CHAPTER-2

LITERATURE SURVEY

Black, F., & Scholes, M. (1973) [24], developed a parabolic PDE of second order for European price call option under the following assumptions:

i. The stock price monitors generalized wiener process with constant expected rate of return and constant volatility of the stock price.

ii. The short trade of securities with full use of proceeds is allowed

iii. There are no operation costs or taxes. All securities are naturally divisible

iv. There are no dividends through the life of the derivatives

v. There are no riskless arbitrage predictions

vi. Security exchange is continuous

vii. The risk free rate of interest is constant and similar for all maturities

The model obtained is:

\[ \frac{1}{2} \sigma^2 S \frac{\partial^2 f}{\partial S^2} + rS \frac{\partial f}{\partial S} + \frac{\partial f}{\partial t} = rf \]

with boundary conditions \( f(S, t) = \begin{cases} S - K, & S \geq K \\ 0, & S < K \end{cases} \), at \( t = T \)

and \( f(0, t) = 0, \quad 0 \leq t \leq T \)

Solution technique: using the transformation \( f(S, t) = \frac{1}{c} e^{-ax-b^2m} \bar{W}(s_K, m) \),

where \( s_K = \ln \left( \frac{S}{K} \right); m = \frac{\sigma^2}{2} (T - t); a = \frac{r}{\sigma^2} - \frac{1}{2}; b = \frac{r}{\sigma^2} + \frac{1}{2} \)
The Black-Scholes equation has been transformed in to \( \frac{\partial W}{\partial m} = \frac{\partial W^2}{\partial s^2} \) with the boundary condition \( W(s_{KT}, 0) = (e^{bs_{KT}} - e^{as_{KT}}) \)

where \( s_{KT} = \ln\left(\frac{S_T}{K}\right) \) when \( t = T \) and \( S_t = S_T \) and \( m = 0 \).

This is well recognized one dimensional Heat equation. Black-Scholes solved this equation analytically and following Conclusion(s) were drawn.

- The option value increases continuously as \( T, r \) or \( \sigma^2 \) rises. In each case, it approaches a supreme value equal to the stock price.
- Option is more impulsive than the stock.

Brennan, M. J., & Schwartz, E. S. (1976) [26], assumed that on the valuation of options embedded in unit-linked (equity-linked) life insurance products the contract advantage was linked directly to the market value of a reference portfolio-the unit-and the embedded guarantee was almost always a maturity guarantee with some specified absolute amount guaranteed to be paid at maturity. Defined in this way, unit-linked life insurance agreements was priced by some adapted version of Black-Scholes option pricing formula.

The model obtained is

\[
\frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial s^2} + rS \frac{\partial f}{\partial s} + \frac{\partial f}{\partial t} = rf
\]

with the following conditions:

i. At termination \( T = t \)

\[
f(S(t), 0, g_b(t)) = \max[S(t) - g_b(t), 0] \quad \text{at } t = T
\]

ii. At any time at which a influence is deemed to be made to the reference portfolio

\[
f(S(T^-), t - T^-, g_b(t)) = f(S(T^+), t - T^+, g_b(t))
\]

iii. At any time previous to maturity

\[
\lim_{S(t) \to \infty} f_S(S(T), t - T, g_b(t)) = 1
\]
iv. When there are no further contributions to be made to the mentioned portfolio \( f(0, t - T, g_b(t)) = 0 \)

v. At any time previous to the final contribution to the reference portfolio, \( f_T(0, t - T, g_b(t)) = r f(0, t - T, g_b(t)) \)

Solution technique: solved using the finite difference scheme

Conclusion(s):

- The put premium rises with the age of the purchaser at admission essentially less is likely to be the operative term of the policy, and of course this effect is more pronounced for longer-term policies which take the policyholder into the years of high mortality.
- Increased the supposed variance rate from 0.01864 to 0.04 and presented that an increase in the variance rate increases the value of a call option and must therefore decrease the value of a put option.
- Measured the ratio between the amount actually invested in the reference portfolio under the riskless strategy, and the amount deemed to be invested in the reference portfolio, and shown that this ratio at different stages in the contract life assuming different rates of return on funds deemed to be financed in the reference portfolio.

**Brennan, M. J., & Schwartz, E. S. (1977) [27],** developed a second order parabolic type PDE for American put options in the similar lines of Black-Scholes model and used this model to evaluate the pricing of put contracts traded in the New York dealer market. Solution of the PDE represents the value of the put option and the put option values found using Finite difference techniques.

