Chapter 1
Definitions and Known Results

1.1 Definitions

Definition 1.1.1. Let $\Omega$ be a non-empty set, $\mathcal{F}$ be the $\sigma$-field of subsets of $\Omega$ and $P$ be a measure defined on $\mathcal{F}$ satisfying $P(\Omega) = 1$. Then the triple $(\Omega, \mathcal{F}, P)$ is called a Probability space.

Definition 1.1.2. A non-decreasing, right continuous function $F$ with finite left limit, defined on $(-\infty, \infty)$ satisfying $F(-\infty) = 0$ and $F(\infty) = 1$ is called a distribution function (d.f.).

Definition 1.1.3. If $F$ is the d.f. of a random variable (r.v.) $X$, then $F(x) = P(X \leq x), x \in (-\infty, \infty)$

Definition 1.1.4. A function $F$ defined by $F(x, y) = P(X \leq x, Y \leq y)$ for all $(x, y) \in \mathbb{R}_2$, is the d.f. of a two dimensional r.v. $(X, Y)$ if and only if (iff) it satisfies the following conditions:
1. $F$ is non-decreasing and right continuous with respect to $x$ for $y$ fixed and with respect to $y$ for $x$ fixed.

2. $F(-\infty, y) = F(x, -\infty) = 0$, $F(\infty, \infty) = 1$, $F(x, \infty) = F_1(x)$, the marginal d.f. of $X$ and $F(\infty, y) = F_2(y)$, the marginal d.f. of $Y$.

3. For every $(x_1, y_1), (x_2, y_2)$ with $x_1 < x_2$ and $y_1 < y_2$ the inequality $F(x_2, y_2) - F(x_2, y_1) + F(x_1, y_1) - F(x_1, y_2) \geq 0$ holds.

**Definition 1.1.5.** A sequence of r.v.s $(X_n)$ is said to converge in probability to a r.v. $X$ if for any given $\epsilon > 0$,

$$\lim_{n \to \infty} P(|X_n - X| > \epsilon) = 0.$$ 

This is denoted by $X_n \overset{P}{\to} X$.

**Definition 1.1.6.** A sequence of r.v.s $(X_n)$ is said to converge almost surely (a.s.) to a r.v. $X$ iff

$$P \left( \omega : \lim_{n \to \infty} X_n(\omega) = X(\omega) \right) = 1.$$ 

This is denoted by $X_n \overset{a.s.}{\to} X$ (That is $X_n \to X$ a.s.).

**Definition 1.1.7.** $X \overset{d}{=} Y$ means the r.v.s $X$ and $Y$ have the same distribution. That is $P(X \leq x) = P(Y \leq x), \ x \in \mathbb{R}$.

**Definition 1.1.8.** A sequence of r.v.s $(X_n), \ n \geq 1$, with corresponding d.f.s $(F_n)$ is said to converge weakly to a r.v. $X$ with the d.f. $F$ if $F_n(x) \to F(x)$, as $n \to \infty$, at all
continuity points of \( F \). Such a convergence is expressed as \( F_n \xrightarrow{w} F \) ((\( F_n \)) converges weakly to \( F \)) or \( X_n \xrightarrow{d} X \) (\( X_n \) converges to \( X \) in distribution).

**Definition 1.1.9.** A d.f. \( F \) is said to be stable iff, for every \( b_1 > 0, b_2 > 0, a_1 \) and \( a_2 \) real, there exists \( b > 0 \) and \( a \) real, such that the following relation holds.

\[
F\left( \frac{x - a_1}{b_1} \right) \ast F\left( \frac{x - a_2}{b_2} \right) = F\left( \frac{x - a}{b} \right).
\]

The characteristic function \( f(t) \) of a stable distribution has the representation,

\[
\log f(t) = i\gamma t - c|t|^\alpha \left\{ 1 - i\beta \frac{t}{|t|}\omega(t, \alpha) \right\}, \quad 0 < \alpha \leq 2,
\]

where

\[
\omega(t, \alpha) = \begin{cases} 
\tan \frac{\pi \alpha}{2} & \text{if } \alpha \neq 1 \\
\frac{2}{\pi} \log |t| & \text{if } \alpha = 1
\end{cases}
\]

where \( \alpha, \beta, \gamma \) and \( c \) are real constants with \( c \geq 0, |\beta| \leq 1, 0 < \alpha \leq 2 \). Here \( \alpha \) is the shape parameter, called the index, \( c \) is the dispersion parameter, \( \beta \) is the skewness parameter and \( \gamma \) is the location parameter.

A stable r.v. is positive (negative) valued whenever \( 0 < \alpha < 1 \) and \( \beta = 1 \) (\( \beta = -1 \)) in the characteristic function representation. A stable r.v. with \( \alpha = 2 \) is a normal r.v.

**Definition 1.1.10.** A d.f. \( G \) is said to be semi-stable if it is either normal or the characteristic function \( f(t) \) of \( G \) is of the form,

\[
\log f(t) = i\gamma t + \int_{-\infty}^{\infty} \left( e^{itx} - 1 - \frac{itx}{1 + x^2} \right) dH(x),
\]
with spectral function,
\[ H(-x) = x^\alpha \theta_1(\log x), x > 0 \]
and
\[ H(x) = -x^\alpha \theta_2(\log x), x > 0, \]
where \(0 < \alpha < 2\) and \(\theta_1\) and \(\theta_2\) are periodic functions with common period such that for all \(x\) and \(h \geq 0\),
\[ e^{\alpha h} \theta_i(x - h) - e^{-\alpha h} \theta_i(x + h) \geq 0, \]
\[ d_i \geq \theta_i(x) \geq c_i, \]
\(i = 1, 2; c_1 + c_2 > 0\).

