Chapter 2

Spacecraft Trajectory Analysis

The discovery of heliocentric model by Copernicus led to the first era of scientific understanding of the motion of the celestial bodies. Tycho Brahe studied the planetary motions in detail and later Kepler utilized those observational data and came up with empirical laws that are since widely referred as the Kepler's laws of planetary motion. Isaac Newton discovered the laws of gravitation and formulated the theory of celestial mechanics which explained the Kepler’s laws of planetary motion verifying the Newton’s theory of gravitation.

The three laws of planetary motion given by Kepler are stated as under:

1.) Planets follow elliptical orbits with Sun at one of the focus.
2.) The radius vector joining the Sun with the planet sweeps out equal area in equal interval of time.
3.) The square of the period of a planet is proportional to the cube of its average radius.

The study of motion of a spacecraft considered as a particle in the gravitational field of a celestial body is called astrodynamics and the subject of the motion of the spacecraft about its center of mass is referred as attitude control. Astrodynamics exploits the Newtonian laws of gravity in most of the cases and would use the Einstein’s general relativity laws for extreme gravity conditions. The Newtonian laws are a special approximation of the Einstein’s general theory
of relativity which is considered by far the most accurate description of laws of motion in the influence of gravity.

2.1 General

Newtonian laws of gravity are a good first hand description of the laws of motion of bodies in the influence of gravitational interactions. The laws of gravitation given by Newton in *Principia* are given as

1.) A body would stay at rest or in a state of uniform motion unless acted upon by an external force.
2.) The rate of change of linear momentum is in direction of and proportional to the force applied.
3.) To every action there is an equal and opposite reaction.

In addition the law of gravitation states that two bodies attract one another with a force proportional to the product of their masses \((m_1, m_2)\) and inversely proportional to the square of distance \((r)\) between them, i.e.

\[
F = G \frac{m_1 m_2}{r^2}
\]  

(2.1)

where, \(G\) is the universal constant of gravitation, \(6.6695 \times 10^{-11}\) \(\text{m}^3/\text{kg.s}^2\).

The concept of work and energy are elaborated using the Newton’s Laws and are required to study the two body problem in detail. The work done on a body is equivalent to the energy stored in the body and is expressed as the scalar sum of product of the force applied and the infinitesimal displacement,

\[
W_{12} = \int_{r_1}^{r_2} F \, dr
\]
\[ \int_{t_1}^{t_2} m \frac{dv}{dt} \cdot \mathbf{v} dt \]
\[ = \frac{1}{2} \int_{t_1}^{t_2} m \frac{d}{dt} (\mathbf{v} \cdot \mathbf{v}) dt \]
\[ = \frac{1}{2} m \left( v^2_{t_2} - v^2_{t_1} \right) \quad (2.2) \]

Thus the work done on a particle is the change in its kinetic energy.

The force is said to be **conservative** if the amount of work done in taking a closed path is zero, i.e.

\[ \oint \mathbf{F} \cdot d\mathbf{r} = 0 \quad (2.3) \]

The potential energy is defined as the work done by a conservative force in going from a point \( \mathbf{r}_1 \) to a reference point \( \mathbf{r}_0 \).

\[ V(\mathbf{r}_1) = \int_{\mathbf{r}_1}^{\mathbf{r}_0} \mathbf{F} \cdot d\mathbf{r} + V(\mathbf{r}_0) \quad (2.4) \]

The force can thus be expressed as the negative gradient of potential,

\[ \mathbf{F} = -\nabla V(\mathbf{r}) \quad (2.5) \]

The principle of conservation of energy preserves the total energy of a particle as it moves in a conservative field. Another quantity that remains conserved is the total angular momentum which is given as

\[ \mathbf{L} = \mathbf{r} \times (m\mathbf{v}) \quad (2.6) \]

The specific angular momentum is defined as \( \mathbf{h} = \mathbf{r} \times \mathbf{v} \).

