Chapter 2

Barndorff - Nielson’s Law of the Iterated Logarithm for the Vector of Extremes

2.1 Introduction

Let \( \{X_n\}, n \geq 1 \) be a sequence of independent, identically distributed (i.i.d), random variables (r.v.s), defined over a common probability space \((\Omega, \mathcal{F}, P)\) and let the common distribution function (d.f.) be \( F \). Suppose that \( F \) is continuous. Let the right extremity of \( F \) be denoted as \( r(F) \). On the same space \((\Omega, \mathcal{F}, P)\), define another sequence \( \{U_n\}, n \geq 1 \), of Uniform (0,1) r.v.s. Let \( M_{j,n} \) denote the \( j^{th} \) highest observation among \( (X_1, X_2, ..., X_n) \) and \( M_{j,n}^* \) denote the \( j^{th} \) highest observation among \( (U_1, U_2, ..., U_n) \), \( 1 \leq j \leq n, n \geq 1 \). \( M_{j,n} \) and \( M_{j,n}^* \) are called as \( j^{th} \) upper extreme or \( j^{th} \) upper order statistic.
Barndorff-Nielsen (1961) established law of the iterated logarithm (L.I.L) for \((M_{1,n}^*)\), which gives almost sure (a.s.) lower bound for the growth of \((M_{1,n}^*)\). Kiefer (1971) established L.I.L. for extremes and intermediate order statistics when \(F\) is \(U(0,1)\). The results of Kiefer gives a.s. upper bound for the \(r^{th}\) extreme \((M_{r,n}^*)\). Peter Hall (1979) extended Kiefer’s results for a class of distributions which includes distributions with exponentially fast right tail. Hall considered the class of distribution functions with 

\[-\log(1 - F(x))\]

regularly varying. We call such a class as the class of d.f.s with exponentially fast right tail.

It may be observed that the class of d.f.s considered by Hall (1979) do not include those with regularly varying right tail and finite right extremity. Vasudeva and Savitha (1992) obtained L.I.L for \((M_{1,n})\), when \(F\) has regularly varying right tail.

In this paper, we obtain the a.s. limit set of \(\{M_{1,n}, M_{2,n}\}\), properly normalized, for (i) the class E of d.f.s with the right tail exponentially fast (ii) the class R of d.f.s with the right tail regularly varying and (iii) the class B of d.f.s with finite right extremity.

In section 2.2, we extend the results of Barndorff-Nielsen (1961) to the vector \(\{M_{1,n}^*, M_{2,n}^*\}\) and present them as Lemmas. In the latter sections, we obtain the a.s. limit set of \(\{M_{1,n}, M_{2,n}\}\) for the three classes of d.f.s E, R and B , mentioned above.
From Hall (1979), one can see that the class E includes exponential, Gumbel and Normal distributions. The class R includes Frechet, Pareto and Burr distributions and the class B includes Weibull and Uniform distributions. In the study of the behaviour of extremes of daily temperature or life expectancy, one can model based on d.f.s of class E; for extremes of insurance claims, loss due to calamities etc., one can develop models based on d.f.s in R. The nearness of extremes play a very important role in these studies. The a.s. limit sets and that of the spacings obtained in this paper throw light on these aspects. It may be noted that in the language of Reliability theory, \(-\log(1 - F(x))\) is the Hazard function and \((1 - F(x))\) is the survival function.

2.2 Lemmas

In this section, we confine to the sequence \(\{U_n\}\) of i.i.d. \(U(0,1)\) r.v.s. Recall that \(M_{j,n}^*\) denotes the \(j^{th}\) maxima. We present some lemmas which are useful in obtaining the a.s. limit set of \(\{M_{1,n}^*, M_{2,n}^*\}\).

**Lemma 2.2.1.** For \(n \geq 1\), \(\limsup_{n} \frac{n(1-M_{1,n}^*)}{\log \log n} = 1\) a.s. and \(\liminf_{n} \frac{n(1-M_{1,n}^*)}{\log \log n} = 0\) a.s.

**Proof.** The proof follows from Barndorff-Nielson (1961).

**Lemma 2.2.2.** For \(n \geq 1\), \(\limsup_{n} \frac{n(1-M_{2,n}^*)}{\log \log n} = 1\) a.s. and \(\liminf_{n} \frac{n(1-M_{2,n}^*)}{\log \log n} = 0\) a.s.
**Proof.** From Barndorff-Nielsen (1961), we know that, \(\limsup \frac{n(1-M_{1,n}^*)}{\log \log n} = 1\) a.s. and
\(\liminf \frac{n(1-M_{1,n}^*)}{\log \log n} = 0\) a.s.

The fact that \(M_{2,n}^* \leq M_{1,n}^*\) implies
\[
\frac{n(1-M_{2,n}^*)}{\log \log n} \geq \frac{n(1-M_{1,n}^*)}{\log \log n}.
\]

To show \(\limsup \frac{n(1-M_{2,n}^*)}{\log \log n} = 1\) a.s., equivalently, we show that for any given \(\epsilon > 0\),

\[
P\left(\frac{n(1-M_{2,n}^*)}{\log \log n} > (1+\epsilon) \ i.o.\right) = 0 \tag{2.2.1}
\]

and

\[
P\left(\frac{n(1-M_{2,n}^*)}{\log \log n} < (1-\epsilon) \ i.o.\right) = 1 \tag{2.2.2}
\]

We have,

\[
P\left(\frac{n(1-M_{2,n}^*)}{\log \log n} > (1+\epsilon)\right) = P\left(M_{2,n}^* < 1 - \frac{\log \log n(1+\epsilon)}{n}\right)
\]