Model: \( \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + r S \frac{\partial f}{\partial S} + \frac{\partial f}{\partial t} = rf \) with the following conditions

i. \( f(S, T) = \max[K_T - S, 0] \)

ii. \( f(S, t) \geq \max[K_t - S, 0] \)
iii. \( f(S,t) \leq K_t \)

iv. \( f(S,t) \geq 0 \)

v. \( \lim_{S \to \infty} f_S(S,t) = 0 \)

vi. \( f(S,T^-) = \max[(K_{T^-}) - S, f(S - D(T), T^+)] \)

Solution technique: solved the above problem using finite difference scheme

Conclusion(s):

- The model scientifically over-value the put contracts comparative to the observed market prices

**Brennan, M. J., & Schwartz, E. S. (1978)** [28], suggested a log transformation of the Black-Scholes PDE to obtain the PDE with constant coefficients which makes it possible to apply the explicit finite difference methods such as Crank-Nicolson method.

Model: \( \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial s^2} + rS \frac{\partial f}{\partial s} + \frac{\partial f}{\partial t} = rf \)

Solution technique: using the transformation \( s = \ln(S) \); \( W(s,t) = f(S,t) \) the model reduced to \( \frac{1}{2} \sigma^2 \frac{\partial^2 W}{\partial s^2} + (r - \frac{\sigma^2}{2}) \frac{\partial W}{\partial s} + \frac{\partial W}{\partial t} = rW \). PDE with constant coefficients which has been solved by implicit-explicit finite difference (Crank-Nicolson method) schemes.

Conclusion(s):

- The implicit finite difference estimate to the log transform of the Black-Scholes PDE is also comparable to approximating the diffusion process by jump process. Jump process is indiscriminate one which allows for the possibility that the stock price will jump to infinity of possible future values.

- The simpler explicit finite difference approximation agrees to a three point jump process while the more complex implicit finite difference
approximation corresponds to a generalized jump process to infinity of possible points.

**Schwartz, E. S. (1997)** [111], offered three models of commodity prices which he named as one-factor, two-factor and three-factor models respectively, and derived the corresponding formulas for pricing futures contract in each model. In the first model he adopts that the logarithm of the spot price of the commodity follows a mean reverting process of the Ornstein-Uhlenbeck type, for the second model, he incorporated a second stochastic factor, the convenience yield in the first model which is mean-reverting and absolutely correlated with the spot price and was further extended the second model by seeing the stochastic interest rates to derive the third model.

One-factor model equation:

\[
\frac{\sigma^2}{2}x^2u_{xx} + k(\mu - \bar{\lambda} - \ln(x))xu_x - u_t = 0
\]

with terminal boundary condition \( u(x, 0) = x \)

The two-factor model equation:

\[
\frac{\sigma_1^2}{2}x^2u_{xx} + \sigma_1\sigma_2\rho_1 xu_{xy} + \frac{\sigma_2^2}{2}u_{yy} + (r - y)xu_x + [k(\alpha - y) - \bar{\lambda}]u_y - u_t = 0
\]

with terminal boundary condition \( u(x, y, 0) = x \)

The three-factor model equation:

\[
\frac{\sigma_1^2}{2}x^2u_{xx} + \frac{\sigma_2^2}{2}u_{yy} + \frac{\sigma_2^2}{2}u_{zz} + \sigma_1\sigma_2\rho_1 xu_{xy} + \sigma_2\sigma_3\rho_2 u_{yz} + \sigma_1\sigma_3\rho_3 xu_{xz} + \\
(z - y)xu_x + k(\hat{\alpha} - y)u_y + a(m^* - z)u_z - u_t = 0
\]

with terminal boundary condition \( u(x, y, z, 0) = x \)

where \( \hat{\alpha} = \alpha - \frac{\bar{\lambda}}{k} \)
The normal Black-Scholes postulation of stock diffusion model with constant volatility was keenly observed by market participants until 1987’s market bang. The crash introduced a new era of market discipline and witnessed to use different volatilities which lead a model through Partial Integro-Differential equation for European option prices which was developed by Andersen, L., & Andersen, J. (2000) [12]

Model: \( \frac{\partial f}{\partial t} + (r - q_d - \lambda_j \cdot \tilde{m}) S \frac{\partial f}{\partial S} + \frac{1}{2} J^2(t,S) \cdot S^2 \frac{\partial^2 f}{\partial S^2} + \lambda_j E[\Delta f] = rf \)

\( E[\Delta f(t,S)] = E[\Delta f(t,J(t)S)] - f(t,S) = \int_0^\infty f(t,Sz) \tilde{\xi}(z;t) dz - f(t,S) \)

with the condition: \( f(S_T, T) = \text{max}(S_T - K, 0) \)

where \( \{J(t)\}_{t \geq 0} \) is the sequence of positive stochastic variables, and \( \tilde{m} = E[J(t) - 1] \)

Solution technique: solved the PIDE using ADI (Alternating Directions Implicit) method.

