Chapter 5

Polar Orbits in Kerr Spacetime

The discovery of Carter’s constant of motion in Kerr spacetime led to the complete analytic solution of the case of a particle motion in rotating curved spacetimes. Polar orbits in Kerr spacetime would be the ones that cross the axis of rotation of the black hole and because of the rotation of spacetime such orbits experience an advancement of the ascending node [46]. The equations of motion for the polar orbits can be deduced for the orbital angle \( \theta = 0 \).

Let \( x^\mu \) be the Boyer Lindquist coordinates in which the Kerr metric is written as

\[
\begin{align*}
    ds^2 &= -\left(1 - \frac{2Mr}{\Sigma}\right) dt^2 - 2\left(\frac{2Mr}{\Sigma}\right) a \sin^2 \theta \; dt \; d\phi + \left(\frac{\Sigma}{\Delta}\right) dr^2 + \Sigma d\theta^2 + \\
    &\left(\frac{A}{\Sigma}\right) \sin^2 \theta \; d\phi^2
\end{align*}
\]  

(5.1)
where

\[ \Sigma = r^2 + a^2 \cos^2 \theta \]  \hspace{1cm} (5.2)  \\
\[ \Delta = r^2 + a^2 - 2Mr \]  \hspace{1cm} (5.3)  \\
\[ A = (r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta \]  \hspace{1cm} (5.4)

The equations of motion obtained by Carter using the Hamilton-Jacobi formulism are given as follows, if \( x^a(\tau) \) is the coordinate image of the timelike geodesic \( C(\tau) \) followed by a particle of rest mass \( \mu \), then the vector \( u^a = \dot{x}^a = dx^a/d\tau \) will satisfy the quadrature equations:

\[ \dot{t} = (\Delta \Sigma)^{-1}(A E - 2Ma\phi) \]  \hspace{1cm} (5.5)  \\
\[ \Sigma^2 \dot{r}^2 = [(r^2 + a^2)E - a\phi]^2 - \Delta(\mu^2 r^2 + K) \]  \hspace{1cm} (5.6)  \\
\[ \Sigma^2 \dot{\theta}^2 = K - \mu^2 a^2 \cos^2 \theta - \left( aE\sin\theta - \frac{\phi}{\sin\theta} \right)^2 \]  \hspace{1cm} (5.7)  \\
\[ \dot{\phi} = \Delta^{-1} \left[ \frac{(2Mr)}{\Sigma} aE + \frac{(1-\frac{2Mr}{\Sigma})\phi}{\sin^2 \theta} \right] \]  \hspace{1cm} (5.8)

Where \( \Phi, E \) and \( K \) are the projection of the angular momentum along the symmetric axis, energy at infinity of the particle, and the Carter’s constant respectively.

For the orbit represented by the \( C(\tau) \) to be polar, it has to intersect the symmetric axis of the Kerr spacetime, since this axis consists of points for which \( \sin\theta = 0 \), hence from the third equation above, \( \Phi = 0 \), is a necessary condition for an orbit to be polar in the Kerr spacetime. Or in other words, for a particle to follow a polar orbit, it is necessary that the particle in Kerr spacetime has a null angular momentum.

The equations of motion for polar orbits are then expressed as
\[ \dot{t} = AE/\Delta \Sigma \]  
\[ \Sigma^2 \dot{r}^2 = R(r) = (r^2 + a^2)^2[E^2 - V^2(r)] \]  
\[ \Sigma^2 \dot{\theta} = Q - a^2(1 - E^2) \cos^2 \theta \]  
\[ \dot{\phi} = 2MaEr/\Delta \Sigma \]

Where the effective potential \( V \) is given as

\[ V^2 = \Delta (K + r^2)/(r^2 + a^2)^2 \]  
\[ Q = K - a^2E^2 \]

The effective potential \( V^2(r) \) is plotted with the radius for different values of \( K \) and it is seen that bound orbits can exist for \( K \) values greater than 8.
Fig 5.1: Effective potential vs log $r/M$ for Kerr spacetime, for four different values of Carter’s constant.