Let \(A_n = \left(M_{2,n}^* < 1 - \frac{(1+\epsilon)\log \log n}{n}\right)\) and let \(n_k = \lfloor \beta^k \rfloor, \beta > 1\). Define

\[
B_k = \left(M_{2,n_k}^* < \left(1 - \frac{(1+\epsilon)\log \log n_{k+1}}{n_{k+1}}\right)\right)
\]
It can be seen that \((A_n \ i.o.) \subseteq (B_k \ i.o.)\). Also

\[
P(B_k) = P \left( M_{2,n_k}^* < \left(1 - \frac{(1 + \epsilon) \log \log n_{k+1}}{n_{k+1}}\right) \right)
\]

\[
= F_{n_k} \left( 1 - \frac{(1 + \epsilon) \log \log n_{k+1}}{n_{k+1}} \right)
+ n_k \left( 1 - F \left( 1 - \frac{(1 + \epsilon) \log \log n_{k+1}}{n_{k+1}} \right) \right) F_{n_k-1} \left( 1 - \frac{(1 + \epsilon) \log \log n_{k+1}}{n_{k+1}} \right)
\]

\[
= \left(1 - \frac{(1 + \epsilon) \log \log n_{k+1}}{n_{k+1}}\right)\left(1 - \frac{(1 + \epsilon) \log \log n_{k+1}}{n_{k+1}}\right)^{n_k-1}
+ n_k \left( \frac{(1 + \epsilon) \log \log n_{k+1}}{n_{k+1}} \right) \left(1 - \frac{(1 + \epsilon) \log \log n_{k+1}}{n_{k+1}}\right)^{n_k-1}
\]

\[
\sim e^{-\frac{(1+\epsilon)}{\beta} \log \log \beta^{k+1}} + (1 + \epsilon_1) \log k e^{-\frac{(1+\epsilon)}{\beta} \log \log \beta^{k+1}}
\]

\[
\geq \frac{1}{(k + 1)^{(1+\epsilon)} / \beta} + (1 + \epsilon_1) \log(k + 1)
\]

Since \(\beta > 1\), is arbitrary, choosing it such that \(\frac{(1+\epsilon)}{\beta} > 1 + \frac{\epsilon_1}{2}\), one gets for \(k\) large,

\[
P(B_k) \leq \frac{\log k}{k^{1+\frac{\epsilon_1}{2}}}
\]

Consequently, \(\sum P(B_k) = \infty\). By Borel-Cantelli (B-C) Lemma, \(P(B_k \ i.o.) = 0\). This

\[
\text{inturn implies that } P(A_n \ i.o.) = 0. \text{ This completes the proof of (2.2.1). Now we show that}
\]

\[
P \left( \frac{n(1 - M_{2,n}^*)}{\log \log n} < (1 - \epsilon) \ i.o. \right) = 1
\]

That is

\[
P \left( M_{2,n}^* < 1 - \frac{(1 - \epsilon) \log \log n}{n} \ i.o. \right) = 1
\]
From Lemma (2.2.1), for any given $\epsilon > 0$, we have

$$P\left(M_{1,n}^* < 1 - \frac{(1 - \epsilon) \log \log n}{n} \ i.o.\right) = 1$$

Since $M_{2,n}^* \leq M_{1,n}^*$, (2.2.2) is immediate. Therefore,

$$\lim \sup \frac{n(1 - M_{2,n}^*)}{\log \log n} = 1 \ a.s.$$ 

Now to show

$$\lim \inf \frac{n(1 - M_{2,n}^*)}{\log \log n} = 0 \ a.s.$$ 

We know that $M_{1,n}^*$ and $M_{2,n}^* \in (0, 1)$. Therefore

$$\frac{n(1 - M_{j,n}^*)}{\log \log n} \geq 0,$$

hence

$$\lim \inf \frac{n(1 - M_{j,n}^*)}{\log \log n} \geq 0 \ a.s.$$ 

For any $x > 0$,

$$P(n(1 - M_{1,n}^*) \leq x) = P\left(M_{1,n}^* \geq 1 - \frac{x}{n}\right) = 1 - P\left(M_{1,n}^* \leq 1 - \frac{x}{n}\right) = 1 - \left(1 - \frac{x}{n}\right)^n$$

Hence, $\lim P(n(1 - M_{1,n}^*) \leq x) = 1 - e^{-x}$

Let $Y_1$ be a r.v. with unit exponential d.f. We have shown that $n(1 - M_{1,n}^*) \overset{d}{\to} Y_1$, which implies that

$$\frac{n(1 - M_{1,n}^*)}{\log \log n} \overset{p}{\to} 0$$
Consequently,
\[
\liminf \frac{n(1 - M_{1,n}^*)}{\log \log n} = 0 \text{ a.s.}
\]

Similarly, we have, for any \( x > 0 \),
\[
P(n(1 - M_{2,n}^*) \leq x) = P\left(M_{2,n}^* \geq 1 - \frac{x}{n}\right)
= 1 - P\left(M_{2,n}^* \leq 1 - \frac{x}{n}\right)
= 1 - \left(1 - \frac{x}{n}\right)^n - n \frac{x}{n} \left(1 - \frac{x}{n}\right)^{n-1}
\]
That is
\[
\lim P\left(n(1 - M_{2,n}^*) \leq x\right) = 1 - e^{-x} - xe^{-x}
\]

Let \( Y_2 \) be a r.v. with d.f. \( G(x) = 1 - e^{-x} - xe^{-x}, \, x > 0 \). We have shown that
\[n(1 - M_{2,n}^*) \overset{d}{\to} Y_2,\] which implies that
\[
\frac{n(1 - M_{2,n}^*)}{\log \log n} \overset{p}{\to} 0
\]
Hence
\[
\liminf \frac{n(1 - M_{2,n}^*)}{\log \log n} = 0 \text{ a.s.}
\]
This completes Lemma (2.2.2). \( \square \)

**Lemma 2.2.3.** For \((x, y) \in [0, 1]^2\), with \(0 < x < y < 1\), for any given \( \epsilon > 0 \),
\[
P(\xi_n \in (x - \epsilon, x + \epsilon) \times (y - \epsilon, y + \epsilon) \text{ i.o.}) = 1,
\]
where
\[
\xi_{k,n} = \frac{n(1 - M_{k,n}^*)}{\log \log n} \quad k = 1, 2, \text{ and } \xi_n = (\xi_{1,n}, \xi_{2,n})
\]
Proof. Let the limit set be $S$. From the above results for $[M_{j,n}^*]$, $j = 1, 2$, we note that $S \in [0, 1]^2$. Further $M_{2,n}^* \leq M_{1,n}^* \Rightarrow \xi_{2,n} \geq \xi_{1,n}$. Consequently,

$$S \subset \{(x, y); 0 \leq x \leq y \leq 1\} = L \text{ (say)}$$

We show that $S = L$.

