Chapter 5

Strong Convergence and \( \alpha \)-Stability

5.1 Introduction

Let \((X_n)\) be a sequence of independent random variables (r.v.s), defined over a common probability space \((\Omega, \mathcal{F}, P)\) and let

\[ S_n = \sum_{j=1}^{n} X_j, \quad n \geq 1. \]

For each \( n > 1 \), let

\[ X_{1,n} \leq X_{2,n} \leq \ldots \leq X_{n-r+1,n} \leq \ldots \leq X_{n,n} \]

be the order statistic of \(X_1, X_2, \ldots, X_n\). We denote \(X_{n-r+1,n}\) by \(M_{r,n}\) and call it as the \(r^{th}\) upper extreme. In particular, \(M_{1,n}\) (or \(M_n\)) is called the maxima, \(n \geq 1\). In this chapter, we discuss some Strong Convergence results for \((S_n)\) and \((M_{r,n})\). Li and Tomkins (1991), in their paper, discuss the Complete Convergence and \(\alpha\)-stability of extreme order statistics, satisfying strong laws. We discuss similar conditions for \((S_n)\)
and relate the concept of $\alpha$-stability with the expected number of boundary crossings.

For $(S_n)$ the Kolmogorov’s Strong Law of Large Numbers (S.L.L.N.s) is well known both under the independent, identically distributed (i.i.d.) setup and independent but not identically distributed set up. When $X_n$s are i.i.d. with a common continuous distribution function (d.f.) $F$, then the following S.L.L.N.s for $(M_n)$ can be seen for example, in Galambose (1978).

**Theorem (A)**

Suppose that the common d.f. $F$ has $r(F) = \infty$. Let $a_n$ be a solution of the equation

$$n(1 - F(a_n)) \simeq 1.$$  

Then $\frac{M_n}{a_n} \to 1$ a.s. if and only if

$$\sum_{n=1}^{\infty} (1 - F(k a_n)) < \infty,$$

for any $k > 1$.

**Remark 5.1.1.**

Whenever $\frac{M_n}{a_n} \to 1$ a.s. as $n \to \infty$, then $(X_n)$ is said to obey the strong law in respect of $(M_n)$.

**Remark 5.1.2.**

Whenever

$$\frac{M_n}{a_n} \to 1 \text{ a.s., } \frac{M_{r,n}}{a_n} \to 1 \text{ a.s.}$$

(see Li and Tomkins (1991)).
When \( \frac{M_{r,n}}{a_n} \rightarrow 1 \) a.s., as \( n \rightarrow \infty \), then \( (M_{r,n}) \) is said to be a.s. stable. Li and Tomkins (1991) introduced the \( \alpha \)-stability (\( \alpha \)-completely, relatively stable) in such a setup as follows. For any \( \alpha \geq -1 \), \( \alpha \)-stability holds for \( (M_{r,n}) \) if for any given \( \epsilon > 0 \),

\[
\sum_{n=1}^{\infty} n^\alpha P \left( \left| \frac{M_{r,n}}{a_n} - 1 \right| > \epsilon \right) < \infty.
\]

(5.1.1)

The above concept of \( \alpha \)-stability can be thought of for any sequence which converges to a constant a.s.. We confine to \( \alpha \geq 0 \). When \( \alpha = 0 \), it would mean Complete Convergence / Complete Stability.

**Definition 5.1.1.**

Let \((Z_n)\) be a sequence of r.v.s with \( Z_n \rightarrow 0 \) a.s. as \( n \rightarrow \infty \). Then \((Z_n)\) is said to satisfy \( \alpha \)-stability condition for some \( \alpha \geq 0 \), provided for any given \( \epsilon > 0 \),

\[
\sum_{n=1}^{\infty} n^\alpha P (|Z_n| > \epsilon) < \infty.
\]

(5.1.2)

**Remark 5.1.3.**

Whenever (5.1.2) holds for \( \alpha = 0 \), then Borel-Cantelli (B-C) Lemma implies that

\[
P(|Z_n| > \epsilon \ i.o.) = 0 \text{ or that } Z_n \rightarrow 0 \text{ a.s.}.
\]

It is well known that, for \((Z_n)\) to converge to 0 a.s., it is not necessary that

\[
\sum_{n=1}^{\infty} P(|Z_n| > \epsilon) < \infty.
\]
Consequently, $\sum P(|Z_n| > \epsilon) < \infty$ is stronger than $Z_n \rightarrow 0$ a.s.. In such a case, it is said that $Z_n \rightarrow 0$, completely. In particular, if $(Z_n)$ is a sequence of independent r.v.s, one can see that a.s. convergence and complete convergence are equivalent.