The constancy of angular momentum requires that \( \mathbf{r} \) and \( \mathbf{v} \) remain in the same plane. An important consequence of which is that the orbits in gravitational field are always planar.
2.2 The Two Body Problem

The motion of two bodies in the influence of gravitational force between them is termed as the two body problem. The two bodies are considered to have spherical mass distribution and the separation between the bodies is assumed to be large enough compared to their dimensions. This assumption allows considering the bodies as point particles. If $\mathbf{R}_1, \mathbf{R}_2, \mathbf{R}_G$ are the radius vectors of the first and second body and the center of mass respectively, in an inertial frame of reference O. Then the position vector $\mathbf{R}_G$ can be expressed as

$$\mathbf{R}_G = \frac{m_1 \mathbf{R}_1 + m_2 \mathbf{R}_2}{m_1 + m_2} \quad (2.7)$$

and the velocity and accelerations will be given as

$$\dot{\mathbf{R}}_G = \frac{m_1 \dot{\mathbf{R}}_1 + m_2 \dot{\mathbf{R}}_2}{m_1 + m_2} \quad (2.8)$$

$$\ddot{\mathbf{R}}_G = \frac{m_1 \ddot{\mathbf{R}}_1 + m_2 \ddot{\mathbf{R}}_2}{m_1 + m_2} \quad (2.9)$$

If $\mathbf{r}$ is the position vector of $m_2$ relative to $m_1$ and $\mathbf{u}_r$ be the unit vector in this direction, then,

$$\mathbf{r} = \mathbf{R}_2 - \mathbf{R}_1 \quad (2.10)$$

The gravitational force exerted on $m_2$ by $m_1$ is
The Newton’s law of motion in gravitational interaction suggests that this force can be expressed as

\[ F_{21} = \frac{Gm_1m_2}{r^2} (-u_r) \]  

(2.11)

By the action-reaction principle, \( F_{21} = -F_{12} \), and so

\[ F_{21} = \frac{Gm_1m_2}{r^2} (-u_r) = m_2 \ddot{R}_2 \]  

(2.12)

\[ F_{21} = \frac{Gm_1m_2}{r^2} (u_r) = m_2 \ddot{R}_1 \]  

(2.13)

Multiplying the first equation by \( m_1 \) and the second equation by \( m_2 \) and adding, gives

\[ m_1m_2 (\ddot{R}_2 - \ddot{R}_1) = -\frac{G(m_1 + m_2)}{r^2} (m_1 + m_2)u_r \]  

(2.14)

This can be written in the form

\[ \ddot{r} = -\frac{\mu}{r^3} r \]  

(2.15)

representing the equation of motion of a two body system under the influence of gravity. Where \( \mu \) is the gravitational parameter given by \( \mu = G(m_1 + m_2) \) with unit \( \text{km}^3\text{s}^{-2} \).

The above equation on cross multiplication with \( r \) becomes,

\[ r \times \ddot{r} = \frac{d}{dt} (r \times \frac{dr}{dt}) = \frac{d}{dt} h = 0 \]  

(2.16)

This is the expression for the constancy of the angular momentum per unit mass derived from the Newton’s laws of gravitation. The cross product of the equation of motion with \( h \) would give,
\[
\frac{d^2r}{dt^2} \times h = -\frac{\mu}{r^3} \cdot r \times h = -\frac{\mu}{r^3} \cdot r \times \left( r \times \frac{dr}{dt} \right) \tag{2.18}
\]

using the identity for triple vector products,

\[
\frac{d^2r}{dt^2} \times h = \mu \frac{d}{dt} \left( \frac{r}{r} \right) \tag{2.19}
\]

since \( h \) is a constant the above equation can be directly integrated to give,

\[
\frac{dr}{dt} \times h = \frac{\mu}{r} \left( r + re \right) \tag{2.20}
\]

Taking dot product of the above equation with \( h \) will give

\[
h \cdot e = 0 \tag{2.21}
\]

The vector \( e \) is referred as the eccentricity vector and it marks the plane of the orbital motion. The dot product with \( r \) of the above equation and using the identity of scalar triple product gives

\[
r + r \cdot e = \frac{h^2}{\mu} \tag{2.22}
\]

\[
r = \frac{h^2/\mu}{1+e \cos \theta} \tag{2.23}
\]

here \( \theta \) is the angle between the eccentricity vector and the radius vector \( r \) and is called the true anomaly. The above equation is known as the orbit equation and it mathematically represents the Kepler’s first law.

