to when $m=2$.

(ii) Further, an increase in $n$ leads to reduced resulting cost.

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**TABLE 3 :** $C_2/C_1$, $N_\ast$, $C(N_\ast,m,n)/C_1$, $m=4$

<table>
<thead>
<tr>
<th>$\frac{C_2}{C_1}$</th>
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<th>$\frac{C(N_\ast,m,n)}{C_1}$</th>
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<th>$\frac{C(N_\ast,m,n)}{C_1}$</th>
<th>$N_\ast$</th>
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REFERENCES


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