When S.L.L.N.s holds for a sequence $(X_n)$ of i.i.d. r.v.s, Slivka and Severo (1971) studied the r.v. associated with the number of boundary crossings. Such a r.v. can be introduced and studied for $(Z_n)$ converging to 0 a.s., as is done below.

The fact that $Z_n \rightarrow 0$ a.s. implies that, for any given $\epsilon > 0$,

$$P(|Z_n| > \epsilon \text{ i.o.}) = 0.$$ 

Define $\xi_n = 1$ if $|Z_n| > \epsilon$, $= 0$ otherwise, and

$$N = \sum_{n=1}^{\infty} \xi_n.$$ 

Note that $N$ is a r.v. giving the number of times $Z_n$ crosses the boundaries $-\epsilon$ or $\epsilon$. Here $(\xi_n)$ is a sequence of Bernoulli r.v.s. Proceeding as in Slivka and Severo (1971), we have the following lemma, the proof of which is omitted.

**Lemma 5.1.1.** For any $\lambda \geq 1$,

$$EN^\lambda < \infty, \text{ whenever } \sum_{n=1}^{\infty} n^{\lambda-1} P(|Z_n| > \epsilon) < \infty.$$ 

In the next section, we discuss the S.L.L.N.s and $\alpha$-stability associated with the partial sum sequence and in section 5.3, we discuss similar problems for the sequence of extremes.
5.2 Strong Laws and $\alpha$-stability for Partial Sum $S_n$

When $(X_n)$ is a sequence of i.i.d. r.v.s with $EX_n = \mu$, it is well known that

$$
\frac{S_n}{n} \to \mu \text{ a.s.}
$$

Define

$$
Z_n = \frac{S_n}{n} - \mu, \quad n \geq 1.
$$

Slivka and Severo (1971) established that whenever $EX_1^\lambda < \infty$, for any $\lambda \geq 1$, then

$$
\sum_{n=1}^{\infty} n^{\lambda-1} P(|Z_n| > \epsilon) < \infty.
$$

In other words, taking $\alpha = \lambda - 1$, the above statement implies that, for any $\alpha \geq 0$, $\alpha$-stability holds for $(S_n)$ provided

$$
EX_1^{\alpha+1} < \infty,
$$

where $\alpha$ is some non-negative integer. In the sequel, we discuss $\alpha$-stability for a sequence $(X_n)$ of independent sub-Gaussian r.v.s.

**Definition 5.2.1.**

A r.v. $X$ is said to be sub-Gaussian if $EX = 0$ and its moment generating function (m.g.f.) satisfies

$$
Ee^{tX} \leq e^{\frac{t^2 \sigma^2}{2}}, \quad -\infty < t < \infty,
$$

where $\sigma^2 > 0$ is some constant. When the equality holds, then $X$ is $N(0, \sigma^2)$. 
Chow (1966) studied sub-Gaussian r.v.s and established some interesting properties listed below.

(i) If $X$ is sub-Gaussian with parameter $\sigma^2$, then for any $\lambda > 0$,

$$P(X > \lambda) \leq e^{-\frac{\lambda^2}{2\sigma^2}}$$  \hspace{1cm} (5.2.2)

(ii) If $X_1, X_2, ..., X_n$ are independent sub-Gaussian r.v.s with parameters $\sigma_1^2, \sigma_2^2, ..., \sigma_n^2$ and if

$$S_n = \sum_{j=1}^{n} X_j,$$

then $S_n$ is sub-Gaussian with parameter,

$$\sigma^2 = \sum_{j=1}^{n} \sigma_j^2, \hspace{0.5cm} n \geq 1.$$  

Taylor and Chung (1987) established the S.L.L.N.s, $\frac{S_n}{n} \to 0$ a.s., under the condition that, for all $n$ large,

$$\sum_{j=1}^{n} \sigma_j^2 < c n^{2-d},$$

where $c > 0, 0 < d < 2$ are some constants. We now establish $\alpha$-stability.