The angular velocity of the position vector is given as \( \dot{\theta} \) and the normal component of the velocity will be

\[
v_\perp = r \dot{\theta} \tag{2.24}
\]
The specific angular momentum can be written as \( h = r v_\perp = r^2 \dot{\theta} \)

since \[ h = r \times v = ru_r \times (v_r u_r + v_\perp u_\perp) = rv_\perp \hat{h} \] (2.25)

therefore

\[ v_\perp = \frac{h}{r} = \frac{\mu}{h} (1 + e \cos \theta) \] (2.26)

The radial component of the velocity vector is given as

\[ v_r = \dot{r} = \frac{dr}{dt} = \frac{\mu}{h} e \sin \theta \] (2.27)

In a differential time \( dt \) the position vector \( r \) sweeps out an area \( dA \) given by

\[ dA = \frac{1}{2} \times \text{base} \times \text{altitude} = \frac{1}{2} \times vdt \times rsin\phi = \frac{1}{2} r v_\perp dt = \frac{1}{2} h dt \] (2.29)

or

\[ \frac{dA}{dt} = \frac{h}{2} = \text{constant} \] (2.30)
\[ \frac{dA}{dt} \] is known as the areal velocity and is a constant of motion and the equation mathematically endorses the second Kepler’s law which is that equal areas are swept in equal times.

The line \( \theta = 0 \) is known as the apse line and the point of closest approach \( r_p \) termed as the periapsis is obtained by setting \( \theta = 0 \) in the orbit equation,

\[
r_p = \frac{h^2}{\mu} \frac{1}{1+e}
\]  

At this point of the trajectory the radial component of the velocity vector vanishes.

### 2.2.1 Energy Equation

The Newton’s equation of motion given as

\[
\ddot{r} = -\frac{\mu}{r^3} r
\]  

is now used to formulate the energy equation by taking dot product with \( \dot{r} \) giving,

\[
\ddot{r} \cdot \dot{r} = -\frac{\mu}{r^3} r \cdot \dot{r}
\]  

taking first the left-hand side of this equation,

\[
\ddot{r} \cdot \dot{r} = \frac{1}{2} \frac{d}{dt} (\dot{r} \cdot \dot{r}) = \frac{1}{2} \frac{d}{dt} (\mathbf{v} \cdot \mathbf{v}) = \frac{d}{dt} \left( \frac{v^2}{2} \right)
\]  

and for the right-hand side

\[
\frac{\mu}{r^3} r \cdot \dot{r} = \mu \frac{\dot{r} \cdot \dot{r}}{r^3} = \mu \frac{\dot{r}}{r^2} = -\frac{d}{dt} \left( \frac{\mu}{r} \right)
\]  

equating the two sides

\[
\frac{d}{dt} \left( \frac{v^2}{2} - \frac{\mu}{r} \right) = 0
\]  

or

29
\[ \frac{v^2}{2} - \frac{\mu}{r} = \epsilon \quad \text{(constant)} \] (2.37)

with \( v^2/2 \) being the relative kinetic energy per unit mass and \(-\mu/r\) the potential energy per unit mass of the body \( m_2 \) in the gravitational field of \( m_1 \). The above equation represents the principle of conservation of energy conserved along the trajectory. The above equation can also be expressed in terms of the orbital parameters \( \mu, h, e \), in which case the equation becomes

\[ \epsilon = -\frac{1}{2} \frac{\mu^2}{h^2} (1 - e^2) \] (2.38)

Thus the orbital energy is not an independent orbital parameter. Depending on different \( e \) values the different types of trajectories will have different forms of specific energies as discussed in the following subsections.

### 2.2.2 Circular Orbits (\( e = 0 \))

The circular orbits are defined by zero eccentricity and thus the orbital equation takes on the simple form

\[ r = \frac{h^2}{\mu} \] (2.39)

now since \( r \) is constant in these orbits \( \dot{r} = 0 \) and hence \( v = v_\perp \) so that the specific angular momentum is \( h = rv \) and we have

\[ v_{\text{circular}} = \frac{\sqrt{\mu}}{\sqrt{r}} \] (2.40)

and since the velocity is constant the time period of the circular orbit will be

\[ T_{\text{circular}} = \frac{\text{circumference}}{\text{speed}} = \frac{2\pi r}{\frac{\mu}{\sqrt{r}}} = \frac{2\pi}{\frac{\mu}{\sqrt{r}}} \frac{r^2}{2} \] (2.41)

The specific energy \( \epsilon \) of a circular orbit will be

30
The energy of a circular orbit is negative and it increases with the increasing radius. Similarly the velocity of a particle decreases with the increasing radius. For a typical geostationary satellite the radius should be 42,164 km and the velocity of 3.075 km/s.

2.2.3 Elliptical Orbits (0 < e < 1)

The orbital equation which is given by

\[ r = \frac{h^2/\mu}{1+e \cos \theta} \]  

will have the denominator on the r.h.s. always positive and never reaching zero for the eccentricity range 0 < e < 1. Thus orbits with this eccentricity range will be bound and are called elliptical or Keplerian orbits. The orbits of all the planets that revolve around the Sun are elliptic and so are the orbits of all the Earth’s satellites.

