Let us consider the nonlinear Black-Scholes equation (Esekon [41]) with

\[ \alpha_b = 1, r > 0 \]

\[ \frac{\partial f}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} \left( 1 + 2 \rho_l S \frac{\partial f}{\partial S} \right) + rS \frac{\partial f}{\partial S} - rf = 0 \tag{4.1} \]

Differentiate the equation (4.1) twice with respect to \( S \), we obtain

\[ \frac{\partial g}{\partial t} + \frac{\sigma^2 S^2}{2} \left( 1 + 4\rho_l S \frac{\partial g}{\partial S} \right) + \frac{2 \rho_l \sigma^2 S^3}{2} \left( \frac{\partial g}{\partial S} \right)^2 + 2\sigma^2 S \left( 1 + 6 \rho_l S \frac{\partial g}{\partial S} \right) + rS \frac{\partial g}{\partial S} \]

\[ + \sigma^2 \left( 1 + 6 \rho_l S \frac{\partial g}{\partial S} \right) g + rg = 0 \tag{4.2} \]

where \( g = \frac{\partial^2 f}{\partial S^2} \)

Using the transformation \( g = \frac{\tilde{g}}{\rho_l S} \) and \( s = \ln(S) \), the equation (4.2) reduces to

\[ \frac{\partial \tilde{g}}{\partial t} + \frac{\sigma^2}{2} \left( 1 + 4 \tilde{g} \right) \frac{\partial^2 \tilde{g}}{\partial s^2} + 2 \sigma^2 \left( \frac{\partial \tilde{g}}{\partial s} \right)^2 + \frac{\sigma^2}{2} \left( 1 + 4 \tilde{g} \right) \frac{\partial \tilde{g}}{\partial s} + r \frac{\partial \tilde{g}}{\partial s} = 0 \tag{4.3} \]

Again using the transformation \( \tilde{g} = \frac{V - 1}{4} \), the equation (4.3) further reduces to

\[ \frac{\partial V}{\partial t} + \frac{\sigma^2}{2} \left( \frac{\partial}{\partial s} \left[ V \frac{\partial V}{\partial s} + \frac{1}{2} V^2 + \frac{2r}{\sigma^2} V \right] \right) = 0 \tag{4.4} \]

Equation (4.4) is a nonlinear PDE with constant coefficients.
4.1 SOLUTION OF NONLINEAR BLACK SCHOLES EQUATION USING FIRST INTEGRAL METHOD

Let us consider the transformation \( \xi = s - ct \) where 'c' is constant. Using (3.25)-(3.28), the equation (4.4) can be written as,

\[
\frac{\sigma^2}{2} XY_\xi + \frac{\sigma^2}{2} Y^2 + (r - c)Y + \frac{\sigma^2}{2} XY = 0
\]

(4.5)

where \( X = V(\xi) \) and \( Y = V_\xi \)

\[
X_\xi = Y
\]

(4.6a)

\[
Y_\xi = \frac{2}{\sigma^2 X} \left[ -\frac{\sigma^2}{2} XY + (r - c)Y - \frac{\sigma^2}{2} Y^2 \right]
\]

(4.6b)

Let us suppose that \( d\xi = X \, d\sigma \), then the equations (4.6a) and (4.6b) becomes

\[
X_{\sigma} = XY
\]

(4.7a)

\[
Y_{\sigma} = -XY - k_\sigma (r - c)Y - Y^2
\]

(4.7b)

where \( k_\sigma = \frac{2}{\sigma^2} \)

Let us assume that \( X = X(\sigma) \) and \( Y = Y(\sigma) \) are non-trivial solutions of equations (4.7a), (4.7b) and

\[
P(X, Y) = \sum_{i=0}^{m} a_i(X)Y^i
\]

is an irreducible polynomial in the complex domain \( \mathbb{C}[X, Y] \) such that

\[
P(X(\sigma), Y(\sigma)) = \sum_{i=0}^{m} a_i(X(\sigma)) (Y(\sigma))^i = 0
\]

(4.8)

where \( a_i(X), (i = 0, 1, ..., m) \) are the polynomials in \( X \) and \( a_m(X) \neq 0 \). Equation (4.8) is called the first integral to equation (4.7a)-(4.7b), due to Division theorem, there exists a polynomial \( h_1(X) + h_2(X)Y \) in the complex domain \( \mathbb{C}[X, Y] \) such that

\[
\frac{\partial P}{\partial \sigma} = \frac{\partial P}{\partial X} \frac{\partial X}{\partial \sigma} + \frac{\partial P}{\partial Y} \frac{\partial Y}{\partial \sigma} = (h_1(X) + h_2(X)Y)\sum_{i=0}^{m} a_i(X(\sigma))(Y(\sigma))^i
\]

(4.9)

Suppose that \( m = 1 \) in the equation (4.8) and compare the coefficients of \( Y^i \) \( (i = 2, 1, 0) \) on both sides of (4.9), we obtain
From (4.10a), we conclude that $a_1(X)$ is constant and $h_2(X) = -1$. For simplicity, let us consider $a_1(X) = 1$.