15.1 Spherical Polar Orbits in Kerr spacetime

The spherical polar orbits for which the radial vector remains constant throughout the orbit can be expressed by replacing the $r$ equation as $\frac{\partial r}{\partial \tau} = 0$.

$$\Sigma^2 \dot{r}^2 = R(r) = (r^2 + a^2)^2 [E^2 - V^2(r)] = 0$$

(5.15)

The above equations are solved numerically for the case $r = 10M$, when the energy of the test particle is $E = 0.956$ and the Carter’s constant $K = 14.783$ for a rotating blackhole with angular momentum $a = 0.8M$. 


Fig 5.2a: Temporal evolution of a spherical polar orbit in the x-y-z space.

The results are plotted in the three dimensional space and the particle is observed to be following a polar orbit which rotates due to the frame dragging. Since the orbit lies on a sphere of fixed radius, such orbits are termed as spherical polar orbits in Kerr spacetime.

The energy and the Carters constants for a spherical polar orbit are given as

\[ E^2 = \Delta (r^2 + K)/(a^2 + r^2)^2 \]  \hspace{1cm} (5.16)

and

\[ K = (Mr^4 - a^2 r^3 - 3Ma^2 r^2 + a^4 r)/(r^3 - 3Mr^2 + a^2 r + Ma^2) \]  \hspace{1cm} (5.17)
Fig 5.2b : Spherical polar orbits in the x-y plane.

The spherical polar orbits have been plotted for different radii with their corresponding E and K values and have been verified to hold good upto large radii of $1000M$, which is the proposed range of orbits for spacecrafts in Kerr spacetime in this study. The variation of $\theta$ and $\phi$ coordinates with respect to the proper time gives the following linear increase for both the coordinates, fig 5.3.

Fig. 5.3: Variation of $\theta, \phi$ coordinates with respect to $\tau$ for an orbit of radius $10M$. 

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5.2 Frame Dragging of Polar Orbits

The rotation of the spacetime inherent in the Kerr geometry will force the lines of nodes of bound orbits of the non-equatorial orbits to advance in the sense of the rotating black hole. Therefore the advancement of the nodal point (where the particle’s orbit intersects the equatorial plane) coincides with the rotation of the black hole. This effect is known as the Lense and Thirring effect [51]. An experiment to measure this effect due to the rotation of the Earth has been proposed and is known as the Stanford Gyroscope Experiment [52] based on the method proposed by Schiff [53].

From the last two equations of motion in polar orbit, the rate of change of the coordinate $\phi$ with the coordinate $\theta$ can be written as

$$\frac{d\phi}{d\theta} = \frac{2MaEr}{\Delta \theta^{1/2}}$$

where

$$k^2 = a^2 \Gamma^2 / Q$$

and

$$\Gamma^2 = 1 - E^2$$

For the cases, $E^2 < 1$, $k$ is also less than 1, and the above equation can be integrated to give the solutions in the form,

$$\phi = \phi_0 + \frac{2MaEr}{\Delta Q^{1/2}} F(\theta, k)$$

Where $F(\theta, k)$ is the elliptical integral of the first kind and is defined as

$$F(\theta, k) = \int_0^\theta (1 - k^2 \cos^2 \theta)^{1/2} \, d\theta$$
In the approximation, $k \ll 1$, the integral can be solved to give the final change of the coordinate $\phi$ per revolution as [49]

$$
\delta \phi = \frac{4\pi Ma Er}{\Delta Q^{1/2}} \left(1 + \frac{k^2}{4} + \frac{3}{26} k^4 + O(k^6)\right) \quad (5.23)
$$

This is a typical result of the dragging effect associated with rotating bodies and the effect vanishes for the non rotating black holes as for such cases $a = 0$ and hence $\delta \phi = 0$.