Conclusion(s):
- They have given the framework for adding Poisson jumps to the standard DVF (Deterministic volatility Function) diffusion models of stock price evolution
- Applied the above PIDE model to the S&P500 market results in a largely constant diffusion volatility overlaid with a substantial jump component

Jensen, B., Jørgensen, P. L., & Grosen, A. (2001) [75], extended the Brennan, M. J., & Schwartz, E. S. [26] model by considering the unit-linked (equity-linked) life insurance products contain a surrender option and with the involvement of any excess return (surplus) generated by the investments—i.e. a bonus option, and obtained the second order PDE
Model: \( \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial s^2} + rS \frac{\partial f}{\partial s} + \frac{\partial f}{\partial t} + \mu \frac{\partial f}{\partial p} = rf \) with the condition \( f_T = P(T) \) for European style and \( f_s \geq P(S) \) for American style for \( 0 \leq S < T \)

Solution technique: solved using the Finite difference explicit method techniques.

Conclusion(s):

- The participating policies can be extremely sensitive to changes in the time to maturity, variations in the spread between the guaranteed interest rate and the market interest rate, and to modifications in the investment policy (volatility).


Model:

\[ \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial s^2} + rS \frac{\partial f}{\partial s} + \frac{\partial f}{\partial t} = rf \quad \text{for } S > \bar{S}(t) \quad \text{and } 0 \leq t < T \quad \text{with the following conditions} \]

\[ f(S, t) = \max(K - S, 0) \quad \text{at } t = T \quad \text{for } S \geq 0 \]

\[ \left. \frac{\partial f}{\partial s} \right|_{S = \bar{S}(t)} = -1 \]

\[ f(S, t) \big|_{S = \bar{S}(t)} = K - \bar{S}(t) \]

\[ \lim_{S \to \infty} f(S, t) = 0 \]

\[ f(S, t) = K - S \quad \text{for } 0 \leq S < \bar{S}(t) \]

\[ f(S, t) \geq \max(K - S, 0) \quad \forall S \geq 0, \text{ and } 0 \leq t < T \quad \text{where } \bar{S}(t) \text{ is called free boundary} \]
Solution technique:

- Solved using front-fixing method. The simple idea of this method is to remove the moving boundary by change of variables, in turns out that this methodology leads to a nonlinear problem defined on a fixed domain. This nonlinear problem has been explained by implicit and upwind explicit difference schemes.

- Penalty method- the basic idea of this technique to add a penalty term to the above problem there by obtained a nonlinear PDE defined on fixed domain. This nonlinear problem has been explained by implicit and upwind explicit difference scheme.

Conclusion(s):

- Computational effectiveness of the schemes differ considerably
- Due to limitations on time steps of upwind explicit scheme, the explicit scheme is much slower than implicit methods.

Oosterlee, C.W. (2003) [100], was replaced the supposed constant volatility with stochastic volatility and achieved a generalization of the Black-Scholes PDE as two dimensional PDE.

Model:

\[
\frac{\partial f}{\partial t} + \frac{1}{2} \left[ S^2 \theta \frac{\partial^2 f}{\partial s^2} + 2 \rho \sigma S \frac{\partial^2 f}{\partial s \partial \sigma} + \sigma^2 \theta \frac{\partial^2 f}{\partial \sigma^2} \right] + r s \frac{\partial f}{\partial s} + \left[ \alpha_1 (\alpha_2 - \theta) - \lambda \gamma \sqrt{\sigma} \right] \frac{\partial f}{\partial \theta} - rf = 0 \text{ with boundary conditions}
\]

\[
f(0, \theta, t) \geq \max(K, 0), \quad \forall \theta \geq 0, t \in [0, T]
\]

\[
f(S, 0, t) \geq \max(K - S, 0), \quad \forall S \geq 0, t \in [0, T]
\]

Solution Technique: solution was achieved with the help of backward difference formula BDF2
Conclusion(s):

- They choose Crank-Nicolson scheme (also called trapezoid rule) discretization, because of its L-stability eccentric and having more advantageous damping properties.
- The time discretization exactness of this implicit scheme is second order.
- With the acceleration technique, fast convergence is attained for an option pricing problem on grids with different grid sizes. The error of the discretization is determined by evaluation with reference solutions.

**Ikonen, S., & Toivanen, J. (2004) [71]**, transformed the generalized Black-Scholes PDE of Oosterlee, C. W. [100], to a linear complementarity problem with initial and boundary conditions.

Solution technique: solved this problem using operator splitting method, in this each time step is divided into two fractional time steps. In the first step a system of linear equations were solved while in the second step the early exercise constraint was prescribed by performing a simple update.