For $0 < a < b < 1$, consider, $P(a - \epsilon \leq \xi_{1,n} \leq a + \epsilon, b - \epsilon \leq \xi_{2,n} \leq b + \epsilon)$. Note that

$$a - \epsilon \leq \xi_{1,n} \leq a + \epsilon \iff a - \epsilon \leq \frac{n(1 - M_{1,n}^*)}{\log \log n} \leq a + \epsilon$$

$$\iff \frac{(a - \epsilon) \log \log n}{n} \leq 1 - M_{1,n}^* \leq \frac{(a + \epsilon) \log \log n}{n}$$

$$\iff -\frac{(a - \epsilon) \log \log n}{n} \geq M_{1,n}^* - 1 \geq -\frac{(a + \epsilon) \log \log n}{n}$$

$$\iff 1 - \frac{(a + \epsilon) \log \log n}{n} \leq M_{1,n}^* \leq 1 - \frac{(a - \epsilon) \log \log n}{n}$$

$$\iff x_{1,n} \leq M_{1,n}^* \leq y_{1,n}$$

where

$$x_{1,n} = 1 - \frac{(a + \epsilon) \log \log n}{n} \quad \text{and} \quad y_{1,n} = 1 - \frac{(a - \epsilon) \log \log n}{n}$$

Similarly,

$$(b - \epsilon) \leq \xi_{2,n} \leq (b + \epsilon) \iff 1 - \frac{(b + \epsilon) \log \log n}{n} \leq M_{2,n}^* \leq 1 - \frac{(b - \epsilon) \log \log n}{n}$$

$$\iff x_{2,n} \leq M_{2,n}^* \leq y_{2,n}$$

where

$$x_{2,n} = 1 - \frac{(b + \epsilon) \log \log n}{n} \quad \text{and} \quad y_{2,n} = 1 - \frac{(b - \epsilon) \log \log n}{n}$$
Choose $\epsilon > 0$, such that $x_{1,n} \geq y_{2,n}$. Then

\[
P \left( M^*_i \in (x_{i,n}, y_{i,n}); i = 1, 2 \right) = n(n-1) \int_{x_{1,n}}^{y_{1,n}} \int_{x_{2,n}}^{y_{2,n}} y^{n-2} dy_2 dy_1
\]

\[
= n(n-1) \left[ \frac{y^{n-1}}{n-1} \right]_{x_{2,n}}^{y_{2,n}} (y_{1,n} - x_{1,n})
\]

\[
= n \left( y_{2,n}^{n-1} - x_{2,n}^{n-1} \right) (y_{1,n} - x_{1,n})
\]

Therefore,

\[
P \left( M^*_i \in (x_{i,n}, y_{i,n}); i = 1, 2 \right) = n \left( y_{2,n}^{n-1} - x_{2,n}^{n-1} \right) (y_{1,n} - x_{1,n}) \quad (2.2.3)
\]

We have

\[
y_{1,n} - x_{1,n} = 1 - \frac{(a - \epsilon) \log \log n}{n} - \left( 1 - \frac{(a + \epsilon) \log \log n}{n} \right)
\]

\[
= 1 - a \frac{\log \log n}{n} + \epsilon \frac{\log \log n}{n} - 1 + a \frac{\log \log n}{n} + \epsilon \frac{\log \log n}{n}
\]

\[
= \frac{2 \epsilon \log \log n}{n}
\]

That is,

\[
y_{1,n} - x_{1,n} = \frac{2 \epsilon \log \log n}{n} \quad (2.2.4)
\]

Also,

\[
y_{2,n}^{n-1} - x_{2,n}^{n-1} = \left( 1 - \frac{(b - \epsilon) \log \log n}{n} \right)^{n-1} - \left( 1 - \frac{(b + \epsilon) \log \log n}{n} \right)^{n-1}
\]

\[
\sim e^{-(b-\epsilon) \log \log n} - e^{-(b+\epsilon) \log \log n}
\]

\[
= \frac{1}{(\log n)^{b-\epsilon}} - \frac{1}{(\log n)^{b+\epsilon}}
\]

\[
\sim \frac{1}{(\log n)^{b-\epsilon}}
\]
That is,
\[ y_{2,n}^{n-1} - x_{2,n}^{n-1} \sim \frac{1}{(\log n)^{b-\epsilon}} \]  \hspace{1cm} (2.2.5)

Using (2.2.4) and (2.2.5) in (2.2.3), we have for some \( C_1 > 0 \),
\[ P\left( M_{i,n}^* \in (x_{i,n}, y_{i,n}); i = 1, 2 \right) \sim \frac{C_1 \log \log n}{(\log n)^{b-\epsilon}} \]  \hspace{1cm} (2.2.6)

We now establish the lemma by appealing to the extended Borel-Cantelli (B-C) Lemma.

Define \( n_k = [\exp k^{1/b}], 0 < b < 1 \), and \( E_{n_k} = (M_{i,n_k}^* \in (x_{i,n_k}, y_{i,n_k}); i = 1, 2) \). We show that \( P(E_{n_k} \ i.o.) = 1 \). From (2.2.6), we have for \( k \) large,
\[ P(E_{n_k}) \geq \frac{C_2 \log k}{(k^{1/b})^{b-\epsilon}} = \frac{C_3 \log k}{k^{1-\epsilon_1}}, \text{ where } \epsilon_1 = \frac{\epsilon}{b} \]

Hence
\[ \sum P(E_{n_k}) = \infty \]

Now consider
\[ \sum_{r=1}^{n-1} \sum_{s=r+1}^{n} P\left( E_{n_r} \cap E_{n_s} \right) \]

We have \( P\left( E_{n_r} \cap E_{n_s} \right) = \)
\[ P\left\{ (x_{1,n_r} \leq M_{1,n_r} \leq y_{1,n_r}, x_{2,n_r} \leq M_{2,n_r} \leq y_{2,n_r}); (x_{1,n_s} \leq M_{1,n_s} \leq y_{1,n_s}, x_{2,n_s} \leq M_{2,n_s} \leq y_{2,n_s}) \right\} \]

where
\[ x_{1,n_r} = \left( 1 - \frac{(a + \epsilon) \log \log n_r}{n_r} \right), \quad y_{1,n_r} = \left( 1 - \frac{(a - \epsilon) \log \log n_r}{n_r} \right), \]
\[ x_{2,n_r} = \left( 1 - \frac{(b + \epsilon) \log \log n_r}{n_r} \right), \quad y_{2,n_r} = \left( 1 - \frac{(b - \epsilon) \log \log n_r}{n_r} \right). \]

Similarly for, \((x_{i,n_s}, y_{i,n_s})\) \(i = 1, 2\).