The relative position vector for elliptical orbits remains bound and its minimum and maximum values are termed as the periapsis (\( r_p \)) and apoapsis (\( r_a \)) respectively,

Periapsis is given at = 0,

\[ r_p = \frac{h^2}{\mu} \frac{1}{1+e} \]  

and the apoapsis is given at \( \theta = \pi \)

\[ r_a = \frac{h^2}{\mu} \frac{1}{1-e} \]  

The semi-major axis \( a \) is given as
\[ 2a = r_a + r_p \]  

(2.45)  

Substituting the values of \( r_a \) and \( r_p \) gives  

\[ a = \frac{h^2}{\mu} \frac{1}{1-e^2} \]  

(2.46)  

And the orbit equation can thus be rewritten in the form  

\[ r = a \frac{1-e^2}{1+ecos\theta} \]  

(2.47)  

The semi-minor axis \( b \) is  

\[ b = a\sqrt{1-e^2} \]  

(2.48)  

If \( x, y \) are the coordinates of a point on the orbit then the equation of the ellipse is given as  

\[ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \]  

(2.49)  

The specific energy of an elliptic orbit is  

\[ \epsilon = -\frac{1}{2} \frac{\mu^2}{h^2} (1 - e^2) \]  

(2.50)  

And will always be negative and can be rearranged to be given in terms of the semi-major axis  

\[ \epsilon = -\frac{\mu}{2a} \]  

(2.51)  

The conservation of energy equation for an elliptical orbit can be written as  

\[ \frac{v^2}{2} - \frac{\mu}{r} = -\frac{\mu}{2a} \]  

(2.52)  

The area of an ellipse given in terms of \( a, b \) is \( A = \pi ab \). To find the time period of the elliptical orbit the second law of Kepler is employed i.e.
\[ \Delta A = \frac{h}{2} \Delta t \quad (2.53) \]

for a complete revolution \( \Delta A = \pi ab \) and \( \Delta t = T \), thus,

\[ T = \frac{2\pi ab}{h} \quad (2.54) \]

which can be re-expressed as

\[ T = \frac{2\pi}{\sqrt{\mu}} a^{3/2} \quad (2.55) \]

This equation mathematically represents the third Kepler’s law which states that the time period of a planet is proportional to the three-half power of the semi-major axis.

**2.2.4 Parabolic Trajectories \((e = 1)\)**

The orbit equation when \( e = 1 \) is given as

\[ r = \frac{h^2}{\mu} \frac{1}{1 + \cos \theta} \quad (2.56) \]

The specific energy of a parabolic trajectory will be

\[ \varepsilon = -\frac{1}{2} \frac{\mu^2}{h^2} (1 - e^2) = 0 \quad (2.57) \]

Such that the conservation of energy for a parabolic trajectory is given as

\[ \frac{v^2}{2} - \frac{\mu}{r} = 0 \quad (2.58) \]

This depicts that the velocity on a parabolic path at any point is given as

\[ v = \sqrt{\frac{2\mu}{r}} \quad (2.59) \]
If a body is on a parabolic trajectory it will continue moving towards infinity where its velocity becomes zero and it escapes the gravitational field of other body. The escape velocity at a distance \( r \) from \( m_1 \) is therefore given as

\[
    v_{\text{esc}} = \sqrt{\frac{2\mu}{r}}
\]  

(2.60)

The velocity of a circular orbit is given as

\[
    v_o = \sqrt{\frac{\mu}{r}}
\]  

(2.61)

Therefore the escape velocity for a circular orbit is

\[
    v_{\text{esc}} = \sqrt{2} \, v_o
\]  

(2.62)

Or a boost of 41.4 percent is required to escape from a circular orbit.

The flight path angle for a parabolic trajectory is given as

\[
    \tan \gamma = \frac{e \sin \theta}{1 + e \cos \theta} = \frac{\sin \theta}{1 + \cos \theta} = \tan \theta / 2
\]  

(2.63)

or the flight path angle of a parabolic trajectory is equal to one half of the true anomaly.

The semi-latus rectum of a conic section is given as

\[
    p = \frac{h^2}{\mu}
\]  

(2.64)

and the periapsis radius for a parabolic trajectory is simply

\[
    r_p = \frac{p}{2}
\]  

(2.65)

and the equation of a parabola in cartesian coordinates written in terms of the semi-latus rectum is,
The parabolic trajectories are rarely found in nature, however some comets have trajectories that approximate a parabola. The parabolic trajectories are interesting from a space-craft point of view as they represent the border line of closed and open trajectories.