Conclusion(s):

- Studied the accuracy of the operators splitting methods in the numerical experiments and found out that their exactness was similar to the exactness of the PSOR method.
- The splitting does not essentially raise the error.
- The computed prices were in good arrangement with the prices available in the literature.
- The time convergence of the Crank-Nicolson method was somewhat irregular while the time convergence for the L-stable BDF2 and Runge-Kutta methods was sturdier.

**Cont, R., & Voltchkova, E. (2005) [31]**, extended the jump-diffusion model with finite jump intensity given by Andersen, L., & Andersen, J. [12], by considering infinite jump intensity (i.e., singular integral kernels) and developed the following model.
In addition, they suggested an analysis of the convergence of the model which was lacking in Andersen, L., & Andersen, J. [12].

Model:
\[
\frac{\partial f}{\partial \tau} = L^* f \quad \text{on} \ (0, T] \times \mathbb{R}; \quad u(0, s_0) = h(s_0), x \in \mathbb{R}; \quad u(\tau, s_0) = g(\tau, s_0), \ s_0 \in \mathbb{R}
\]
where
\[
L^* f = \frac{\sigma^2}{2} \left[ \frac{\partial^2 f}{\partial s_0^2} - \frac{\partial f}{\partial s_0} \right] + \int_{-\infty}^{+\infty} \nu \left[ f(s_0 + \zeta) - f(s_0) - (e^{\zeta} - 1) \frac{\partial f}{\partial s_0} \right] d\zeta
\]

\[
s_0 = \ln \left( \frac{S}{s_0} \right) \quad \text{and} \quad \tau = T - t, \ S_0 \ initial \ stock \ price
\]

Solution technique: solved using explicit-implicit finite difference scheme

Conclusion(s):
- When the number of time/space steps is amplified. The performance is quite similar to the case of Black-Scholes model
- The performance of the error (for a fixed grid size) as a function of maturity for a smooth one (forward contract) and a non-smooth one (put option).
- A non-smooth initial condition leads to a lack of small T
- Numerical convergence of a double barrier put price as the number N of space steps rises.

Rodrigoa, M. R., & Mamon, R. S. (2006) [108], developed a model for the price of an option on a time dependent dividend-paying equity.

Model:
\[
\frac{\sigma^2}{2} S^2 \frac{\partial^2 f}{\partial s^2} + \frac{\partial f}{\partial s} + [r - D(t)]S \frac{\partial f}{\partial s} - rf = 0
\]

with the conditions \( f(S, T) = (S_T - K)^+ \equiv \max(S_T - K, 0) \)
Solution technique:

Using the transformations \( f(S, t) = h_0(t)\tilde{f}(\tilde{S}, \tilde{t}), \tilde{S} = \phi_0(t)S, \tilde{t} = \psi_0(t) \).

\( \tilde{t} = T \) when \( t = T \), the above model has transformed to PDE with constant coefficients

\[
\frac{1}{2} \sigma^2 \tilde{S}^2 \frac{\partial^2 f}{\partial \tilde{S}^2} + \frac{\partial f}{\partial \tilde{t}} + r_c \tilde{S} \frac{\partial f}{\partial \tilde{S}} = r_c \tilde{f}
\]

with terminal condition \( \tilde{f}(\tilde{S}, \tilde{T}) = \max(\tilde{S}_T - K, 0) \)

It was solved analytical in the similar lines of Black-Scholes model

Conclusion(s):

- Results indicates that the price of a European call option on a non-dividend paying equity is decomposed as a product of three simple terms involving of a Black–Scholes price for the constant-coefficient case in a non-dividend-paying set-up, the ratio of two strike prices, and a modified factor reflecting the parametrised time.
- This offered method can also be applied to other European-type options such as puts

Cen, Z., Le, A., & Xi, L. (2007) [29], were applied hybrid finite difference scheme on a piecewise uniform mesh for a class of Black-Scholes equations governing option pricing which is path-dependent.

Model:

\[
\frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + \frac{\partial f}{\partial t} + [rS - D(t)]\frac{\partial f}{\partial S} = rf \quad \text{for} \ (S,t) \in \Omega \quad \text{with the following conditions}
\]

\[
f(0, t) = g_1(t); \quad f(S_T, t) = g_2(t), \quad t \in [0, T); \quad f(S, T) = g_3(S), \quad S \in \mathbb{R}_0
\]

Solution technique:

- Solved using hybrid finite difference scheme on a piecewise uniform mesh. In spatial discretization a hybrid finite difference scheme linking a
central difference method with an upwind difference method on a piecewise uniform mesh was used.

- For time discretization, they used an implicit difference method on a uniform mesh.