**Theorem 5.2.1.** Let $(X_n)$ be a sequence of independent sub-Gaussian r.v.s, with the respective sequence $(\sigma_n^2)$ of parameters. If for some $a > 0$ and $0 < d < 2$,

$$\sum_{j=1}^{n} \sigma_j^2 < a n^{2-d}, \hspace{0.5cm} n \geq 1,$$

then $\alpha$-stability holds for $(S_n)$. 

Proof. When for some $c > 0$, $0 < d < 2$,

$$\sum_{j=1}^{n} \sigma_j^2 < c \ n^{2-d}, \ n \geq 1,$$

by Taylor and Chung (1987), we have the strong law,

$$\frac{S_n}{n} \to 0 \ a.s.$$

Also, from (5.2.2), we have for any given $\epsilon > 0$,

$$P(|S_n| > \epsilon n) \leq e^{-\frac{\epsilon^2 n^2}{2 \sigma^2(n)}},$$

where

$$\sigma^2(n) = \sum_{j=1}^{n} \sigma_j^2.$$

From the fact that $\sigma^2(n) < c \ n^{2-d}$, we have

$$P(|S_n| > \epsilon n) \leq e^{-\frac{\epsilon^2 n^d}{2 \epsilon}}.$$

Consequently, for any $\alpha \geq 0$,

$$\sum_{n=1}^{\infty} n^\alpha P(|S_n| > \epsilon n) \leq \sum_{n=1}^{\infty} n^\alpha e^{-\frac{\epsilon^2 n^d}{\epsilon^2}} < \infty,$$

and the proof is complete.

When $(X_n)$ is a sequence of i.i.d. positive stable r.v.s, with parameter $\gamma$, $0 < \gamma < 1$, then it is known that $EX_1$ does not exist. Hence $(X_n)$ does not obey S.L.L.N.s. However, the following strong convergence holds.
Theorem 5.2.2. Let \((X_n)\) be a sequence of i.i.d. positive stable r.v.s with exponent \(\gamma\), \(0 < \gamma < 1\). Then

\[
\lim \left( \frac{S_n}{n^{\gamma}} \right)^{\frac{1}{\log n}} = 1 \text{ a.s.}
\]

Proof. From Pakshirajan and Vasudeva (1977), we have

\[
\limsup \left( \frac{S_n}{n^{\gamma}} \right)^{\frac{1}{\log \log n}} = e^{\frac{1}{\gamma}} \text{ a.s.}
\]

and

\[
\liminf \left( \frac{S_n}{n^{\gamma}} \right)^{\frac{1}{\log \log n}} = 1 \text{ a.s.}
\]

The above result implies that for any given \(\epsilon > 0\),

\[
P \left( S_n > n^{\frac{1}{\gamma}} (\log n)^{-\frac{1+\epsilon}{\gamma}} \ i.o. \right) = 0 \tag{5.2.3}
\]

and

\[
P \left( S_n < n^{\frac{1}{\gamma}} (\log n)^{-\epsilon} \ i.o. \right) = 0 \tag{5.2.4}
\]

In turn (5.2.3) and (5.2.4) imply that

\[
P \left( S_n > n^{\frac{1+\epsilon}{\gamma}} \ i.o. \right) = 0
\]

and

\[
P \left( S_n > n^{\frac{1-\epsilon}{\gamma}} \ i.o. \right) = 0
\]

or

\[
P \left( \left( \frac{S_n}{n^{\frac{1}{\gamma}}} \right)^{\frac{1}{\epsilon}} > e^{\frac{\epsilon}{\gamma}} \ i.o. \right) = 0 \tag{5.2.5}
\]
and

\[ P \left( \left( \frac{S_n}{n^{\frac{1}{7}}} \right)^{\frac{1}{\log n}} > e^{-\frac{i}{7}} \text{ i.o.} \right) = 0 \quad (5.2.6) \]

One can trivially note that (5.2.5) and (5.2.6) together imply that

\[ \lim \left( \frac{S_n}{n^{\frac{1}{7}}} \right)^{\frac{1}{\log n}} = 1 \text{ a.s.} \]

\[ \square \]

**Theorem 5.2.3.** Under the setup of the previous theorem, let

\[ Z_n = \left( n^{-\frac{1}{7}} S_n \right)^{\frac{1}{\log n}} - 1, \quad n \geq 2. \]

Then for no \( \alpha \geq 0 \), \((Z_n)\) is \( \alpha \)-stable.