For \(n_r\) and \(n_s\) sufficiently large,

\[ x_{1,n_s} > y_{1,n_r} \iff \left( 1 - \frac{(a + \epsilon) \log \log n_s}{n_s} \right) > \left( 1 - \frac{(a - \epsilon) \log \log n_r}{n_r} \right) \]
\[ \iff \frac{\log \log n_s (a + \epsilon)}{n_s} < \frac{\log \log n_r (a - \epsilon)}{n_r} \]
\[ \iff \frac{1}{b} \log \frac{s}{e^{s/b}} (a + \epsilon) < \frac{1}{b} \log \frac{r}{e^{r/b}} (a - \epsilon) \]
\[ \iff \frac{\log s}{\log r} e^{s/b} < \frac{a - \epsilon}{a + \epsilon} \]

For \(s = r + 1\)

\[ \frac{\log s}{\log r} e^{r/b} = \frac{\log(r + 1)}{\log r} e^{r/(r+1)} \]
\[ = \frac{\log r \left( 1 + \frac{1}{r} \right)}{\log r} \frac{1}{e^{r/(r+1)}} \]
\[ = \frac{\log r + \log \left( 1 + \frac{1}{r} \right)}{\log r} \frac{1}{e^{r/(r+1)}} \]
\[ = \left( 1 + \frac{\log \left( 1 + \frac{1}{r} \right)}{\log r} \right) \frac{1}{e^{(1/r-1)}} \]
\[ \rightarrow 0 \quad (since \frac{1}{b} > 1) \]

Consequently, for \(r\) large, say \(r \geq r_0\), \(x_{1,n_{r+1}} > y_{1,n_r}\). Since \((n_r)\) is increasing, for all \(s \geq r + 1\), and for all \(r \geq r_0\), \(x_{i,n_s} > y_{1,n_r}\). Define \(M'_{i,n_r,n_s}\) as the \(i^{th}\) highest (maxima) among \((X_{n_r+1}, X_{n_r+2}, ..., X_{n_s})\). Then for \(r \geq r_0\),

\[ P(E_r \cap E_s) = P \left( M_{i,n_r} \in (x_{i,n_r}, y_{i,n_r}), M'_{i,n_r,n_s} \in (x_{i,n_s}, y_{i,n_s}); i = 1, 2 \right) \]
\[ = P (M_{i,n_r} \in (x_{i,n_r}, y_{i,n_r}); i = 1, 2) \quad P(M'_{i,n_r,n_s} \in (x_{i,n_s}, y_{i,n_s}); i = 1, 2) \]
Proceeding as in (2.2.6), one can show that for \( s > r \) and \( r \geq r_0 \), and for \( \delta > 0 \),

\[
P(M'_{i,n,x} \in (x_{i,n,x}, y_{i,n,x}); i = 1, 2) \simeq \frac{C \log \log n_x}{(\log n_x)^{b-\epsilon}} \simeq (1+\delta) P(M_{i,n,x} \in (x_{i,n,x}, y_{i,n,x}); i = 1, 2),
\]

since \( \frac{n_x}{n_s} \to 0 \) as \( r \to \infty \) \((s \geq r + 1)\) or \( \frac{n_s-n_r}{n_s} \to 1 \) as \( r, s \to \infty \).

Consequently, for \( r \geq r_0, s \geq r \), and for some arbitrary \( \delta > 0 \),

\[
P(E_r \cap E_s) \leq (1 + \delta) P(E_r) P(E_s)
\]

In turn,

\[
\sum_{r=1}^{n} \sum_{s=1}^{n} P(E_{nr} \cap E_{ns}) \left( \sum_{r=1}^{n} P(E_{nr}) \right)^2 = \frac{2 \sum_{r=1}^{n-1} \sum_{s=r+1}^{n} P(E_{nr} \cap E_{ns})}{\left( \sum_{r=1}^{n} P(E_{nr}) \right)^2}
\]

\[
= \frac{2 \sum_{r=1}^{r_0} \sum_{s=r+1}^{n} P(E_{nr} \cap E_{ns})}{\left( \sum_{r=1}^{n} P(E_{nr}) \right)^2} + \frac{2(1 + \delta) \sum_{r=r_0+1}^{n-1} \sum_{s=r+1}^{n} P(E_{nr}) P(E_{ns})}{\left( \sum_{r=1}^{n} P(E_{nr}) \right)^2}.
\]

This implies that

\[
\lim \frac{\sum_{r=1}^{n} \sum_{s=1}^{n} P(E_{r} \cap E_{s})}{\left( \sum_{r=1}^{n} P(E_{r}) \right)^2} \leq (1 + \delta) \tag{2.2.7}
\]

By (2.2.6) and (2.2.7), and by the Extended B-C Lemma, one gets

\[
P(E_{nr} \ i.o.) \geq \frac{1}{(1 + \delta)}.
\]

Since \( \delta \) is arbitrary, as \( \delta \to 0 \), one can show that

\[
P(E_{nr} \ i.o) = 1,
\]

which in turn completes the proof of the Lemma.
2.3 Limit Set of \( \{M_{1,n}, M_{2,n}\} \) when \( F \in E \)

Define \( U(x) = -\log(1 - F(x)) \) and let \( V(.) \) denote the inverse of \( U(.) \). Hall (1979) assumed that \( V(.) \) satisfies

\[
\lim_{y \to \infty} \frac{V(y(1 + a(y))) - V(y)}{a(y)V(y)} = \frac{1}{\gamma},
\]

where \( \gamma > 0 \) is some constant and \( a(.) \) is a real valued function with \( a(y) \to 0 \) as \( y \to \infty \). Hall observed that \( U(x) = x^\gamma L(x) \), where \( L \) is a slowly varying function satisfying (2.3.1). We now establish the a.s. limit set of \( \{M_{1,n}, M_{2,n}\} \), properly normalized, when \( F \in E \). We denote \( \log \log \log n \) by \( \log_3 n \).