Conclusion(s):

- On applying the discrete maximum principle and barrier function technique they proved that their scheme was second-order convergent in space for the arbitrary volatility and the arbitrary asset price.
- For $K = 1024$ a sufficiently large special value they obtained second-order convergence in space.

Xi, L., Cen, Z., & Chen, J. (2008) [132], presented a numerical method combining the Crank-Nicolson method in the time discretization with a hybrid finite difference scheme on a piecewise uniform mesh in the spatial discretization to solve Black-Scholes PDE.

Model: they considered the model obtained by Cen, Z., Le, A., & Xi, L. [29]

Solution technique: solved by combining the Crank-Nicolson method in the time discretization with a hybrid finite difference scheme on a piecewise uniform mesh in the spatial discretization.

Conclusion(s):

- The difference scheme is steady for the arbitrary volatility and arbitrary asset price.
- They showed that the scheme was second-order convergent with respect to both time and spatial variables
- This difference scheme can handle the degeneracy of the Black-Scholes differential operator at $S = 0$ without truncating the domain


Model:
\[ \frac{1}{2} \sigma_0^2 \left( 1 + \varphi_0 \left[ \exp \left( r(T - t) a_0^2 S^2 \frac{\partial^2 f}{\partial S^2} \right) \right] \right) S^2 \frac{\partial^2 f}{\partial S^2} + r S \frac{\partial f}{\partial S} + \frac{\partial f}{\partial t} = rf, \quad S > 0, \]

\[ t \in [0, T] \]

with following conditions

\[ f(S, t) |_{t=T} = \max(0, S - K), \quad S > 0; \quad f(0, t) = 0; \quad \lim_{S \to \infty} \frac{f(S, t)}{S - K e^{-r(T-t)}} = 1 \]

where \( a_0 = \mu_0 \sqrt{\gamma_0 N_0} \),

\[ \sigma_n^2 = \sigma_0^2 \left( 1 + \varphi_0 \left[ \exp \left( r(T - t) a_0^2 S^2 \frac{\partial^2 f}{\partial S^2} \right) \right] \right). \]

Solution technique:
The above nonlinear problem transformed into another simpler nonlinear parabolic problem with bounded domain.

Transformation:
Using the substitution \( s_n = e^{r(T-t)} S; \quad \tau_n = \frac{\sigma_0^2}{2} (T - t); \quad U = e^{r(T-t)} f \), the above problem reduced to

\[ \frac{\partial U}{\partial \tau_n} - \left[ 1 + \varphi_0 \left( a_0^2 s_n^2 \frac{\partial^2 U}{\partial s_n^2} \right) \right] \frac{\partial^2 U}{\partial s_n^2} = 0; \quad 0 < s_n < \infty, \quad 0 < \tau_n \leq \frac{\sigma_0^2 T}{2} \text{ with initial-boundary conditions} \]

\[ U(0, \tau_n) = 0; \quad \lim_{s_n \to +\infty} U(s_n, \tau_n) = s_n - K; \quad U(s_n, 0) = \max(0, s_n - K) \]

The transformed partial differential equation has been solved using an explicit finite difference scheme.

Conclusion(s):
- The solution of scheme is positive, monotonically increasing in the space index
- The parameter \( a_0 \) has a direct influence in the steadiness condition
- For \( a_0 = 0 \), the model becomes well known Black-Scholes PDE
The numerical scheme was reliable i.e.: the exact hypothetical solution of the partial differential equation approximates well to the exact solution of scheme as the step sizes tends to zero

Jump-diffusion mathematical models lead to partial integro-differential operators that are non-local, owed to the integral part. That their discretization yields full matrices makes various methods computationally too expensive. Andersen, L., & Andersen, J. [12], and Ikonen, S., & Toivanen, J. [71], have considered numerical methods for jump-diffusion mathematical models based on the linear complementarity problem and variational inequality formulations, through finite difference discretization. One of the central objectives of those studies has to advance computational efficiency by using second-order exact discretizations and faster ways to handle the integral operator. Toivanen, J. (2010) [122], derived a numerical method based on the free-boundary formulation for pricing American options in jump-diffusion models with finite jump activity. For easiness, he considered only American put options; similar methods can easily be resultant as well for American call options when the underlying asset paying dividends constantly. His front-tracking method achieves an implicit finite difference discretization on time-dependent non-uniform grids refined near the expiry and free boundary. For interpolations amongst grids and the construction of finite difference stencils, Lagrange interpolation polynomials were used. It gives an easy way to implement fourth-order accurate discretization as well. A non-linear system of equations is solved using Brent’s root-finding method, which is easy to use, robust and efficient at each time step. An improvement of that formulation is that it was easy to develop higher-order methods by tracking the location of the free boundary and then by refining grids sufficiently near the free boundary, where the solution is less steady. Also, suggested second-order and fourth-order perfect discretization with respect to the number of time and space steps. The numerical tests confirmed that these convergence rates are attainable under the Black–Scholes model.
Model:

\[
\frac{\partial f}{\partial t} = -\frac{1}{2} \left( \sigma^2 S^2 \right) \frac{\partial^2 f}{\partial S^2} - (r - \bar{\lambda} \xi_0) S \frac{\partial f}{\partial S} + \left( r + \bar{\lambda} \right) f - \bar{\lambda} \int_{\mathbb{R}^+} f(t, S_{s_j}) f_0(s_j) \, ds_j
\]

\[(t, S) \in [0, T) \times \mathbb{R}^+\]

If \(\bar{\lambda} = 0\), the above model will reduces to the standard Black-Scholes PDE [24]

Solution technique:
Solved based on the free-boundary formulation for pricing American options under jump-diffusion models with finite jump activity. For simplicity, he considered only American put options; similar methods can easily be derived as well for American call options when the underlying asset paying dividends continuously. His front-tracking method performs an implicit finite difference discretization on time-dependent non-uniform grids refined near the expiry and free boundary.

For interpolations between grids and the construction of finite difference stencils, Lagrange interpolation polynomials were used.

Conclusion(s):
- An advantage of this formulation is that it is easy to develop higher-order methods by tracking the location of the free boundary and then by refining grids sufficiently.
- This gives an easy way to implement fourth-order accurate discretization as well.
- At each time step, a non-linear system of equations is solved using Brent’s root-finding method, which is easy to use, robust and efficient near the free boundary, where the solution is less regular.

**Sophocleous, C., & Leach, P. G. L. (2010) [115],** solved the commodity price models analytically

Model: considers the one-factor, two-factor, and three-factor commodity price models given by Schwartz, E. S. [111]
Solution technique: solved these equations using Lie point symmetries and obtained analytical solution

**Tangman, D. Y., & et al. (2011) [119]**, developed a model for pricing fixed-strike arithmetic Asian options under the Black–Scholes model and obtained the following model

Model:

\[
\frac{\partial f}{\partial t} = \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + (r - q_d)S \frac{\partial f}{\partial S} - rf + \left( \frac{S-a_s}{T-t} \right) \frac{\partial f}{\partial a_s} \quad 0 \leq S < \infty, \; 0 \leq a_s < \infty, \; 0 \leq t \leq T
\]

with the following conditions

\[
\bar{f}(S, a_s, 0) = \max[\varphi_0(a_s - K), 0]; \frac{\partial \bar{f}}{\partial t} = -rf \; \text{as} \; S \to 0; \frac{\partial^2 \bar{f}}{\partial S^2} = 0 \; \text{as} \; S \to \infty
\]

where \( \varphi_0 = 1 \) for fixed strike call and \( \varphi_0 = -1 \) for fixed strike put

Solution technique:

- The methodology uses the exponential time integration (ETI) scheme in combination with a dimensional splitting technique. They have chosen to implement this time stepping scheme though ETI can be expensive over very refined meshes.

Conclusion(s):

- Showed that precise Asian option prices can be obtained by using dimensional splitting, which involves a spectral or a central discretization in the asset price, a Hermite interpolation beside the average quantity and a Strang splitting strategy within the ETI framework developed.
- They have designated how to obtain at least second-order convergent solutions for Asian options with multiple features using the Black–Scholes model, the jump-diffusion model and CGMY processes.
Yun, T. (2011) [136], developed a model for price changing of commodities under the assumption that the rising price of commodity immediately affect the price of its relying products without any delay.

Model:

\[
\frac{\partial u}{\partial t} = p_c \frac{\partial^2 u}{\partial x^2}
\]

where; \( t < t_0 \) is an equilibrium state

The initial-boundary conditions are:

(i) \( u_j(0,0^-) = u_{j0} \) at \( t = t_0 = 0 \) and \( u_j = 0 \)

(ii) \( \frac{\partial u_j}{\partial t} = u_j(0,0^+) - u(0,0^-) = \dot{u}_{j0} \) at \( t = t_0 = 0 \) and \( u_j = 0 \)

(iii) \( \frac{\partial u}{\partial x} = \frac{[u(x,0^-) - u(0,0^-)]}{x-0} = u'_{j0} \) at \( t = t_0 = 0 \) and \( u_j = 0 \)

Solution technique: solved the above problem using the substitution \( u(x,t) = A e^{c_1 x + c_2 t} \) and hence obtained the solution is \( u(x,t) = u_{j0} e^{\left(\frac{u'_{j0}}{u_{j0}}\right)x + \left(\frac{u_{j0}}{u'_{j0}}\right)t} \)