2.2.5 Hyperbolic Trajectories \((e > 1)\)

Meteors that strike the earth and the interplanetary space-probes that leave the earth travel with hyperbolic trajectories relative to the earth. A hyperbolic trajectory is important if we want the escaping body to have some excess speed when it escapes the influence of the other body.

If \(e > 1\), the orbit formula

\[
r = \frac{h^2}{\mu} \frac{1}{1 + e \cos \theta}
\]

(2.67)

describes the geometry of a hyperbola.

The hyperbolic system consists of two symmetric curves, the curve on the left represents the orbiting body and the curve on the right is its mathematical image. In the case of repulsive forces the right curve will represent the motion of the second body with the first body at the focus of the left curve.

The denominator of the above equation reaches zero when \(\cos \theta = -\frac{1}{e}\). This value \(\theta_{\infty}\) is known as the true anomaly of the asymptote as the radial distance at \(\theta_{\infty}\) reaches infinity,

\[
\theta_{\infty} = \cos^{-1}(1/e)
\]

(2.68)

The periapsis \(P\) lies on the apse line where the left hyperbola meets the apse line and the apoapsis \(A\) lies on intersection of the right hyperbola with the apse line. The point halfway between periapsis and apoapsis is called the center of the
The asymptotes of the hyperbola are the straight lines which the curve approaches at infinity, meeting at C, and making an angle $\beta$ with the apse line,

where

$$\beta = \pi - \theta_\infty = \cos^{-1}(1/e) \quad (2.69)$$

The periapsis point is given by the equation

$$r_p = \frac{h^2}{\mu} \frac{1}{1+e} \quad (2.70)$$

and the apoapsis point is at

$$r_a = \frac{h^2}{\mu} \frac{1}{1-e} \quad (2.71)$$

since $e > 1$ then $r_a$ comes out to be negative that signifies that apoapse lies on the right of the focus F.

The semimajor axis $a$ is given by as

$$2a = -r_a - r_p = -\frac{h^2}{\mu} \left( \frac{1}{1-e} + \frac{1}{1+e} \right) \quad (2.73)$$

or,

$$a = \frac{h^2}{\mu} \frac{1}{e^2-1} \quad (2.74)$$

The semiminor axis is the distance from the periapsis P to the asymptote, measured perpendicular to the apse line and is calculated to be

$$b = a \sqrt{e^2 - 1} \quad (2.75)$$
Another feature of a hyperbola is the aiming radius which is the distance between a asymptote and a parallel line to the focus and is given as

\[ \Delta = (r_p + a) \sin \beta = a e \sin \theta_\infty = b \]  

(2.76)

Thus the aiming radius equals the length of the semiminor axis of the hyperbola.

The specific energy of a hyperbolic trajectory can be calculated using the energy equation

\[ \epsilon = \frac{1}{2} \frac{\mu^2}{h^2} (e^2 - 1) \]  

(2.77)

Substituting the value of \( a \) in the above equation gives

\[ \epsilon = \frac{\mu}{2a} \]  

(2.78)

Therefore the energy of the hyperbolic trajectory is always positive and the conservation of energy for a hyperbolic trajectory is

\[ \frac{v^2}{2} - \frac{\mu}{r} = \frac{\mu}{2a} \]  

(2.79)

We can find now the speed at which a body will escape to infinity by letting \( r = \infty \) in the above equation to obtain the hyperbolic excess speed \( v_\infty \),

\[ v_\infty = \sqrt{\frac{\mu}{a}} \]  

(2.80)

Therefore the energy equation can be alternatively written in the form

\[ \frac{v^2}{2} - \frac{\mu}{r} = \frac{v_\infty^2}{2} \]  

(2.81)

Substituting the expression for the escape velocity we can further write the energy equation in the form

\[ v^2 = v_{\text{esc}}^2 + v_\infty^2 \]  

(2.82)
Thus hyperbolic excess speed is the velocity with which a body escapes the gravity of another body. For interplanetary missions this excess velocity is required with which the probe travels farther as it leaves the sphere of influence of the primary planet.

### 2.2.5 Perifocal Frame

This is the most natural frame of reference for the study of orbits. The origin the $xy$ plane of the frame lies in the plane of the orbit, with the $x$ axis pointing in the direction of periapsis and the $y$ axis perpendicular to the apse line lying in the plane of the orbit. The $z$ axis lies perpendicular to the plane of the orbit and coincides with the direction of the angular momentum $h$.