**Theorem 2.3.1.** Let \( F \in E \) with \( U(x) = x^\gamma L(x), \gamma > 0 \), and let

\[
\eta_{j,n} = \gamma \log n \left\{ \frac{M_{j,n}}{V(\log n)} - 1 \right\} + \log_3 n.
\]

Then \( \limsup \eta_{j,n} = \infty \) a.s. and \( \liminf \eta_{j,n} = 0 \) a.s., \( j = 1, 2 \).

**Proof.** Given that \( X_n \) has d.f. \( F \), define \( U_n = F(X_n), n \geq 1 \), and observe that \( (U_n) \) is distributed as i.i.d. \( \text{Uniform}(0, 1) \) r.v.s. Also, observe that \( M_{j,n}^* = F(M_{j,n}) \), \( 1 \leq j \leq n \), where \( M_{j,n}^* \) is the \( j^{th} \) highest among \( (U_1, U_2, ..., U_n), n \geq 1 \). From Lemmas 2.2.1 and 2.2.2, we have for \( j = 1, 2 \),

\[
\limsup \frac{n(1 - M_{j,n}^*)}{\log \log n} = 1 \text{ a.s.}
\]

and

\[
\liminf \frac{n(1 - M_{j,n}^*)}{\log \log n} = 0 \text{ a.s.}
\]
Let $x \in (0, 1)$ and let $\epsilon > 0$ be small such that $x - \epsilon > 0$ and $x + \epsilon < 1$. Then for $j = 1, 2$,

$$(x - \epsilon) \frac{n(1 - M_{j,n}^*)}{\log \log n} < (x + \epsilon) \iff \frac{(x - \epsilon) \log \log n}{n} < 1 - F(M_{j,n}) < \frac{(x + \epsilon) \log \log n}{n}$$

$$\iff \log n - \log_3 n - \log (x + \epsilon) < U(M_{j,n}) < \log n - \log_3 n - \log (x - \epsilon)$$

$$\iff V (\log n(1 + a_{1,n})) < M_{j,n} < V (\log n(1 + a_{2,n}))$$

where

$$a_{1,n} = \frac{-\log_3 n - \log (x + \epsilon)}{\log n} \quad \text{and} \quad a_{2,n} = \frac{-\log_3 n - \log (x - \epsilon)}{\log n}$$

Consequently,

$$P\left( (x - \epsilon) < \frac{n(1 - M_{j,n}^*)}{\log \log n} < (x + \epsilon) \text{ i.o.} \right)$$

$$= P\left( V(\log n(1 + a_{1,n})) - V(\log n) < M_{j,n} - V(\log n) < V(\log n(1 + a_{2,n})) - V(\log n) \text{ i.o.} \right)$$

$$= P\left( a_{1,n}V(\log n) < \gamma (M_{j,n} - V(\log n)) < a_{2,n}V(\log n) \text{ i.o.} \right)$$

$$= P\left( \log \frac{1}{x + \epsilon} < \gamma \log n \left( \frac{M_{j,n}}{V(\log n)} - 1 \right) + \log_3 n < \log \frac{1}{x - \epsilon} \text{ i.o.} \right)$$

Therefore,

$$P\left( (x - \epsilon) < \frac{n(1 - M_{j,n}^*)}{\log \log n} < (x + \epsilon) \text{ i.o.} \right)$$
\[ P \left( \log \frac{1}{x + \epsilon} < \gamma \log n \left( \frac{M_{j,n}}{V(\log n)} - 1 \right) + \log_3 n < \log \frac{1}{x - \epsilon} \text{ i.o.} \right) \quad (2.3.2) \]

Suppose that \( x \) is close to 0. Let \( x = \epsilon \). Then, we get by Lemmas 2.2.1 and 2.2.2 for \( j = 1, 2, \)

\[ 1 = P \left( \frac{n(1 - M_{j,n}^*)}{\log \log n} < 2 \epsilon \text{ i.o.} \right) = P \left( \gamma \log n \left( \frac{M_{j,n}}{V(\log n)} - 1 \right) + \log_3 n > \log \frac{1}{2\epsilon} \text{ i.o.} \right) \quad (2.3.3) \]

For \( \epsilon \) small, observe that \( \log \frac{1}{2\epsilon} \) is arbitrarily large and denote it by \( M \). From (2.3.3),

we have

\[ P \left( \gamma \log n \left( \frac{M_{j,n}}{V(\log n)} - 1 \right) + \log_3 n > M \text{ i.o.} \right) = 1 \quad (2.3.4) \]

Similarly, suppose that \( x \) is close to 1. Then for \( j = 1, 2, \)

\[ 1 = P \left( \frac{n(1 - M_{j,n}^*)}{\log \log n} > (1 - \epsilon) \text{ i.o.} \right) = P \left( \gamma \log n \left( \frac{M_{j,n}}{V(\log n)} - 1 \right) + \log_3 n < \log \frac{1}{(1 - \epsilon)} \text{ i.o.} \right) \]

Put \( \log \frac{1}{(1 - \epsilon)} = \delta \) and note that \( \delta > 0 \) and small whenever, \( \epsilon > 0 \) but small. Hence

\[ P \left( \gamma \log n \left( \frac{M_{j,n}}{V(\log n)} - 1 \right) + \log_3 n < \delta \text{ i.o.} \right) = 1 \quad (2.3.5) \]

From (2.3.4) and (2.3.5), we conclude that for \( j = 1, 2, \)

\[ \limsup \eta_{j,n} = \infty \text{ a.s. and } \liminf \eta_{j,n} = 0 \text{ a.s.} \]

and the proof of the theorem is complete. \( \square \)

In the next theorem, we give the a.s. limit set of \( \{M_{1,n}, M_{2,n}\} \), properly normalized.
Theorem 2.3.2. Let \( \eta_n = (\eta_{1,n}, \eta_{2,n}), n \geq e^2 \). The set of all a.s. limit points of \( (\eta_n) \) is given by

\[
L = \{(x, y), 0 < y < x < \infty\}
\]

Proof. Define \( U_n = F(X_n), n \geq 1, \) and observe that \( (U_n) \) is a sequence of i.i.d. \( \text{Uniform}(0, 1) \) r.v.s. Also observe that \( M_{j,n}^* = F(M_{j,n}), j = 1, 2. \) From Theorem 2.3.1, and from the fact that \( \eta_{1,n} \geq \eta_{2,n} \), it is seen that the set of a.s. limit points of \( (\eta_n) \) is a subset of \( L \). We establish that the limit set is \( L \) itself. Recall from Lemma 2.2.3 that the set of a.s. limit points of \( (\xi_n) \) is

\[
\{(a, b); 0 \leq a \leq b \leq 1\}.
\]