5.3 Non Spherical Polar Orbits

The radial equation corresponding to the polar orbits in Kerr spacetime is given as

$$
\rho^4 \dot{r} = R(r) = (r^2 + a^2)(E^2 - V^2(r)) \quad (5.24)
$$

If $r_0$ is the radius of a spherical orbit and a double root of the function $R(r)$,

$$
R(r) = (r - r_0)^2 G(r) \quad (5.25)
$$

where

$$
G(r) = -\Gamma^2 r^2 + 2(M - \Gamma^2 r_0) r - a^2 Q / r_0^2 \quad (5.26)
$$

If the energy of the particle now varies from $E$ to $E_0$, then the radial equation can be written as

$$
R(r) = (r^2 + a^2)[E^2 - E_0^2 + E_0^2 - V^2(r)]
$$

$$
= (r^2 + a^2)(E^2 - E_0^2) + (r - r_0)^2 G(r) \quad (5.27)
$$

The turning points of the radial coordinate can be given by

$$
(r_1 - r_0)^2 = (r_1^2 + a^2)(E^2 - E_0^2) / G(r) \quad (5.28)
$$
For small differences in the energy the points $r_1$ will lie close to $r_0$, and the radial function can be expressed as

$$R(r) \approx -G(r_0)[(r_1 - r_0)^2 - (r - r_0)^2]$$

such that

$$\int \frac{dr}{\sqrt{R(r)}} \approx [G(r_0)]^{-1/2} \frac{\sin^{-1}(r-r_0)}{|r_1 - r_0|}$$

The above equations yield the approximate solution for the radial motion of the particle as its orbit oscillates between the radial values $r_0 + \Delta r$ and $r_0 - \Delta r$ where $(\Delta r = |r_1 - r_0|)$ [50].

Alternatively, when the double root of the equation $R(r) = 0$, is associated with the interval $r_1 < r_0 < r_2$, and the function $G(r_0) > 0$

$$R(r) = \Gamma^2 (r - r_0)^2(r - r')(r'' - r)$$

substituting

$$x = (r - r_0)^{-1}$$

$$l_0(r) = \int \frac{dr}{R^{1/2}} = \int dr \ [(r - r_0)^2 G]^{-1/2} = -\int dx/X^{1/2}$$

where

$$X(x) = \alpha + \beta x + \gamma x^2$$

$$\alpha = -\Gamma^2$$

$$\beta = 2(M - 2\Gamma^2 r_0) = \Gamma^2 (r' + r'' - 2r_0)$$

$$\gamma = G(r_0) = \Gamma^2 (r_0 - r')(r'' - r_0)$$

and

$$l_0 = -(1/\gamma)^{1/2} \ln(\beta + 2\gamma x + \gamma^2 X^{1/2}) \text{ if } G(r_0) > 0$$
\[ (-1/\gamma)^{1/2} \sin^{-1}(2\gamma x + \beta)/\Gamma^2(r'' - r') \text{ if } G(r_0) < 0 \]  

(5.38)

If \( G(r) \) vanishes at \( r_0 \) it would imply that \( r_0 \) coincides with either one value of \( r' \) or \( r'' \), which would give

\[ I_0 = \pm \frac{2}{|\Gamma(r'' - r_0)|} \left( \frac{r'' - r}{r - r_0} \right)^{1/2} \text{ if } r'' = r_0 \text{ and similarly,} \]

\[ I_0 = \pm \frac{2}{|\Gamma(r' - r_0)|} \left( \frac{r' - r}{r - r_0} \right)^{1/2} \text{ if } r'' = r_0. \]

(5.39)

The above integrals give the analytic solutions for the near spherical polar orbits and can be evaluated to identify the non spherical polar orbits associated with a spherical polar orbit of radius \( r_0 \).

### 5.5 Precession of Gyroscope

The calculations for the change in the spin of the gyroscope as it completes a rotation of the Kerr black hole has been possible using the equations of parallel transport in Kerr geometry by J.-A. Marck [54]. A further study of parallel transport in polar orbits has been given by Tsoubelis et al. [54] For a gyroscope falling freely along the path \( \mathcal{C}(\tau) \), that of the polar orbit, the spin vector \( S(\tau) \) will be parallel transported along the geodesic. If an orthonormal tetrad is constructed, \( \{\lambda_{(a)}\} \), which is parallelly transported along the geodesic then the spin \( S(0) = 0 \) of the gyroscope in this reference frame would stay constant.