If $\hat{p}, \hat{q}, \hat{w}$ are the respective unit vectors in the $x, y, z$ directions in the perifocal frame then the radius vector can be written as

$$ \mathbf{r} = x\hat{p} + y\hat{q} \quad (2.83) $$

and the $z$ unit vector is

$$ \hat{w} = \frac{h}{h} \quad (2.84) $$

Now $x = r\cos\theta$, $y = r\sin\theta$ and the magnitude of $r$ is given by the orbit equation

$$ r = \left(\frac{h^2}{\mu}\right)[1/(1 + e\cos\theta)] \quad (2.85) $$

and the radius vector can be expressed as

$$ \mathbf{r} = \frac{h^2}{\mu} \frac{1}{1 + e\cos\theta} (\cos\theta \hat{p} + \sin\theta \hat{q}) \quad (2.86) $$

The velocity can be found by taking the time derivative of the radius vector $\mathbf{r}$

$$ \mathbf{v} = \dot{\mathbf{r}} = \dot{x}\hat{p} + \dot{y}\hat{q} \quad (2.87) $$
where $\dot{x}$ and $\dot{y}$ can be found as

$$\dot{x} = \dot{r} \cos \theta - r \dot{\theta} \sin \theta \quad (2.88)$$

$$\dot{y} = \dot{r} \sin \theta + r \dot{\theta} \cos \theta \quad (2.89)$$

giving

$$v = \frac{\mu}{h} \left[ -\sin \theta \hat{p} + (e + \cos \theta) \hat{q} \right] \quad (2.90)$$

This describes the orbital kinematics in the perifocal frame. The next section utilizes the concept of perifocal frame to calculate the Lagrange coefficients which provide a direct method to calculate the radial vector and velocity at later time once the initial values of these two quantities are known.

### 2.2.6 The Lagrange Coefficients

The position and velocity of a body in an orbit at a later time can be calculated from the initial values of these quantities and the coefficients of such a multiplication matrix are known as the Lagrange coefficients. The Lagrange coefficients can be derived in the following manner,

If $(r_0, v_0)$ are the initial set of values at time $t_0$, and $(r, v)$ the values at a later time $t$. Then in the perifocal frame,

$$r_0 = x_0 \hat{p} + y_0 \hat{q} \quad (2.91)$$

$$v_0 = x_0 \hat{p} + y_0 \hat{q} \quad (2.92)$$

and

$$r = x \hat{p} + y \hat{q} \quad (2.93)$$

$$v = \dot{x} \hat{p} + \dot{y} \hat{q} \quad (2.94)$$
The angular momentum \( h \) can be written as

\[
h = r_0 \times v_0 = \begin{vmatrix} \hat{p} & \hat{q} & \hat{w} \\ x_0 & y_0 & 0 \\ \dot{x} & \dot{y} & 0 \end{vmatrix} = \hat{w}(x_0\dot{y} - y_0\dot{x}) = h\hat{w}
\]  \hspace{1cm} (2.95)

Unit vectors \( \hat{p}, \hat{q} \) can be expressed in terms of the \( r_0, v_0, x, y, x_0, y_0 \) and \( h \) as,

\[
\hat{p} = \frac{y_0}{h}r_0 - \frac{y}{h}v_0
\]  \hspace{1cm} (2.96)

\[
\hat{q} = \frac{x_0}{h}r_0 + \frac{x_0}{h}v_0
\]  \hspace{1cm} (2.97)

substituting these values of \( \hat{p}, \hat{q} \) in the expressions of \( r, v \) yields,

\[
r = x\left(\frac{y_0}{h}r_0 - \frac{y}{h}v_0\right) + y\left(\frac{x_0}{h}r_0 + \frac{x_0}{h}v_0\right) = \frac{xy_0 - yx_0}{h}r_0 + \frac{-yx_0 + xy_0}{h}v_0
\]  \hspace{1cm} (2.98)

\[
v = \dot{x}\left(\frac{y_0}{h}r_0 - \frac{y}{h}v_0\right) + y\left(\frac{x_0}{h}r_0 + \frac{x_0}{h}v_0\right) = \frac{\dot{y}y_0 - \dot{y}x_0}{h}r_0 + \frac{-\dot{x}y_0 + \dot{x}x_0}{h}v_0
\]  \hspace{1cm} (2.99)

or

\[
r = fr_0 + gv_0
\]  \hspace{1cm} (2.100)

\[
v = \dot{fr}_0 + \dot{gv}_0
\]  \hspace{1cm} (2.101)

where

\[
f = \frac{xy_0 - yx_0}{h}
\]  \hspace{1cm} (2.102)

\[
g = \frac{-yx_0 + xy_0}{h}
\]  \hspace{1cm} (2.103)
and

\[
\dot{f} = \frac{\dot{x}y_0 - \dot{y}x_0}{h} \quad (2.104)
\]

\[
\dot{g} = \frac{-\dot{x}y_0 + \dot{y}x_0}{h} \quad (2.105)
\]