Conclusion(s):

- Equivalence to heat diffusion equation, the price changing diffusion equation was obtained via the description of Newton’s second law
- The major dissimilarity between the above equation and heat diffusion equation was that the constant can be measured and is known as a given constant in heat equation, while herein the constant \( p_c \) is tough to be measured and is treated to be an unknown constant and has given by \( p_c = \frac{u_{j0} u_{j0}}{(u'_{j0})^2} \)
- When the price varying declines then the substitution can be replaced by \( u(x,t) = A e^{c_1 x - c_2 t} \) to get the solution.
Fadugba, S., Nwozo, C., & Babalola, T. (2012) [48], they underwent to the comparative study of the convergence of the two numerical methods to the Black-Scholes price of European options.

Model:
Considered the standard Black-Scholes PDE [24]

Conclusion(s):
- Both the numerical methods have its advantages and disadvantages of use: finite difference method converges faster and more accurate, they are fairly robust and good for pricing vanilla option. They can also require sophisticated algorithms for solving large sparse linear systems of equations and are relatively difficult to code.
- Monte Carlo method works effectively for pricing both European and exotic options, it is flexible in handling varying and even high dimensional financial problems, hence in spite of its significant progress, an early exercise is problematic.
- Crank Nicolson method is unconditionally steady, more précised and converges faster than Monte Carlo method when pricing European option.

Nwozo, C. R., & Fadugba, S. E. (2012) [98], they underwent to the comparative study of the convergence of the three methods to the Black-Scholes price of European options.

Model: Considered the standard Black-Scholes PDE [24] and Binomial model
Solution technique: Monte Carlo method and Finite difference method (Crank-Nicolson) and with the results of Binomial model

Conclusion(s):
- Binomial models are good for pricing options with early exercise opportunities and they are relatively easy to implement but can be quite tough to adjust to more complex functions.
• Finite difference methods converge quicker and more accurate; they are fairly tough and good for pricing vanilla options where there are possibilities of early exercise.

• Monte Carlo method works perfectly for pricing European options, approximates every arbitrary exotic options, it is flexible in handling varying and even high dimensional financial problems.

• Crank Nicolson method is unconditionally steady, more perfect and converges faster than Binomial model and Monte Carlo method when pricing European option.

• Monte Carlo method is good for pricing the path dependent options

Kumar, A., Waiko, A., & Chakrabarty, S. P. (2012) [79], developed a model for the pricing of arithmetic average strike Asian call option and obtained the following model:

Model:

\[ \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + rS \frac{\partial f}{\partial S} + \frac{\partial f}{\partial t} + S \frac{\partial f}{\partial t} = rf \]

where \( f(S, I, t) = \text{Asian call option price} \); \( I(t) = \int_0^t S(\zeta) d\zeta \)

The above problem is three dimensional which leads to greater computational expenses. This inspires the reduction of higher dimension problem into lower dimension.

Transformation:

\[ \bar{R}(t) = \frac{1}{S(t)} \int_0^t S(u) du \]

Let \( f(S, I, t) = S \ bar{H}(\bar{R}, t) \) for some function \( \bar{H}(\bar{R}, t) \) by substituting this the above equation reduces to

\[ \frac{1}{2} \sigma^2 \bar{R}^2 \frac{\partial^2 \bar{H}}{\partial \bar{R}^2} + (1 - r\bar{R}) \frac{\partial \bar{H}}{\partial \bar{R}} + \frac{\partial \bar{H}}{\partial t} = 0 \]

with the following conditions:

\[ \bar{H}(\bar{R}(T), T) = \max \left( 1 - \frac{1}{T} \bar{R}(T), 0 \right) \]
\( \bar{H}(\bar{R}, t) = 0 \) for \( \bar{R} \to \infty \)

\[ \frac{\partial H}{\partial t} + \frac{\partial H}{\partial \bar{R}} = 0 \] as \( \bar{R} \to 0 \)

Solution technique:

Crank-Nicolson Implicit Method (CNIM) and Higher Order Compact (HOC) which is fourth order finite difference scheme

Conclusion(s):

The results attained by both the methods were excellent agreement with Monte Carlo results

For very small values of \( \sigma \), the results attained using HOC Scheme are more accurate compared with the Crank-Nicolson Implicit method.

Yun, T. (2012) [137], studied the application of the instant diffusion equation to the calculation of strategy on changing of owning shares or currencies. The strategy of selling share(s) with maximum altering rate of price-ratio and purchasing share(s) with lowest altering rate of price-ratio (SMaPMi) was calculated by instant diffusion equation with multiple sources of stock-price changing.