Considering the base \( \{e_a\} \) such that,

\[ e_0 = \left( \frac{A}{\Sigma \Delta} \right)^{1/2} \partial_t + \left[ \frac{2Mar}{(\Lambda \Sigma \Delta)^2} \right] \partial_\phi \]  

(5.40)

\[ e_1 = \left( \frac{A}{\Sigma} \right)^{1/2} \partial_r \]  

(5.41)

\[ e_2 = \left( \frac{1}{\Sigma} \right)^{1/2} \partial_\theta \]  

(5.42)
\[ e_3 = \left( \frac{\Delta}{a^2} \right)^{1/2} \left( \frac{1}{\sin \theta} \right) \partial_\phi \]  

(5.43)

The Kerr metric in this base would now take the form

\[ ds^2 = \eta_{ab} e^a \otimes e^b \]  

(5.44)

where \( \eta_{ab} = \text{diag}(-1,+1,+1,+1) \) and \( e^a \) are the one-form dual to \( e_a \). Now according to the set of geodesic equations in Kerr spacetime for polar orbits, the vectors

\[ e_0 = P e_0 + Q e_2 \]  

(5.45a)

\[ e_\tilde{1} = e_1 \]  

(5.45b)

\[ e_\tilde{2} = Q e_0 + P e_2 \]  

(5.45c)

\[ e_\tilde{3} = e_3 \]  

(5.45d)

would form a comoving frame along the geodesic \( \mathcal{C}(\tau) \), where

\[ P = \left( \frac{A}{\Sigma a} \right)^{1/2} E \]  

(5.46)

and

\[ Q = \Sigma^{1/2} \dot{\theta} \]  

(5.47).

This base is not defined well on the symmetry axis as \( \sin \theta = 0 \) and thus there is a coordinate singularity in the Boyer Lindquist coordinate system. This coordinate singularity can be avoided by choosing another coordinate system which is known as the Kerr-Schild coordinate system \( (x_0, x, y, z) \) which is well behaved on the symmetry axis. In this coordinate system the Kerr metric on the symmetric axis is given as
\[ ds^2 = -\left[1 - \frac{2Mz}{z^2 + a^2}\right] d(x^0)^2 + \left[1 - \frac{2Mz}{z^2 + a^2}\right]^{-1} d\zeta^2 + dx^2 + dy^2 \quad (5.48) \]

Therefore as \( \sin \theta \to 0 \)

\[
\begin{align*}
\mathbf{e}_0 & \to \left[1 - 2Mz/(z^2 + a^2)\right]^{-1/2} \partial_\zeta \\
\mathbf{e}_1 & \to \left[1 - 2Mz/(z^2 + a^2)\right]^{1/2} \partial_z \\
\mathbf{e}_2 & \to \cos \phi \partial_\zeta + \sin \phi \partial_\zeta \\
\mathbf{e}_3 & \to -\sin \phi \partial_\zeta + \cos \phi \partial_\zeta
\end{align*}
\quad (5.49a-b)
\]

Thus if the initial direction along which the orbit emerges from the \( z \)-axis is set, it is possible to join the orthonormal base \( \{ \mathbf{e}_a \} \) to a unique coordinate-tied tetrad there. Assuming that initially \( \phi = 0 \), the gyroscope will return after one rotation to the starting point on the \( z \)-axis along a direction which is given as

\[
\phi = \frac{4\pi M a E r}{\Delta Q^{1/2}} \left(1 + k^2/4 + \frac{3}{26} k^4 + O(k^6)\right) \quad (5.50)
\]

which is derived earlier, equation (5.23).

Making use of the Marck’s [47] construction of parallelly transported orthonormal tetrad along a geodesic in Kerr spacetime to express the set of vectors \( \{ \lambda_{(a)} \} \) in terms of the base \( \{ \mathbf{e}_a \} \).

\[
\lambda_{(0)} = \mathbf{e}_6 \quad (5.51a)
\]

\[
\lambda_{(1)} = \cos \Psi(\tau) \lambda_{(1)}' - \sin \Psi(\tau) \lambda_{(3)}' \quad (5.51b)
\]

\[
\lambda_{(2)} = P \left( \frac{1}{KA} \right)^{1/2} (r^2 + a^2) a \cos \theta \mathbf{e}_1 - \left( \frac{\Delta}{KA} \right)^{1/2} a r \sin \theta \mathbf{e}_2 - Q \left( \frac{1}{KA} \right)^{3/2} (r^2 + a^2) r \mathbf{e}_3 \quad (5.51c)
\]