Consider a point \((a, b)\), with \( 0 < a < b < 1 \) and choose \( \epsilon > 0 \), but small, such that \( a - \epsilon > 0 \) and \( b + \epsilon < 1 \). By Lemma 2.2.3, note that

\[
\begin{align*}
P(\xi_n \in (a - \epsilon, a + \epsilon) \times (b - \epsilon, b + \epsilon) \ i.o.) &= 1 \quad (2.3.6)
\end{align*}
\]

Proceeding as in Theorem 2.3.1, one can show that when (2.3.6) holds,

\[
P(\xi_n \in (a - \epsilon, a + \epsilon) \times (b - \epsilon, b + \epsilon) \ i.o.)
\]

\[
= P\left(\eta_n \in \left(\log \frac{1}{a + \epsilon}, \log \frac{1}{a - \epsilon}\right) \times \left(\log \frac{1}{b + \epsilon}, \log \frac{1}{b - \epsilon}\right) \ i.o.\right) = 1 \quad (2.3.7)
\]

Put \( x = \log \frac{1}{a} \) and \( y = \log \frac{1}{b} \). Observe that \( 0 < a < b < 1 \) implies that \( 0 < y < x < \infty \).

We claim from (2.3.7), that \( (x, y) \) is a limit point of \( (\eta_n) \). Note that

\[
\begin{align*}
\log \frac{1}{a + \epsilon} &= x - \delta_1, \quad \log \frac{1}{a - \epsilon} = x + \delta_2, \quad \log \frac{1}{b - \epsilon} = y - \delta_3, \text{ and } \log \frac{1}{b + \epsilon} = y + \delta_4,
\end{align*}
\]
where $\delta_1, \delta_2, \delta_3$ and $\delta_4$, are positive constants. Define

$$\delta = \min_{1 \leq i \leq 4} \delta_i.$$ 

Then (2.3.7) implies that

$$P(\eta_n \in (x - \delta, x + \delta) \times (y - \delta, y + \delta) \text{i.o.}) = 1$$

or that $(x, y)$ is a limit point of $(\eta_n)$ a.s.. Points along the boundary can be trivially claimed to be limit points, since all the interior points are a.s. limit points. Consequently, the set of a.s. limit points of $(\eta_n)$ coincides with $L$. $\square$

**Remark 2.3.1.** We give here examples of some well known d.f.s $F \in E$ and give the form of $(\eta_n)$.

**Example 1:** $F$ is Unit Exponential or Gumbel. In this case $\gamma = 1$ and $V(x) = x$. Hence

$$\eta_{j,n} = \log n \left( \frac{M_{j,n}}{\log n} - 1 \right) + \log_3 n = M_{j,n} - \log n + \log_3 n, \ j = 1, 2$$

and $\eta_n = (\eta_{1,n}, \eta_{2,n})$

**Example 2:** $F$ is Standard Normal. In this case $\gamma = 2$ and $V(x) = \sqrt{2x}$. Hence

$$\eta_{j,n} = 2\log n \left( \frac{M_{j,n}}{2\log n} - 1 \right) + \log_3 n = \sqrt{2\log n} \ M_{j,n} - 2 \log n + \log_3 n, \ j = 1, 2$$

and $\eta_n = (\eta_{1,n}, \eta_{2,n})$
Remark 2.3.2. Let \((X_n)\) be a sequence of i.i.d. r.v.s, with a common d.f. \(F\) and with \(r(F) = \infty\). Let \(a_n\) be a solution of the equation \(n(1 - F(a_n)) \approx 1\). Then we know that, \(\frac{M_{j,n}}{a_n} \to 1\) a.s. if and only if \(\sum(1 - F(ka_n)) < \infty\), for any \(k > 1\). This result is known as Strong Law of Large Numbers (S.L.L.N.s) for the extremes.

If \(F\) is Unit Exponential or Gumbel, then one can see that \(a_n = \log n\) and that \(\frac{M_{j,n}}{\log n} \to 1\) a.s.. For any given \(\epsilon > 0\), \(P\left(\left|\frac{M_{j,n}}{\log n} - 1\right| > \epsilon \text{ i.o.}\right) = 0\), which gives the lower sequence for \((M_{j,n})\) as \(((1 - \epsilon) \log n)\). From Example 1, we notice that an a.s. lower sequence for \((M_{j,n})\) is \((\log n - (\log_3 n + \epsilon))\), which is sharper.

Similarly, if \(F\) is a Standard Normal d.f., then \(a_n = \sqrt{2 \log n}\) and \(\frac{M_{j,n}}{\sqrt{2 \log n}} \to 1\) a.s.. Hence a lower sequence obtained from S.L.L.N.s is \(((1 - \epsilon) \sqrt{2 \log n})\). From Example 2, an a.s. lower sequence for \((M_{j,n})\) is \((\sqrt{2 \log n} - (\log_3 n + \epsilon))\)

2.4 Limit Set of \(\{M_{1,n}, M_{2,n}\}\) when \(F \in R\)

Let \(\{X_n\}\) be a sequence of i.i.d. r.v.s, with a common d.f. \(F\), such that \(1 - F(x) = x^{-\gamma}L(x)\), as \(x \to \infty\), where \(\gamma > 0\) is some constant and \(L(.)\) is a S.V. function. Then \((M_{1,n})\), properly normalized, converges in distribution to a Frechet law with index \(\gamma\). Define \(U_1(x) = 1 - F(x), x > 0\). Let \(V_1\) be its inverse. Then we know that \(V_1(x) = x^{-\frac{1}{\gamma}} \ell(\frac{1}{x})\), as \(x \to \infty\), where \(\ell(.)\) is S.V.. Let \(B_n\) be a solution of the equation
Then it is well known that $B_n = n^{-\frac{1}{2}} \ell(n) = V_1(\frac{1}{n})$. We have the following theorems.

