CHAPTER - II

TANDEM QUEUE MODEL WITH
THREE STAGES
APPLICABLE TO HOSPITAL SERVICES
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2.1 INTRODUCTION

In real life situations arriving customers pass through many stages to get specialized services. Hunt (1956) treated a modified series model using finite difference operator to solve a two-station sequential series queue in which no queue is allowed between stations, but where a queue with no limit is allowed in front of station I. He obtained the steady state equations for this model. Perros (1994) considered queueing networks with blocking. Singh (2001) in her thesis studied a single channel queueing model consisting of two stages of service and obtained expression for expected waiting time. The present chapter is an extension of Singh (2001) for the case of three stages.

In the present chapter, we are considering queues in series at three stages with blocking and no possibility of queueing at any stage. One may conceive of a situation where stage I consists of a physician, stage II consists of immunization server and stage III consists of a specialist. A patient coming for service has to go through all the stages one by one. At first, patient has to go to the general physician. Then physician may recommend for some types of immunization services viz. X-ray, pathological tests, dressings of infection etc. done by a different server. At last, patient will go to a specialist depending upon his/her disease, viz. cardiologist or orthopaedic. The situation may be deemed as queue with blocking, as shown in the diagram below.
### DIAGRAM:

Arrival $\rightarrow^\lambda \mu_1 \rightarrow^{} \mu_2 \rightarrow^{} \mu_3 \rightarrow$ Departure

Stage I is said to be blocked if the patient got service completed at this stage but unable to get service immediately from stage II, since server (immunization server) in stage II is busy or blocked. Similarly stage II will be blocked if patient got service completed service at this stage but unable to get service immediately from stage III, since server (doctor) in stage III is busy. It may be noted that there will be no blocking at stage III. This kind of situation gives rise to tandem queues with blocking. The model developed here may be applicable to other fields also. In the above figure the symbols $\lambda$ denotes arrival rate and $\mu_s$ is the service rate, where $s=1,2,3$.

Let the symbols $0,1,b$ denote free state, busy state and blocked state respectively. Let $i$ represents possible states of stage I, $j$ of stage II and $k$ of stage III respectively. Then $i$ and $j$ can have states $0,1,b$ where $k$ can have only two states $0$ and $1$. The system can be in either one of the following states.

$$[i,j,k]= [ (0,0,0); (1,0,0); (0,1,0); (0,0,1); (1,0,1); (1,1,0); (b,1,0); (0,1,1); (0,b,1); (1,1,1); (b,1,1); (b,1,b); (b,b,1) ]$$

### 2.2 MATHEMATICAL FORMULATION

We assume that arrival occurs according to Poisson distribution with rate $\lambda$ and service time at each stage is exponentially distributed with rate
where $s=1,2,3$. In each and every stage patients will be served on FCFS basis.

Let $P(t)_{ij}$ be the probability that system is in state $(i,j,k)$ at time $t$. Then all the related probabilities for different states with three stages of services could be derived as follows:

1) The probability that the system would be in state $(0,0,0)$ at $(t+\Delta t)$ will be the sum of following probabilities:

- $P_{000}(t)(1-\lambda \Delta t) = P(\text{all stages are free at } t \text{ and no arrival during } \Delta t)$
- $P_{001}(t)(\mu_3 \Delta t) = P(\text{only server 3 of stage III is busy at } t \text{ and completed service during } \Delta t)$

Hence, $P_{000}(t+\Delta t) = P_{000}(t)(1-\lambda \Delta t) + P_{001}(t)(\mu_3 \Delta t)$

Likewise,

13) The probability that the system would be in state $(b,b,1)$ at $(t+\Delta t)$ will be the sum of following probabilities:

- $P_{b01}(t)(1-\mu_2 \Delta t) = P(\text{stage I and stage II blocked at time } t \text{ and server in stage III has not completed service during } \Delta t)$
- $P_{b11}(t)(\mu_3 \Delta t) = P(\text{servers in stage II and III are busy but stage I is blocked at time } t \text{ and server in stage II has completed service during } \Delta t)$

- $P_{1b1}(t)(\mu_1 \Delta t) = P(\text{stage I and III busy but stage II is blocked at time } t \text{ and server in stage I has completed service during } \Delta t)$

Hence, $P_{b01}(t+\Delta t) = P_{b01}(t)(1-\mu_3 \Delta t) + P_{b11}(t)(\mu_2 \Delta t) + P_{1b1}(t)(\mu_1 \Delta t)$
Probability equations for all possible states are summarized below:

1) $P_{000}(t+\Delta t) = P_{000}(t)(1-\lambda \Delta t) + P_{011}(t)(\mu_3 \Delta t)$

2) $P_{100}(t+\Delta t) = P_{100}(t)(1-\mu_1 \Delta t) + P_{000}(t)\lambda \Delta t + P_{101}(t)\mu_3 \Delta t$

3) $P_{010}(t+\Delta t) = P_{010}(t)(1-(\lambda + \mu_2 \Delta t)) + P_{000}(t)\mu_1 \Delta t + P_{011}(t)\mu_3 \Delta t$

4) $P_{001}(t+\Delta t) = P_{001}(t)(1-(\lambda + \mu_3 \Delta t)) + P_{010}(t)\mu_2 \Delta t + P_{001}(t)\mu_3 \Delta t$

5) $P_{101}(t+\Delta t) = P_{101}(t)(1-(\mu_1 + \mu_3 \Delta t)) + P_{001}(t)\lambda \Delta t + P_{110}(t)\mu_2 \Delta t + P_{101}(t)\mu_3 \Delta t$

6) $P_{110}(t+\Delta t) = P_{110}(t)(1-(\mu_1 + \mu_2 \Delta t)) + P_{010}(t)\lambda \Delta t + P_{110}(t)\mu_3 \Delta t$

7) $P_{b10}(t+\Delta t) = P_{b10}(t)(1-\mu_2 \Delta t) + P_{110}(t)\mu_1 \Delta t + P_{b11}(t)\mu_3 \Delta t$

8) $P_{011}(t+\Delta t) = P_{011}(t)(1-(\lambda + \mu_2 + \mu_3 \Delta t)) + P_{b11}(t)\mu_3 \Delta t + P_{011}(t)\mu_1 \Delta t + P_{b10}(t)\mu_2 \Delta t$

9) $P_{011}(t+\Delta t) = P_{011}(t)(1-(\lambda + \mu_3 \Delta t)) + P_{001}(t)\mu_2 \Delta t$

10) $P_{111}(t+\Delta t) = P_{111}(t)(1-(\mu_1 + \mu_3 \Delta t)) + P_{011}(t)\lambda \Delta t + P_{111}(t)\mu_2 \Delta t$

11) $P_{b21}(t+\Delta t) = P_{b21}(t)(1-(\mu_1 + \mu_3 \Delta t)) + P_{111}(t)\lambda \Delta t + P_{011}(t)\mu_3 \Delta t$

12) $P_{111}(t+\Delta t) = P_{111}(t)(1-(\mu_2 + \mu_3 \Delta t)) + P_{111}(t)\mu_1 \Delta t$

13) $P_{b01}(t+\Delta t) = P_{b01}(t)(1-\mu_3 \Delta t) + P_{101}(t)\mu_1 \Delta t + P_{b11}(t)\mu_2 \Delta t$
Let $\mu_1, \mu_2, \mu_3, \mu_4$ be the steady state equations in terms of $\rho = \lambda / \mu$ are found to be:

1) $-\rho P_{000} + P_{001} = 0$

2) $P_{100} + \rho P_{000} + P_{101} = 0$

3) $-(\rho + 1) P_{010} + P_{100} + P_{011} = 0$

4) $-(\rho + 1) P_{000} + P_{010} + P_{011} = 0$

5) $-2P_{100} + \rho P_{000} + P_{110} + P_{1b1} = 0 \quad \{2.2.2\}$

6) $-2P_{110} + \rho P_{100} + P_{111} = 0$

7) $-P_{b10} + P_{110} + P_{b11} = 0$

8) $-(\rho + 2) P_{011} + P_{101} + P_{b11} + P_{b10} = 0$

9) $-(\rho + 1) P_{1b1} + P_{011} = 0$

10) $-2P_{1b1} + \rho P_{001} + P_{111} = 0$

11) $-3P_{111} + \rho P_{011} = 0$

12) $-2P_{b11} + P_{111} = 0$

13) $-P_{b11} + P_{1b1} + P_{b11} = 0$

The necessary condition for steady state solution is:

$$P_{000} + P_{100} + P_{010} + P_{001} + P_{101} + P_{110} + P_{b10} + P_{011} + P_{001} + P_{b11} + P_{1b1} + P_{b11} = 1, \quad P_{ijk} > 0 \quad \{2.2.3\}$$

2.3 SOLUTION USING D OPERATOR

The advantage of D Operator method here over other methods is that 13*13 matrix can be reduced to 13*5 matrix and solution will be obtained quickly. The details of linear operator can be found in Gross and Harris (1974)
Expressions for expected number of patients in the system, in the cue, expected waiting time in the system, in the queue, expected length non-empty queue and average number of idle servers can be written in right forward manner.