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\[ \lambda_{(3)} = \sin \Psi(\tau) \lambda'_{(1)} + \cos \Psi(\tau) \lambda'_{(3)} \]  

(5.51d)

where the angle \( \Psi(\tau) \) is given by the equation

\[ \Psi = EK^{1/2}(K - a^2)/(r^2 + K)(K - a^2 \cos^2 \theta) \]  

(5.52)

and

\[ \lambda'_{(1)} = \alpha P \left( \frac{1}{KA} \right)^{1/2} (r^2 + a^2) r e_1 + \beta \left( \frac{\Delta}{KA} \right)^{1/2} a^2 \sin \theta \cos \theta \ e_2 + \beta Q \left( \frac{1}{KA} \right)^{1/2} (r^2 + a^2) a \cos \theta \ e_3 \]  

(5.53)

\[ \lambda'_{(3)} = \beta Q \left[ \frac{\Sigma}{A(K + a^2)} \right]^{1/2} (r^2 + a^2) e_2 - \beta \left[ \frac{\Sigma \Delta}{A(K + r^2)} \right]^{1/2} a \sin \theta \ e_3 \]  

(5.54)

where

\[ \alpha^2 = \beta^{-2} = (K - a^2 \cos^2 \theta)/(K + r^2) \]  

(5.55)

If \( \Psi(0) = 0 \) and \( S^i(0) \) are the components of the gyroscope’s spin at the beginning, then when the gyroscope returns to the starting point after one complete latitude oscillation, the components of the spin vector in the comoving frame \( \{e_i\} \) will have changed to \( S^i(T_\tau) \) where

\[ S^i(T_\tau) = R^i_j S^j(0) \]  

(5.56)

where the matrix \( R^i_j \) given as

\[ R^i_j = \begin{pmatrix} 
1 + (\cos \Psi - 1) \cos^2 Z & -\sin \Psi \cos Z & (\cos \Psi - 1) \sin Z \cos Z \\
\sin \Psi \cos Z & \cos \Psi & \sin \Psi \sin Z \\
(\cos \Psi - 1) \sin Z \cos Z & -\sin \Psi \sin Z & 1 + (\cos \Psi - 1) \sin^2 Z 
\end{pmatrix} \]  

(5.57)

and

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\[ \cot Z = \left( \frac{a}{r} \right) \frac{(K + r^2)^{3/2}}{(K - a^2)^{1/2}} \]  

(5.58)

The interpretation as given in [55] is that in each revolution of the gyroscope about the gravitating center its spin rotates by an angle \( \Psi \) around an axis which is inclined by an angle \( Z \) relative to \( e_1 \) and lies in the \( e_1 - e_3 \) plane of the comoving frame \( \{e_i\} \). During the same interval the frame \( \{e_i\} \) itself rotates with respect to the \( x - y - z \) coordinate system by an angle \( \phi \) about the \( z \)-axis which coincides with the vector \( e_1 \) at the beginning and end of the cycle.

The precession of the gyroscope in the polar orbit can be a measure of the properties of the black hole if a satellite is made to revolve in such geodesics. This effect can be directly measured with respect to the asymptotic Lorentz frame and can provide a useful technique for the navigation of spacecrafts in the Kerr spacetime.