**Proof.** We first show that

\[ \sum_{n=1}^{\infty} P(|Z_n| > \epsilon) = \infty. \]

We have,

\[ P(|Z_n| > \epsilon) \geq P \left( \left( \frac{S_n}{n^{\frac{1}{7}}} \right)^{\frac{1}{\log n}} > 1 + \epsilon \right) \]

Put \((1 + \epsilon) = e^\delta\), and note that \(\delta > 0\). Then for \(n\) large,

\[ P \left( \left( \frac{S_n}{n^{\frac{1}{7}}} \right)^{\frac{1}{\log n}} > e^\delta \right) = P \left( S_n > n^{\left(\frac{1}{7} + \delta\right)} \right) \]

\[ = P \left( X_1 > n^\delta \right) \geq \frac{c}{n^{\delta \alpha}}, \text{ since } \frac{S_n}{n^{\frac{1}{7}}} \overset{d}{=} X_1. \]
Consequently, for $n$ large,
\[ P(|Z_n| > \epsilon) \geq \frac{c}{n^{\delta \alpha}}. \]

For $\epsilon$ sufficiently small, one can choose $\delta$ such that $\delta \alpha < 1$. Hence

\[ \sum_{n=1}^{\infty} P(|Z_n| > \epsilon) = \infty \]

and complete convergence fails. For $\alpha > 0$,

\[ n^\alpha P(|Z_n| > \epsilon) > P(|Z_n| > \epsilon), \]

implies that $\alpha$-stability fails for any $\alpha > 0$.

## 5.3 Strong Law and $\alpha$-stability for Extremes

Let $(X_n)$ be a sequence of i.i.d. r.v.s with a common continuous d.f. $F$. Suppose that $F(x) < 1$ for all $x \in (-\infty, \infty)$ or $r(F) = \infty$.

Let $M_{j,n}$ be the $j^{th}$ highest among $(X_1, X_2, ..., X_n)$, $n \geq 1$. Then Theorem (A) gives a necessary and sufficient condition for the S.L.L.N.s to hold for $(M_n)$. We give a sufficient condition under which for a fairly large class of distribution S.L.L.N.s holds.

**Theorem 5.3.1.** Suppose that

\[ -\log(1 - F(x)) \simeq x^\gamma L(x), \quad x > 0, \]
where \( \gamma > 0 \) is a constant and \( L \) is a slowly varying (S.V.) function. Then

\[
\frac{M_n}{a_n} \to 1 \text{ a.s.}
\]

**Proof.** From Theorem (A), note that \( a_n \) is a solution of the equation

\[
n(1 - F(a_n)) = 1.
\]

Under the condition of the theorem, we have

\[
1 - F(a_n) \simeq \exp \left\{ -a_n^\gamma L(a_n) \right\}.
\]

Again, by Theorem (A), we note that, for any \( k > 1 \),

\[
\frac{M_n}{a_n} \to 1 \text{ a.s. provided } \sum_{n=1}^{\infty} (1 - F(k a_n)) < \infty.
\]

We have for any \( k > 1 \),

\[
1 - F(k a_n) = \exp \left\{ -k^\gamma a_n^\gamma L(k a_n) \right\} = \exp \left\{ -k^\gamma a_n^\gamma L(a_n) \frac{L(k a_n)}{L(a_n)} \right\}
\]

The condition,

\[
n(1 - F(a_n)) = 1 \implies \exp \left\{ -a_n^\gamma L(a_n) \right\} = \frac{1}{n}
\]

or

\[
a_n^\gamma L(a_n) = \log n.
\]

On substitution, one gets

\[
1 - F(k a_n) \simeq \exp \left\{ -k^\gamma \log n \frac{L(k a_n)}{L(a_n)} \right\}.
\]
Since $L$ is S.V., we have

$$\frac{L(k a_n)}{L(a_n)} \to 1 \text{ as } n \to \infty.$$ 

For $n$ large, one can find a $\delta > 0$ such that,

$$\frac{L(k a_n)}{L(a_n)} > 1 - \delta \text{ and } k'^{(1 - \delta)} = k_1 > 1.$$ 

Consequently, for $n$ large,

$$1 - F(k a_n) \leq e^{k_1 \log n} = \frac{1}{n^{k_1}}, \quad (5.3.1)$$

which implies that for any $k > 1$,

$$\sum_{n=1}^{\infty} (1 - F(k a_n)) < \infty.$$ 

In turn, by Theorem (A),

$$\frac{M_n}{a_n} \to 1 \text{ a.s.}$$ 

Remark 5.3.1.