The functions \( f \) and \( g \) are known as Lagrange coefficients. The Lagrange coefficients and their time derivatives are functions of time and initial conditions. The conservation of total angular momentum imposes a condition on \( f, g, \dot{f} \) and \( \dot{g} \).

\[
f \dot{g} - \dot{f} g = 1 \quad (2.106)
\]

Thus we require any three of the four functions to calculate the position and velocity in an orbit at a later time once the initial values of the position and velocity are given. The Lagrange coefficients can also be expressed in terms of the radius vector and the change in true anomaly,

\[
f = 1 - \frac{\mu r}{h^2} (1 - \cos \Delta \theta) \quad (2.107)
\]

\[
g = \frac{rr_0}{h} \sin \Delta \theta \quad (2.108)
\]

\[
\dot{f} = \frac{\mu 1 - \cos \Delta \theta}{h \sin \Delta \theta} \left[ \frac{\mu}{h^2} (1 - \cos \Delta \theta) - \frac{1}{r_0} - \frac{1}{r} \right] \quad (2.109)
\]

\[
\dot{g} = 1 - \frac{\mu r_0}{h^2} (1 - \cos \Delta \theta) \quad (2.110)
\]

where \( r \) is given as

\[
r = \frac{h^2}{\mu} \frac{1}{1 + \left( \frac{h^2}{\mu r_0} - 1 \right) \cos \Delta \theta - \frac{h \nu_0 \sin \Delta \theta}{\mu}} \quad (2.111)
\]

And the initial radial velocity \( v_{r_0} \) is the projection of \( \mathbf{v}_0 \) onto the direction of \( \mathbf{r}_0 \),

\[
v_{r_0} = \mathbf{v}_0 \cdot \frac{\mathbf{r}_0}{r_0} \quad (2.112)
\]
with

$$r_0 = \frac{h^2}{\mu} \frac{1}{1+e \cos \theta_0}$$  \hspace{1cm} (2.113)$$

and

$$v_{r0} = \frac{\mu}{h} e \sin \theta_0$$  \hspace{1cm} (2.114)$$

This provides a method to calculate the position and velocity vectors at a given time with the knowledge of the eccentricity and the true anomaly of the initial point.

### 2.3 Restricted Three Body Problem

The two body problem can be completely solved analytically whereas the scenario is not similar for a general three body problem where there exist no analytical solutions. However for special cases it is possible to describe the problem analytically. One such case is the problem of two bodies of masses $m_1, m_2$ circling about a common center of mass and a third body of mass $m$, negligible compared to $m_1$ and $m_2$, moving under the combined gravitational influence of $m_1$ and $m_2$. The reference frame is chosen such that the origin lies at the center of mass of the first two bodies with the $x$ axis pointing towards $m_2$ and the $xy$ plane coinciding with the plane of the orbit. In this frame of reference the two bodies $m_1, m_2$ will appear to be at rest, ref Fig 8. Since the mass of the third body is considered to be negligible the problem is referred as restricted three body problem.
The inertial angular momentum that is a constant of motion is given by,

$$\Omega = \Omega \hat{k}$$  \hspace{1cm} (2.115)

where $\Omega = 2\pi/T$ and $T$ is the time period of the circular orbit of the primary bodies, given by the Kepler law,

$$T = 2\pi \frac{r_{12}^{3/2}}{\sqrt{\mu}}$$  \hspace{1cm} (2.116)

If $M$ is the total mass of the primary bodies and $x_1, x_2$ their distances from the origin, and if we denote $\pi_1, \pi_2$ as the mass ratios then

$$m_1x_1 + m_2x_2 = 0$$  \hspace{1cm} (2.117)

and

$$x_1 = -\pi_2 r_{12}$$  \hspace{1cm} (2.118)

$$x_2 = \pi_1 r_{12}$$  \hspace{1cm} (2.119)
where \( \pi_1 = m_1/(m_1 + m_2) \) and \( \pi_2 = m_2/(m_1 + m_2) \).