### 5.6 Numerical Simulation of Spherical Polar Orbits

Orbits of different radii ranging from \((5 - 2000)M_\odot\), are shown in figure 5.4 (a)-(f), for a black hole of 1 \( M_\odot \) and \( a = 0.8M_\odot \). The first graph of the set plots \( \theta \) vs \( \tau \) and \( \phi \) vs \( \tau \), and the second 3D plot traces the trajectory of the particle in the spherical polar orbit. With the increase in radius of the spherical polar orbit, the trajectory spans lesser spherical surface but gets confined to a ring of decreasing width.
Fig:5.4a: The above plots show (i) variation of $\theta$ and $\phi$ coordinates of the Boyer Lindquist coordinates system for a spherical polar orbit at $r = 5M_\odot$, the top line represents $\theta(\tau)$ and the second line represents $\phi(\tau)$, the graph indicates that the variations are significant for both the cases of $\theta(\tau)$ and $\phi(\tau)$ in lower orbits in Kerr spacetime. (ii) Trajectory of a particle in Kerr orbit at $r = 5M_\odot$. 
Fig: 5.4b  The above plots show (i) variation of $\theta$ and $\phi$ coordinates of the Boyer Lindquist coordinates system for a spherical polar orbit at $r = 10M_\odot$, the top line represents $\theta(\tau)$ and the second line represents $\phi(\tau)$, the graph indicates that the variations are significant for the cases of $\theta(\tau)$ and lesser variation for $\phi(\tau)$ on increasing the size of orbit from $r = 5M_\odot$-10$M_\odot$ in Kerr spacetime. (ii) Trajectory of a particle in Kerr orbit at $r = 10M_\odot$. The orbit are suggested to rotate greater in lower spherical polar orbit.

![Graph](image1)

Fig: 5.4c  The above plots show (i) variation of $\theta$ and $\phi$ coordinates of the Boyer Lindquist coordinates system for a spherical polar orbit at $r = 50M_\odot$, the top line represents $\theta(\tau)$ and the second line represents $\phi(\tau)$, the graph indicates that the variation is significant for the case of $\theta(\tau)$ and further less variation in $\phi(\tau)$ on increasing the size of orbit from $r = 10M_\odot$-50$M_\odot$ in Kerr spacetime. (ii) Trajectory of a particle in Kerr orbit at $r = 50M_\odot$. The orbits are very less rotated in the $\phi$ direction, implying decrease in the Lense-Thirring effect.

![Graph](image2)
Fig 5.4d: The above plots show (i) variation of $\theta$ and $\phi$ coordinates of the Boyer Lindquist coordinates system for a spherical polar orbit at $r = 100M_\odot$, the top line represents $\theta(\tau)$ and the second line represents $\phi(\tau)$, the graph indicates that the variation is significant for the cases of $\theta(\tau)$ and further less variation in $\phi(\tau)$ on increasing the size of orbit from $r = 50M_\odot$-100$M_\odot$ in Kerr spacetime.(ii) Trajectory of a particle in Kerr orbit at $r = 100M_\odot$. The orbits are very less rotated in the $\phi$ direction.

Fig 5.4e: The above plots show (i) variation of $\theta$ and $\phi$ coordinates of the Boyer Lindquist coordinates system for a spherical polar orbit at $r = 500M_\odot$, the
top line represents $\theta(\tau)$ and the second line represents $\phi(\tau)$, the graph indicates that the variation is significant for the cases of $\theta(\tau)$ and further less in $\phi(\tau)$ on increasing the size of orbit from $r = 100M_\odot - 500M_\odot$ in Kerr spacetime.\(\text{(**i**)}\) Trajectory of a particle in Kerr orbit at $r = 500M_\odot$. The orbits are less rotated in the $\phi$ direction.

![Graph](image1)

![Graph](image2)

\textbf{Fig 5.4f :} The above plots show \(\text{(i) variation of } \theta \text{ and } \phi \text{ coordinates of the Boyer Lindquist coordinates system for a spherical polar orbit at } r = 1000M_\odot, \) the top line represents $\theta(\tau)$ and the second line represents $\phi(\tau)$, the graph indicates that the variation is significant for the cases of $\theta(\tau)$ and further less variation in $\phi(\tau)$ on increasing the size of orbit from $r = 500M_\odot - 1000M_\odot$ in Kerr spacetime.\(\text{(ii)}\) Trajectory of a particle in Kerr orbit at $r = 1000M_\odot$. The orbits are very less rotated in the $\phi$ direction.

The above simulations are run for greater time steps with increasing radii. The energy and Carter’s constants and corresponding time periods is shown in table 6.1. The polar orbits at large radii in Kerr spacetime approach the Keplerian polar orbits in the field of a non rotating gravitational source.