One may observe that Exponential, Gumbel and Normal are some well known distributions satisfying the conditions of the above theorem.

Remark 5.3.2. Under the conditions of Theorem 5.3.1, $\alpha$-stability need not hold.

Proof. Let $1 - F(x) = e^{-x}, x > 0$. Then one can see that, $a_n = \log n$ and that

$$\frac{M_n}{\log n} \to 1 \text{ a.s.}$$
For any given $\epsilon$ with $0 < \epsilon < 1$, for $n$ large,

\[
P\left( \left| \frac{M_n}{\log n} - 1 \right| > \epsilon \right) \geq P \left( M_n > (1 + \epsilon) \log n \right) \geq \frac{1}{2} n^\epsilon
\]

Consequently,

\[
\sum_{n=1}^{\infty} P \left( \left| \frac{M_n}{\log n} - 1 \right| > \epsilon \right) = \infty.
\]

Also, for $\alpha \geq 0$,

\[
n^\alpha P \left( \left| \frac{M_n}{\log n} - 1 \right| > \epsilon \right) \geq P \left( \left| \frac{M_n}{\log n} - 1 \right| > \epsilon \right) \Rightarrow \sum_{n=1}^{\infty} n^\alpha P \left( \left| \frac{M_n}{\log n} - 1 \right| > \epsilon \right) = \infty.
\]

Hence the $\alpha$-stability fails.

Remark 5.3.3.

For a class of d.f.s satisfying $\alpha$-stability conditions, see Li and Tomkins (1991).

Remark 5.3.4.

Let

\[
1 - F(x) \simeq x^{-\alpha} L(x), \text{ as } x \to \infty,
\]

where $\alpha > 0$ is some constant and $L(.)$ is a S.V. function. In this case $a_n$ becomes $n^{\frac{1}{2}} \ell(x)$, where $\ell(.)$ is another S.V. function.
For any \( k > 0 \),
\[
1 - F(k a_n) = \frac{L(k a_n)}{k^\alpha a_n^{\alpha n}} \geq \frac{n L(a_n)}{k^\alpha a_n^{\alpha n}} \frac{L(k a_n)}{n L(a_n)}.
\]

Note that \( n(1 - F(a_n)) \approx 1 \) implies that \( \frac{n L(a_n)}{a_n^{\alpha n}} \to 1 \) and that \( L(.) \) is S.V. implies that \( \frac{L(k a_n)}{L(a_n)} \to 1 \) as \( n \to \infty \). Hence for \( n \) large,
\[
1 - F(k a_n) \geq \frac{1}{2 k^\alpha n}.
\]

As a result, for any \( k > 1 \),
\[
\sum_{n=1}^{\infty} (1 - F(k a_n)) = \infty
\]
or S.L.L.N.s fails to hold.

We now give a strong limit theorem for \( (M_n) \), when \( 1 - F(x) \) is regularly varying.

**Theorem 5.3.2.** Let \( (X_n) \) be i.i.d. with a common d.f. \( F \) such that \( 1 - F(x) \) is regularly varying and let \( a_n \) be a solution of the equation \( n(1 - F(a_n)) = 1, \ n \geq 1 \).

Then
\[
\lim \left( \frac{M_n}{a_n} \right)^{1/\log n} = 1 \ a.s.
\]

**Proof.** Given that \( r(F) = \infty \), one can easily see that \( M_n > 0 \) a.s.. The theorem is proved by showing that, for any given \( \epsilon > 0 \),
\[
P \left( \left( \frac{M_n}{a_n} \right)^{\frac{1}{\log n}} > \epsilon \ i.o. \right) = 0
\]
or
\[
P (M_n > a_n n^\epsilon \ i.o.) = 0 \quad (5.3.2)
\]
and
\[ P \left( \left( \frac{M_n}{a_n} \right)^{\frac{1}{\log n}} < e^{-\epsilon} \ i.o. \right) = 0 \]
or
\[ P \left( M_n < a_n n^{-\epsilon} \ i.o. \right) = 0 \quad (5.3.3) \]