By making use of linear operator $D^1 P_{n,m,r} = P_{n+1,m,r}$ and $D^b P_{n,m,r} = P_{n+1,m,r}$ steady state equations can be put as:

1) $-\rho P_{000} + P_{001} = 0$

2) $(\rho - D^1)P_{000} + D^1 P_{001} = 0$

3) $-(\rho + 1)P_{010} + D^1 P_{000} + P_{011} = 0$

4) $-(\rho + 1)P_{001} + P_{010} + P_{011} = 0$

5) $(\rho - 2D^1)P_{001} + D^1 P_{010} + D^1 P_{011} = 0$ \{2.3.1\}

6) $(\rho - 2D^1)P_{010} + D^1 P_{011} = 0$

7) $(D^1 - D^b)P_{010} + D^b P_{011} = 0$

8) $-(\rho + 2)P_{011} + D^1 P_{001} + D^b P_{001} + D^b P_{010} = 0$

9) $-(\rho + 1)P_{001} + P_{011} = 0$

10) $(\rho - 2D^1)P_{011} + D^1 P_{011} = 0$

11) $(\rho - 3D^1)P_{011} = 0$

12) $(D^1 - 2D^b)P_{011} = 0$

13) $(D^1 - D^b)P_{011} + D^b P_{011} = 0$
The above equations can be represented in matrix form as follows:

\[
\begin{bmatrix}
-p & 0 & 1 & 0 & 0 \\
p-D^b & 0 & D^b & 0 & 0 \\
D^b & -(1+p) & 0 & 1 & 0 \\
0 & 1 & -(p+1) & 0 & 1 \\
0 & D^b & p-2D^b & 0 & D^b \\
0 & p-2D^b & 0 & D^b & 0 \\
0 & D^b - D^b & 0 & D^b & 0 \\
0 & D^b & D^b & -(p+2) & D^b \\
0 & 0 & 0 & 1 & -(1+p) \\
0 & 0 & 0 & D^b & p-2D^b \\
0 & 0 & 0 & p-3D^b & 0 \\
0 & 0 & 0 & D^b - 2D^b & 0 \\
0 & 0 & 0 & D^b & D^b - 2D^b
\end{bmatrix}
\begin{bmatrix}
P_{000} \\
P_{010} \\
P_{001} \\
P_{011} \\
P_{0b1}
\end{bmatrix} = 0 \quad \{2.3.2\}
Corresponding augmented matrix is:

\[
\begin{pmatrix}
-\rho & 0 & 1 & 0 & 0 & \mid & 0 \\
\rho-D^l & 0 & D^l & 0 & 0 & \mid & 0 \\
D^l & -(1+\rho) & 0 & 1 & 0 & \mid & 0 \\
0 & 1 & -(\rho+1) & 0 & 1 & \mid & 0 \\
0 & D^l & \rho-2D^l & 0 & D^l & \mid & 0 \\
0 & \rho-2D^l & 0 & D^l & 0 & \mid & 0 \\
0 & D^l-D^a & 0 & D^a & 0 & \mid & 0 \\
0 & D^a & D^l & -(\rho+2) & D^a & \mid & 0 \\
0 & 0 & 0 & 1 & -(1+\rho) & \mid & 0 \\
0 & 0 & 0 & D^l & \rho-2D^a & \mid & 0 \\
0 & 0 & 0 & \rho-3D^l & 0 & \mid & 0 \\
0 & 0 & 0 & D^l-2D^a & 0 & \mid & 0 \\
0 & 0 & 0 & D^a & D^l-D^a & \mid & 0
\end{pmatrix}
\]
By using elementary operations the final form of the matrix is given below:

\[
\begin{pmatrix}
-\rho & 0 & 1 & 0 & 0 \\
0 & 1 & -(1+\rho) & 0 & -1 \\
\rho-D^1 & 0 & D^1 & 0 & 0 \\
0 & 0 & (1+D^1)-(p+1) \cdot (p-1) & -1 & -(1+\rho) \\
0 & 0 & -D^1(1+p)-(p-2D^1) & 0 & 0 \\
0 & \rho-2D^1 & 0 & -D^1 & 0 \\
0 & D^1-D^b & 0 & -D^b & 0 \\
0 & 0 & -D^b(1+p)-D^1 & -(p+2) & 0 \\
0 & 0 & 0 & -1 & (1+p) \\
0 & 0 & 0 & -D^1 & -(p-2D^1) \\
0 & 0 & 0 & -(p-3D^1) & 0 \\
0 & 0 & 0 & -(D^1-2D^b) & 0 \\
0 & 0 & 0 & -D^b & -(D^1-D^b)
\end{pmatrix}
\]

\[\begin{pmatrix}
P_{000} \\
P_{010} \\
P_{001} \\
P_{011} \\
P_{011}
\end{pmatrix} = 0\]
From the above matrix, solutions can be obtained in terms of $P_{000}$ which itself can be determined from necessary condition (2.2.3)

1) $P_{100} = \rho P_{000}/(1-\rho)$
2) $P_{010} = \rho(-\rho^3 - \rho^2 + \rho + 2) P_{000}/2(1-\rho^2)$
3) $P_{001} = \rho P_{000}$
4) $P_{101} = \rho^2 P_{000}/(1-\rho)$
5) $P_{011} = \rho^2(\rho^2 + \rho - 1) P_{000}/2(1-\rho)$
6) $P_{000} = \rho^2(\rho^2 + \rho - 1) P_{000}/2(1-\rho^2)$
7) $P_{110} = \rho^2(-\rho^4 - 5\rho^3 - 3\rho^2 + 4\rho + 6) P_{000}/12(1-\rho^2)$
8) $P_{110} = \rho^2(-2\rho^4 - 7\rho^3 - 3\rho^2 + 5\rho + 6) P_{000}/12(1-\rho^2)$
9) $P_{111} = \rho^3(\rho^2 + \rho - 1) P_{000}/6(1-\rho)$
10) $P_{011} = \rho^3(\rho^2 + \rho - 1) P_{000}/12(1-\rho)$
11) $P_{111} = \rho^3(\rho^2 + \rho - 1)(\rho + 4) P_{000}/12(1-\rho^2)$
12) $P_{011} = \rho^3(\rho^2 + \rho - 1)(2\rho + 5) P_{000}/12(1-\rho^2)$
13) $P_{000} = 4(1-\rho^2)/(\rho^6 + 4\rho^5 + 4\rho^4 - \rho^3 + 6\rho^2 + 12\rho + 4)$

2.4 RATE TRANSITION DIAGRAM

Another method for obtaining probability equations is by flow balance procedure also known as stochastic balance procedure. The flow rates are all in terms of $\lambda$ and $\mu$ such that the mean total flow out of state $n$ equals $\lambda P_n + \mu P_n$ while mean total flow into state $n$ equals $\mu P_{n+1} + \lambda P_{n-1}$. The rate transition diagrams of different probability equations are given below.
\[ \lambda P_{000} = \mu_3 P_{001} \]

\[ \mu_1 P_{100} = \lambda P_{000} + \mu_3 P_{101} \]

\[ \lambda P_{010} + \mu_2 P_{010} = \mu_1 P_{100} + \mu_3 P_{011} \]
\[ \lambda P_{001} + \mu_3 P_{001} = \mu_2 P_{010} + \mu_3 P_{0b1} \]

\[ \mu_1 P_{101} + \mu_3 P_{101} = \lambda P_{001} + \mu_2 P_{110} + \mu_3 P_{1b1} \]
$$\mu_1 P_{110} + \mu_2 P_{110} = \lambda P_{010} + \mu_3 P_{111}$$

$$\mu_2 P_{010} = \mu_1 P_{110} + \mu_3 P_{011}$$
\[ \lambda p_{011} + \mu_2 p_{011} + \mu_3 p_{011} = \mu_2 p_{101} + \mu_1 p_{101} + \mu_3 p_{101} \]

\[ \lambda p_{0b1} + \mu_3 p_{0b1} = \mu_2 p_{011} \]
\[ \mu_1 P_{1b1} + \mu_3 P_{1b1} = \lambda P_{0b1} + \mu_2 P_{111} \]

\[ \mu_1 P_{111} + \mu_2 P_{111} + \mu_3 P_{111} = \lambda P_{011} \]
12) \[ \mu_1 \]

\[ \mu_2 P_{bb1} + \mu_3 P_{b11} = \mu_1 P_{111} \]