Model: considered the same model of Yun, T. [136]

where \( u_j(x,t) \) represents the price of commodity ‘x’ at time ‘t’ due to a raising price changing source at \( x_j \); The diffusion with beginning at time \( 0^- \) and end at the time \( 0^+ \) changes an old equilibrium state to a new equilibrium state.

(i) \( u_j(x_j,0^-) = u_{j0} \) at \( t = 0^- \)

(ii) \( \frac{\partial u_j}{\partial t} = \frac{[u_j(x_j,0^+)-u_j(x_j,0^-)]}{\Delta t} = \frac{[u_j(x_j,0^+)-u_j(x_j,0^-)]}{1} = u_{j0} \) at \( t = 0^- \)

(iii) \( \frac{\partial u_j}{\partial x} = \lim_{x \to x_j} \frac{[u_j(x,0^-)-u_j(x_j,0^-)]}{x-x_j} = u_{j0}' \) at \( t = 0^- \)

Solution technique: solved the above problem using the substitution \( u_j(x,t) = A e^{c_{j1}(x-x_j)+c_{j2}t} \) and hence obtained the solution is
$$u_j(x, t) = u_{j0} e^{\left(\frac{u_{j0}}{u_{j0}}(x-x_j)\right) + \left(\frac{u_{j0}}{u_{j0}}\right)t}$$

Conclusion(s):

- SMaPMi is well-matched for short term speculation, if operator is proper.
- Diffusion is a process from the beginning of a breaking of an old stability state to the end of a new stability state due to inertia. The calculation of approach of SMaPMi based on diffusion equation of multiple sources, was suited for time $t \geq 0^+$ (the end of the new equilibrium state) if no new breaking facto acting.
- SMaPMi is also suited for changing of currencies.

**Esekon, J. E. (2013)** [41], studied the hedging of derivatives in illiquid markets and derived a nonlinear Black-Scholes equation given by

$$\frac{\partial f}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} \left(1 + 2 \rho_t S \frac{\partial^2 f}{\partial S^2}\right) + \frac{1}{2} \rho_t (1 - \alpha_b^2) \sigma^2 S^4 \left(\frac{\partial^2 f}{\partial S^2}\right)^3 + r S \frac{\partial f}{\partial S} - rf = 0$$

If $\alpha_b = 1, r > 0$ this corresponds to no slippage and the model moderates to

$$\frac{\partial f}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} \left(1 + 2 \rho_t S \frac{\partial^2 f}{\partial S^2}\right) + r S \frac{\partial f}{\partial S} - rf = 0$$

If $r = 0$ then the model reduces to

$$\frac{\partial f}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} \left(1 + 2 \rho_t S \frac{\partial^2 f}{\partial S^2}\right) = 0$$

If $\rho_t = 0$ then the asset’s price monitors the standard Black-Scholes model with constant volatility.

Solution technique: assumed the solution of a forward wave, a classical solution for the nonlinear Black-Scholes equation was found.

Conclusions:

The solution of this model supports all the suppositions of Black-Scholes model [24], that the option is more volatile than the stock.
Allahviranloo, T., & Behzadi, Sh. (2013) [8], solved the standard Black-Scholes, using non-discretization techniques.

Model: considered the standard Black-Scholes [24] equation


Conclusion(s): Homotopy analysis method was the faster convergent method than the other considered methods.
2.1 CONCLUSION

In the literature it is found that Black-Scholes (BS) equation (used to find option price in securities market) has been solved by discretization techniques such as Finite Difference and Finite Element Methods and non-discretization techniques such as Adomian Decomposition Method, Variational Iteration Method, Homotopy Perturbation Method and Homotopy Analysis Method.

It has also been observed that solution of the commodity price models (used to find future prices for commodity products) have been solved using discretization techniques. In these techniques the calculations become cumbersome and these techniques are quite difficult to handle by market traders. It has been observed that modeling of spot and future commodity price models have been used in the international commodity products. Very few Indian commodity products are covered under these models.

2.2 OBJECTIVE

In this investigation we shall try to solve the nonlinear Black Scholes equation (Esekon [41]) using the analytical methods like (i) First Integral Method, (ii) Tanh-Coth Method and (iii) Sine-Cosine Method. These methods are found to be powerful in solving nonlinear partial differential equations (NPDE).

We shall also study the applicability of approximate solution techniques like (i) Adomian Decomposition Method, (ii) Variational Iteration Method, (iii) Homotopy Perturbation Method and (iv) Homotopy Analysis Method to solve PDE equations occurring in commodity price models. These techniques give results what are in close agreement with the exact solutions. These techniques would be easy to understand and apply at market traders’ level.