**Theorem 2.4.1.** Let

$$
\tau_{j,n} = \frac{M_{j,n}}{B_{\lfloor n/\log \log n \rfloor}}, j = 1, 2.
$$

Then

$$
\liminf \tau_{j,n} = 1 \text{ a.s. and } \limsup \tau_{j,n} = \infty \text{ a.s., } j = 1, 2
$$

Proof. Define $U_n = F(X_n), n \geq 1$, and note that $U_n$ is Uniform over $(0, 1)$ and $M_{j,n}^* = F(M_{j,n})$. Let $x \in (0, 1)$ and let $\epsilon > 0$ be such that $x - \epsilon > 0$ and $x + \epsilon < 1$.

Recall that

$$
\xi_{j,n} = \frac{n(1 - M_{j,n}^*)}{\log \log n}, j = 1, 2.
$$

Proceeding as in Theorem (2.3.1), we have for $j = 1, 2$

$$(x - \epsilon) < \xi_{j,n} < (x + \epsilon)$$

$$
\iff \frac{(x - \epsilon) \log \log n}{n} < 1 - F(M_{j,n}) < \frac{(x + \epsilon) \log \log n}{n}
$$

$$
\iff V_1 \left( \frac{(x + \epsilon) \log \log n}{n} \right) < M_{j,n} < V_1 \left( \frac{(x - \epsilon) \log \log n}{n} \right)
$$

(2.4.1)
We have,

\[
V_1 \left( \frac{(x - \epsilon) \log \log n}{n} \right) = \frac{n^{1/\gamma}}{(x - \epsilon)^{1/\gamma} (\log \log n)^{1/\gamma}} \ell \left( \frac{n}{(x - \epsilon) \log \log n} \right)
\]

\[
\sim \frac{1}{(x - \epsilon)^{1/\gamma} (\log \log n)^{1/\gamma}} \ell \left( \frac{n}{\log \log n} \right)
\]

\[
\simeq \frac{1}{(x - \epsilon)^{1/\gamma}} B_{[n/\log \log n]}
\]

Similarly,

\[
V_1 \left( \frac{(x + \epsilon) \log \log n}{n} \right) \simeq \frac{1}{(x + \epsilon)^{1/\gamma}} B_{[n/\log \log n]}
\]

Consequently, from (2.4.1), one gets for \( j = 1, 2 \),

\[
P((x - \epsilon) < \xi_{j,n} < (x + \epsilon) \text{ i.o.}) = P \left( \frac{1}{(x + \epsilon)^{1/\gamma}} < \frac{M_{j,n}}{B_{[n/\log \log n]}} < \frac{1}{(x - \epsilon)^{1/\gamma}} \text{ i.o.} \right)
\]

Considering \( x \) close to 0 and \( x \) close to 1 respectively, and arguing as in Theorem (2.3.1), one can show that for \( j = 1, 2 \),

\[
\liminf \tau_{j,n} = 1 \text{ a.s. and } \limsup \tau_{j,n} = \infty \text{ a.s.}
\]

\[\square\]

**Theorem 2.4.2.** Let \( \tau_n = (\tau_{1,n}, \tau_{2,n}), n \geq e \). Then the set of all a.s. limit points of \( \tau_n \) is given by

\[
L_1 = \{(x, y); 1 \leq y \leq x < \infty \}
\]

**Proof.** Define \( U_n = F(X_n), n \geq 1 \), and \( M_{j,n}^* = F(M_{j,n}), j = 1, 2 \), as in the previous theorem. From Theorem 2.4.1, and from the fact that \( M_{1,n} \geq M_{2,n} \), the set of a.s.
limit points of \((\tau_n)\) can be seen to be contained in \(L_1\). We now show that it coincides with \(L_1\). As in the proof of Theorem 2.3.2, consider a point \((a, b)\) with \(0 < a < b < 1\) and choose \(\epsilon\) such that \(a - \epsilon > 0\) and \(b + \epsilon < 1\). Proceeding on lines of arguments similar to those in Theorem 2.3.2, one can see that for some \(\delta > 0\),

\[
1 = P(\xi_n \in (a - \epsilon, a + \epsilon) \times (b - \epsilon, b + \epsilon) \ i.o.)
\]

\[
= P(\tau_n \in (x - \delta, x + \delta) \times (y - \delta, y + \delta) \ i.o.),
\]

where \(x = \frac{1}{a}\) and \(y = \frac{1}{b}\). Consequently, any point \((x, y)\) \(\in L_1\) with \(1 < y < x < \infty\) turns out to be an a.s. limit point. Points on the boundary turn out to be limit points trivially. \(\square\)

**Remark 2.4.1.** Let \(F\) be Pareto with \(1 - F(x) = \frac{1}{x^\gamma}\) if \(x \geq 1\), \(= 0\) if \(x < 1\) or let \(F\) be Fréchet with d.f. \(F(x) = e^{-x^{-\gamma}}\), if \(x > 0\), \(= 0\) if \(x \leq 0\). In both the cases \(B_n = n^{1/\gamma}\) and hence

\[
\tau_n = \left\{ \frac{M_{1,n} \text{(loglogn)}}{n^{1/\gamma}}, \frac{M_{2,n} \text{(loglogn)}}{n^{1/\gamma}} \right\}
\]

**Remark 2.4.2.** When \(1 - F(x)\) is regularly varying, S.L.L.N.s fails to hold for \((M_{j,n})\). However, when \(F\) is either Pareto or Fréchet,

\[
M_{j,n} \geq \frac{(1 - \epsilon)n^{1/\gamma}}{(\log \log n)^{1/\gamma}} \ a.s.
\]

which gives an a.s. lower sequence \(\left(\frac{(1 - \epsilon)n^{1/\gamma}}{(\log \log n)^{1/\gamma}}\right)\) for \((M_{j,n})\), \(j = 1, 2\).
2.5 Limit Set when $F$ has $r(F) < \infty$ or $F \in B$

Let $\{X_n\}$ be sequence of i.i.d. r.v.s with a common continuous d.f. $F$ having $r(F) < \infty$. Assume that $F$ belongs to the max-domain of attraction of a Weibull law with index $\gamma$. Define

$$Y_n = \frac{1}{r(F) - X_n}, n \geq 1.$$  

Let $M_{j,n}$ denote the $j^{th}$ highest among $(X_1, X_2, ..., X_n)$ and $\widetilde{M}_{j,n}$ be the $j^{th}$ highest among $(Y_1, Y_2, ..., Y_n)$, $n \geq 1, j = 1, 2$. Then it is well known that (see, Galambos (1978)), $(\widetilde{M}_{1,n})$, properly normalized, converges to a Frechet law with index $\gamma$. Further,

$$\widetilde{M}_{j,n} = \frac{1}{r(F) - M_{j,n}}, j = 1, 2.$$  

We have the following theorem, where $\widetilde{F}$ stands for the d.f. of $Y_n, n \geq 1$.