13) \[ \mu_2 \]

\[ \mu_3 P_{bb1} = \mu_1 P_{1b1} + \mu_2 P_{b11} \]
Probability equations in terms of $\rho = \lambda / \mu$ are as follows:

1) $-\rho P_{000} + P_{101} = 0$

2) $-P_{010} + \rho P_{001} + P_{101} = 0$

3) $-(\rho + 1)P_{010} + P_{001} + P_{011} = 0$

4) $-(\rho + 1)P_{001} + P_{010} + P_{011} = 0$

5) $-2P_{101} + \rho P_{001} + P_{110} + P_{111} = 0$

6) $-2P_{110} + \rho P_{010} + P_{111} = 0$

7) $-P_{110} + P_{110} + P_{111} = 0$

8) $-(\rho + 2)P_{011} + P_{101} + P_{111} = 0$

9) $-(\rho + 1)P_{001} + P_{011} = 0$

10) $-2P_{101} + \rho P_{011} + P_{111} = 0$

11) $-3P_{111} + \rho P_{011} = 0$

12) $-2P_{111} + P_{111} = 0$

13) $-P_{111} + P_{111} + P_{111} = 0$

It is observed the same equations are obtained by using this method, hence solutions will be the same.

2.5 ILLUSTRATIONS AND DISCUSSIONS

For illustration we consider a situation described in the introductory section for which arrival rate is 10/hour. Servers in each stage takes 5 minutes to observe a patient, let $\lambda = 10$/hour, $\mu = 60/5 = 12$/hour, $\rho = \lambda / \mu = 0.83333$
The probabilities $P_{nk}$ work out to be:

- $P_{000} = 0.058$
- $P_{001} = 0.049$
- $P_{010} = 0.122$
- $P_{100} = 0.28$
- $P_{101} = 0.235$
- $P_{011} = 0.061$
- $P_{010} = 0.033$
- $P_{110} = 0.04$
- $P_{b10} = 0.034$
- $P_{111} = 0.017$
- $P_{b11} = 0.009$
- $P_{121} = 0.02$
- $P_{b21} = 0.03$

Expected number of patients in the system can be calculated from the equation

$$L_s = 0.058 + 1 \cdot (0.049 + 0.122 + 0.28) + 2 \cdot (0.235 + 0.061 + 0.033 + 0.04 + 0.034 + 0.017 + 0.009 + 0.02 + 0.03)$$

$$L_s = 1.5$$

Expected waiting time in the system will be

$$W_s = L_s / \lambda$$

$$= 0.15 \text{ hour} = 9 \text{ min.}$$

Expected waiting time in the queue

$$W_q = W_s - 1 / \mu$$

$$= 0.0666 \text{ hour} = 4 \text{ min.}$$
Expected number of patients in the queue

\[ L_q = \lambda W_q = 0.67 \]

Expected length of non-empty queue \((L/L>0)\)

\[ (L/L>0) = \frac{L_s}{\text{Probability (an arrival has to wait, } L>0)} \]

\[ = \frac{L_s}{(1-P_{000})} \text{(since probability of an arrival not to wait is } P_{000}) \]

\[ = 1.5/(1-0.058) = 1.5924 \]

Average number of idle servers = \(s-\) (average number of customers served)

\[ = 3-1.5 = 1.5 \]

We would like to compare these values for two different cases, firstly when there are two stages of service with blocking for which solutions already exist in the literature.

For two stages of service with blocking \(p=.833, L_s = .9175, W_s = 5.51 \text{ min.}, W_q = .51 \text{ min.}, L_q = .0847 \)

Here it is seen that waiting time in two stages with blocking is less as compared to that of three stages with blocking since in the latter case blocking increases threefold. Then also, three stages with blocking is more advantageous to the extent that patients get appropriate treatment at the cost of waiting.

In case of three stages where infinite queue is allowed in front of each stage, \(L_s = 3p/(1-p) = 15, W_s = 1\text{hour}30\text{min.}, W_q = 1\text{hour}25\text{min.}, L_q = 14.2 \)

It is observed that due to blocking not only expected duration of waiting time decreases but also patients get appropriate treatment. This model can be modified for analyzing data pertaining to emergency service in hospitals.