In the co-moving reference frame, the position of the \( m \) relative to \( m_1, m_2 \) is

\[
\mathbf{r}_1 = (x - x_1)\mathbf{i} + y\mathbf{j} + z\mathbf{k} = (x + \pi_2 r_{12})\mathbf{i} + y\mathbf{j} + z\mathbf{k} \tag{2.120}
\]

and

\[
\mathbf{r}_2 = (x - x_2)\mathbf{i} + y\mathbf{j} + z\mathbf{k} = (x - \pi_1 r_{12})\mathbf{i} + y\mathbf{j} + z\mathbf{k} \tag{2.121}
\]

The position vector of the secondary body in the co-moving reference frame is

\[
\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \tag{2.122}
\]

However, since the co-moving reference frame is rotating with a constant angular velocity the absolute velocity of the secondary mass can be derived as

\[
\dot{\mathbf{r}} = \mathbf{v}_G + \mathbf{\Omega} \times \mathbf{r} + \mathbf{v}_{\text{rel}} \tag{2.123}
\]

Where \( \mathbf{v}_G \) is the inertial velocity of the center of mass of the primary bodies and \( \mathbf{v}_{\text{rel}} \) is the velocity of the secondary body in the moving reference frame,

\[
\mathbf{v}_{\text{rel}} = \dot{x}\mathbf{i} + \dot{y}\mathbf{j} + \dot{z}\mathbf{k} \tag{2.124}
\]

The absolute acceleration of the secondary body will be given as

\[
\ddot{\mathbf{r}} = \mathbf{a}_G + \dot{\mathbf{\Omega}} \times \mathbf{r} + \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}) + 2\mathbf{\Omega} \times \mathbf{v}_{\text{rel}} + \mathbf{a}_{\text{rel}} \tag{2.125}
\]

Since the velocity of the moving frame is constant and so is the angular velocity of the center of mass (\( \mathbf{\Omega} \)) in this frame, therefore \( \mathbf{a}_G = \dot{\mathbf{\Omega}} = 0 \) and

\[
\mathbf{a}_{\text{rel}} = \ddot{x}\mathbf{i} + \ddot{y}\mathbf{j} + \ddot{z}\mathbf{k} \tag{2.126}
\]

Therefore

\[
\ddot{\mathbf{r}} = (\ddot{x} - 2\Omega \dot{y} - \Omega^2 x)\mathbf{i} + (\ddot{y} + 2\Omega \dot{x} - \Omega^2 y)\mathbf{j} + \ddot{z}\mathbf{k} \tag{2.127}
\]

The Newton’s second law for the secondary body gives us
\[ m \ddot{r} = F_1 + F_2 \quad (2.127) \]

where

\[ F_1 = -\frac{\mu_1 m}{r_1^3} r_1 \quad (2.128) \]

and

\[ F_2 = -\frac{\mu_2 m}{r_2^3} r_2 \quad (2.129) \]

Or

\[ \ddot{r} = -\frac{\mu_1}{r_1^3} r_1 - \frac{\mu_2}{r_2^3} r_2 \quad (2.130) \]

Finally substituting the corresponding values,

\[
(\ddot{x} - 2\Omega \dot{y} - \Omega^2 x) \hat{i} + (\ddot{y} + 2\Omega \dot{x} - \Omega^2 y) \hat{j} + \ddot{z} \hat{k} = -\frac{\mu_1}{r_1^3} [(x + \pi_2 r_{12}) \hat{i} + y \hat{j} + z \hat{k}] - \frac{\mu_2}{r_2^3} [(x - \pi_1 r_{12}) \hat{i} + y \hat{j} + z \hat{k}] \]

Equating coefficients,

\[
\ddot{x} - 2\Omega \dot{y} - \Omega^2 x = -\frac{\mu_1}{r_1^3} (x + \pi_2 r_{12}) - \frac{\mu_2}{r_2^3} (x - \pi_1 r_{12})
\]

\[
\ddot{y} + 2\Omega \dot{x} - \Omega^2 y = -\frac{\mu_1}{r_1^3} y - \frac{\mu_2}{r_2^3} y
\]

\[
\ddot{z} = -\frac{\mu_1}{r_1^3} z - \frac{\mu_2}{r_2^3} z \quad (2.132)
\]

These correspond to the analytically derived equation of motions for a restricted three-body problem.