From (4.10b), it concludes that $\deg(h_1(X)) \leq \deg(a_0(X))$.

Using $a_1(X)$ and $h_2(X)$ values, from (4.10b), it can be written as,

$$Xa'_0 = X + (r - c)k_\sigma + h_1 - a_0 \tag{4.10d}$$

From (4.10d), it can be concluded that $a_0(X)$ is not a polynomial.

Hence, due to lack of polynomial $a_0(X)$, FIM will not be applied to solve the nonlinear Black-Scholes equation.

**4.2 SOLUTION OF NONLINEAR BLACK SCHOLES EQUATION USING Tanh-Coth METHOD**

Let us consider the transformation $\xi = s - ct$ where $'c'$ is constant

Using (3.30)-(3.33), the equation (4.4) can be written as

$$\frac{\sigma^2}{2} V \frac{\partial^2 V}{\partial \xi^2} + \frac{\sigma^2}{2} \left( \frac{\partial V}{\partial \xi} \right)^2 + (r - c) \frac{\partial V}{\partial \xi} + \frac{\sigma^2}{2} V \frac{\partial V}{\partial \xi} = 0 \tag{4.11}$$

Integrating the equation (4.11) on both sides,

$$\frac{\sigma^2}{2} V \frac{\partial V}{\partial \xi} + (r - c)V + \frac{\sigma^2}{4} V^2 = 0 \tag{4.12}$$

From (4.12), by balancing the nonlinear term $\left(V \frac{\partial V}{\partial \xi}\right)$ with the highest order liner term it can be concluded that $M$ will not be a positive integer.

Hence, due to lack of a positive integer value of $M$, the tanh-coth method will not be applied to solve the nonlinear Black-Scholes equation.
4.3 SOLUTION OF NONLINEAR BLACK SCHOLES EQUATION USING Sine-Cosine METHOD

Let us consider $V = \lambda \cos^\beta (\mu_s \xi)$ and from the equation (4.11), we obtain

\[
\lambda \beta \mu_s c \cos^{\beta-1}(\mu_s \xi) \sin(\mu_s \xi) + \lambda^2 \beta^2 \mu_s^2 \sigma^2 \cos^{2\beta-2}(\mu_s \xi) -
\]

\[
\lambda^2 \beta^2 \mu_s^2 \sigma^2 \cos^{2\beta}(\mu_s \xi) - \frac{\lambda^2}{2} \beta \mu_s \sigma^2 \cos^{2\beta-1}(\mu_s \xi) \sin(\mu_s \xi) -
\]

\[
\frac{\lambda^2}{2} \beta \mu_s^2 \sigma^2 \cos^{2\beta-2}(\mu_s \xi) - r\lambda \beta \mu_s \cos^\beta - 1(\mu_s \xi) \sin(\mu_s \xi) = 0
\]

(4.13)

From (4.13), equating the exponents $2\beta - 2$ and $\beta - 1$ yields $2\beta - 2 = \beta - 1$, so that $\beta = 1$

It needs to be noted that, on equating the exponent pairs $\beta - 1 = 2\beta$ we obtain the same value of $\beta = 1$.

Setting the coefficients to zero yields,

\[
\lambda \beta \mu_s c - r\lambda \beta \mu + \lambda^2 \beta^2 \mu_s^2 \sigma^2 - \frac{\lambda^2}{2} \beta \mu_s^2 \sigma^2 = 0
\]  

(4.14a)

\[
\lambda \beta \mu_s c - r\lambda \beta \mu_s - \lambda^2 \beta^2 \mu_s^2 \sigma^2 = 0
\]  

(4.14b)

\[
- \frac{\lambda^2}{2} \beta \mu_s \sigma^2 = 0
\]  

(4.14c)

From (4.14c), we obtain either $\lambda = 0$ or $\mu_s = 0$

In both the cases we obtain the zero solution.

4.4 CONCLUSION

Literature survey shows that nonlinear equations may be solved using Tanh-Coth, Sine-Cosine, and FIM methods. As a part of this study, a particular non-linear Black-Scholes equation (Esekon [41]) is selected to examine its validation using above methods.

The results show that, while balancing of exponents using Tanh-Coth method, the positive integer (M) could not be obtained, hence, Tanh-Coth method is not suitable for its application. Secondly, while using Sine-Cosine method, during
balancing exponents, \( \beta \) happens to be 1, that supports application of method, however, the solution happens to be trivial that challenging efficacy of the process of validation. Similarly, during the application of FIM method, nonexistence of integral polynomials happens to be the distinct short coming of the method.