Let
\[ 1 - F(x) = x^{-\alpha} L(x), \]
where \( \alpha > 0 \) is some constant and \( L(x) \) is S.V. Then recall that \( a_n = n^{\frac{1}{\alpha}} \ell(n) \), where \( \ell(.) \) is S.V. Consequently, \((a_n n^\epsilon)\) is increasing. By Galambose (1978), (5.3.2) follows provided
\[ P(X_n > a_n n^\epsilon \ i.o.) = 0 \quad (5.3.4) \]

We have
\[ P(X_n > a_n n^\epsilon) = 1 - F(a_n n^\epsilon) \simeq \frac{L(a_n n^\epsilon)}{a_n^\alpha n^{\epsilon \alpha}} = \frac{n L(a_n)}{a_n^\alpha} \frac{L(a_n n^\epsilon)}{L(a_n)} \frac{1}{n^{1+\epsilon \alpha}} \quad (5.3.5) \]

Since \( \frac{n L(a_n)}{a_n^\alpha} \rightarrow 1 \) as \( n \rightarrow \infty \), and for any \( \delta > 0 \), \( n^{-\delta} \frac{L(a_n n^\epsilon)}{L(a_n)} \rightarrow 0 \) as \( n \rightarrow \infty \), for \( n \) large, from (5.3.5) one can get for some \( \epsilon_1 > 0 \), such that
\[ P(X_n > a_n n^\epsilon) \leq \frac{c}{n^{1+\epsilon_1}}. \]

Consequently,
\[ \sum_{n=1}^{\infty} P(X_n > a_n n^\epsilon) < \infty \]
and (5.3.4) becomes a consequence of Borel-Cantelli (B-C) Lemma, which in turn establishes (5.3.2).

We now show that (5.3.3) holds. We have

\[ P(M_n \leq a_n n^{-\epsilon}) = F^n(a_n n^{-\epsilon}) = \left(1 - (1 - F(a_n n^{-\epsilon}))\right)^n. \]

For \( n \) large, note that

\[ 1 - F(a_n n^{-\epsilon}) \approx \frac{L(a_n n^{-\epsilon})}{a_n^{\alpha n} n^{-\alpha}} = \frac{n L(a_n)}{a_n^{\alpha}} \frac{L(a_n n^{-\epsilon})}{L(a_n)} \frac{1}{n^{1-\epsilon\alpha}}. \]

Since \( \frac{n L(a_n)}{a_n^{\alpha}} \to 1 \) as \( n \to \infty \) and for any \( \delta > 0 \), \( \frac{n^{-\delta} L(a_n)}{L(a_n n^{-\epsilon})} \to 0 \) as \( n \to \infty \), one can find a \( n_0 \) such that for all \( n \geq n_0 \) and for \( \epsilon_1 \in (0, 1) \), for some \( \epsilon_2 > 0 \),

\[ 1 - F(a_n n^{-\epsilon}) \geq \frac{(1 - \epsilon_1)}{n^{1-\epsilon\alpha+\delta}} = \frac{(1 - \epsilon_1)}{n^{1-\epsilon_2}}. \]

Consequently, for \( n \) large, one can find a \( c > 0 \) such that,

\[ P(M_n \leq a_n n^{-\epsilon}) \leq \left(1 - \frac{(1 - \epsilon_1)}{n^{1-\epsilon_2}}\right)^n \leq e^{-c n^{\epsilon_2}}. \]

We have

\[ \sum_{n=1}^{\infty} P(M_n \leq a_n n^{-\epsilon}) < \infty \]

and (5.3.3) follows from B-C Lemma.

\[ \square \]

**Remark 5.3.5.**

By taking \( 1 - F(x) = \frac{1}{x} \), if \( x \geq 1 \), one gets \( a_n = n \). For \( n \) large,

\[ P(M_n > a_n n^\epsilon) = P(M_n > n^{1+\epsilon}) \geq \frac{c}{n^\epsilon}. \]
For $\epsilon \in (0, 1)$,

$$\sum_{n=1}^{\infty} P(M_n > a_n n^\epsilon) = \infty$$

and for any $\alpha > 0$,

$$\sum_{n=1}^{\infty} n^\alpha P(M_n > a_n n^\epsilon) = \infty,$$

which shows that $\alpha$-stability fails to hold for any $\alpha \geq 0$. 