**Theorem 2.5.1.** The set of all a.s. limit points of

$$W_n = \{(r(F) - M_{1,n})B_{n/\log\log n}, (r(F) - M_{2,n})B_{n/\log\log n}\}$$

coincides with the set $S = \{(x, y), 0 \leq x \leq y \leq 1\}$, where $B_n$ is a solution of the equation $n(1 - \widetilde{F}(B_n)) \simeq 1$.

**Proof.** From Theorem 2.4.2, note that the set of all a.s. limit points of

$$\frac{1}{B_{n/\log\log n}} \{\widetilde{M}_{1,n}, \widetilde{M}_{2,n}\} is L_1.$$

Using the relation between $M_{j,n}$ and $\tilde{M}_{j,n}$, we find that a.s. limit set of

\[
\frac{1}{B_{[n/\log \log n]}} \left\{ \frac{1}{r(F) - M_{1,n}}, \frac{1}{r(F) - M_{2,n}} \right\} \text{ is } L_1.
\]

This in turn implies that the set of a.s. limit points of

\[
B_{[n/\log \log n]} \{r(F) - M_{1,n}, r(F) - M_{2,n}\} \text{ is } S.
\]

\[\square\]

**Remark 2.5.1.** When $F$ is Weibull with d.f. $F(x) = \exp\left(-(-x)^\gamma\right)$ if $x < 0$, = 1 if $x \geq 0$, then $r(F) = 0$ and $B_n = n^{1/\gamma}$. The a.s. limit set of

\[
\frac{n^{1/\gamma}}{(\log \log n)^{1/\gamma}}(-M_{1,n}, -M_{2,n}) = \frac{-n^{1/\gamma}}{(\log \log n)^{1/\gamma}}(M_{1,n}, M_{2,n}) \text{ is } S.
\]

**Remark 2.5.2.** Let $F(x) = 0$ if $x < 0$, = $x^p$ if $0 \leq x \leq 1$, = 1 if $x > 1$, $p \geq 1$. Note that when $p = 1$, $F$ is Uniform$(0, 1)$. Here

\[
W_n = \left\{ \frac{p}{\log \log n}(1 - M_{1,n}), \frac{p}{\log \log n}(1 - M_{2,n}) \right\}
\]

has the set of a.s. limit points given by $S$. When $F$ is Uniform$(0, 1)$, that is, $p = 1$, the results coincides with that of Lemma 2.2.3.

### 2.6 L.I.L. Results for the Spacings $(M_{1,n} - M_{2,n})$

**Theorem 2.6.1.** Let

\[
Q_n^* = \frac{n(M_{1,n}^* - M_{2,n}^*)}{\log \log n}.
\]
Then

\[ \limsup Q_n^* = 1 \text{ a.s. and } \liminf Q_n^* = 0 \text{ a.s.} \]

and all points in \([0, 1]\) are a.s. limit points of \(Q_n^*\).

**Proof.** From Lemma 2.2.3, we know that the a.s. limit set of \((\xi_n = (\xi_{1,n}, \xi_{2,n}))\) is \\{\((x, y); 0 \leq x \leq y \leq 1\)\}, where \(\xi_{j,n} = \frac{n(1-M_{j,n}^*)}{\log \log n}, j = 1, 2\). Consider a point \((x, y)\) with \(x < y\) and \(\epsilon > 0\) such that \(x + \epsilon < y - \epsilon\). Since \((x, y)\) is a limit point of \((\xi_n)\), we have

\[ P(x - \epsilon < \xi_{1,n} < x + \epsilon, y - \epsilon < \xi_{2,n} < y + \epsilon) = 1 \quad (2.6.1) \]

Observe that \((x - \epsilon < \xi_{1,n} < x + \epsilon, y - \epsilon < \xi_{2,n} < y + \epsilon)\)

\[ \Rightarrow \left( y - x - 2\epsilon < \frac{n(M_{1,n}^* - M_{2,n}^*)}{\log \log n} < y - x + 2\epsilon \right) \]

Hence, from (2.6.1), we note that \((y - x)\) is an a.s. limit point of \(Q_n^*\). When \(y\) is close to 1 and \(x\) is close to 0, the limit point will be close to 1. Similarly, when \(x\) and \(y\) are close to each other, limit points will be close to 0. Consequently,

\[ \limsup Q_n^* = 1 \text{ a.s. and } \liminf Q_n^* = 0 \text{ a.s.} \]

and the a.s. limit set of \((Q_n^*)\) is \([0, 1]\). \(\square\)

One can have results for spacings when \(F \in E, F \in R\) and \(F \in B\). The results are stated without proof, as proofs are similar to that of Theorem 2.6.1.
Theorem 2.6.2. Let $F \in E$ and let

$$Z_n = \gamma \log n \left( \frac{M_{1,n} - M_{2,n}}{V(\log n)} \right).$$

Then $\lim \inf Z_n = 0$ a.s. , $\lim \sup Z_n = \infty$ a.s. and every point in $[0, \infty]$, is an a.s.
limit point of $(Z_n)$.

Theorem 2.6.3. Let $F \in R$ and let

$$T_n = \frac{M_{1,n} - M_{2,n}}{B_{[n/\log \log n]}}.$$ 

Then $\lim \inf T_n = 0$ a.s. , $\lim \sup T_n = \infty$ a.s. and all points in $[0, \infty]$ are a.s. limit
points of $(T_n)$.

Theorem 2.6.4. Let $F \in B$ and let

$$S_n = B_{[n/\log \log n]}(M_{1,n} - M_{2,n}).$$

Then $\lim \inf S_n = 0$ a.s., $\lim \sup S_n = 1$ a.s. and all points in $[0, 1]$ are a.s. limit
points.

Remark 2.6.1. One may observe that in all the cases, the limit set of the spacings, properly normalized, stretches over the entire region, making these results uninteresting. However, the fact that they are obtained as by-products of the limit sets of vector of extremes is interesting.