**Definition 1.1.11.** Let \(\{X_n, n \geq 1\}\) denote the sequence of independent identically distributed (i.i.d.) r.v.s with a common c.d.f. \(F\) and let \(M_n = \max\{X_1, X_2, ..., X_n\}\). Then \(F\) is said to belong to the max domain of attraction of a max stable d.f. \(H(\cdot)\), if there exists sequences \((A_n) \in \mathbb{R}\) and \(B_n > 0, B_n \to \infty\) as \(n \to \infty\), such that,
\[ \lim_{n \to \infty} P(M_n \leq xB_n + A_n) = H(x), -\infty < x < \infty. \]
This is denoted by \(F \in max - DA(H)\).

The only possible classes of d.f.s, \(H\) can take are

Frechet :
\[ H_{1,\gamma}(x) = \begin{cases} \exp(-x^{-\gamma}) & \text{if } x > 0, \gamma > 0 \\ 0 & \text{otherwise} \end{cases} \]

Weibull:
\[ H_{2,\gamma}(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ \exp(-(-x)^\gamma) & \text{otherwise, } \gamma > 0 \end{cases} \]
Gumbel:

\[ H_3(x) = \exp(-e^{-x}), \ -\infty < x < \infty. \]

called as the extreme limiting laws.

Suppose that there exists a constant \( \gamma \) positive, such that, for all \( x > 0 \), as \( t \to \infty \)

\[ \lim \frac{1 - F(tx)}{1 - F(t)} = x^{-\gamma} \quad (A). \]

Let

\[ B_n = \inf \left\{ x : 1 - F(x) \leq \frac{1}{n} \right\}. \]

Then,

\[ \lim_{n \to \infty} P(M_n \leq B_n x) = H_{1,\gamma}(x), \ -\infty < x < \infty. \]

Here \( F \) is said to belong to the max domain of attraction of \( H_{1,\gamma} \), denoted by \( F \in DA(H_{1,\gamma}). \)

Suppose that \( F \) has finite right extremity, denoted by \( \omega(F) \). Define

\[ F^*(x) = F \left( \omega(F) - \frac{1}{x} \right), \ x > 0. \]

If \( F^* \) satisfies the condition \( (A) \), and if with \( A_n = \omega(F) \) and

\[ B_n = \omega(F) - \inf \left\{ x : 1 - F(x) \leq \frac{1}{n} \right\}, \]

then

\[ \lim_{n \to \infty} P(M_n \leq A_n + B_n x) = H_{2,\gamma}(x), \ -\infty < x < \infty. \]
Here $F$ is said to belong to max domain of attraction of $H_{2,\gamma}$, denoted by $F \in DA(H_{2,\gamma})$.

A d.f. $F$ is said to belong to max domain of attraction of $H_3$, denoted by $F \in DA(H_3)$, if there exist constants $A_n = R(B_n)$ and $B_n = \inf \{ x : 1 - F(x) \leq \frac{1}{n} \}$, such that

$$\lim_{n \to \infty} P(M_n \leq A_n + B_n x) = H_3(x), \quad -\infty < x < \infty,$$

where

$$R(t) = (1 - F(t))^{-1} \int_t^{w(F)} (1 - F(y)) dy.$$

For $F$ to belong to $DA(H_3)$, it is necessary and sufficient that,

$$\lim_{t \to \infty} \frac{1 - F(t + xR(t))}{1 - F(t)} = \exp(-x).$$

**Definition 1.1.12.** $F$ is said to belong to the max-domain of partial attraction of a d.f. $G(.)$, if there exists a sequence of integers $(k_n), (a_n)$ and $(b_n)$ of real constants such that

$$\lim_{n \to \infty} P(M_{k_n} \leq b_n x + a_n) = G(x)$$

at all continuity points of $G$. This is denoted by $F \in DPA(G)$.

**Definition 1.1.13.** A positive valued function $L(x)$ is said to be slowly varying (S.V.) at infinity iff for each $t > 0$,

$$\lim_{x \to \infty} \frac{L(tx)}{L(x)} = 1.$$

Any S.V. function has the representation,

$$L(x) = a(x) \exp \left( \int_0^x \frac{\epsilon(y)}{y} dy \right),$$
where \( a(x) \to c > 0, \epsilon(x) \to 0 \) as \( x \to \infty \). This representation is known as Karamata’s representation.

**Definition 1.1.14.** A d.f. \( F \) is said to belong to the domain of attraction of a stable law with index \( \alpha, 0 < \alpha \leq 2 \), denoted by \( \text{sum} – DA(\alpha), 0 < \alpha \leq 2 \), if there exists sequences \( A_n \in R, B_n > 0, B_n \to \infty \) as \( n \to \infty \) such that,

\[
\frac{S_n - A_n}{B_n} \xrightarrow{d} Y_\alpha
\]

where \( Y_\alpha \sim G_\alpha \) and \( G_\alpha \) is stable. (Note that the weak convergence is over the whole sequence \( (n) \)). Here

\[
B_n \simeq n^{\frac{1}{\alpha}} L(n),
\]

where \( L(.) \) is S.V.. When \( \alpha = 2 \), the d.f. \( F \) is said to belong to the domain of attraction of normal distribution, denoted by \( DA(2) \).

**Note 1:** When \( 0 < \alpha < 1 \) and \( \beta = 1 \), the d.f. \( F \) is said to belong to the domain of attraction of a positive stable law with index \( \alpha, 0 < \alpha < 1 \), denoted by \( \text{sum} – DPA(\alpha), 0 < \alpha < 1 \). When \( L(.) \equiv C, C > 0 \), the d.f. \( F \) is said to belong to the domain of normal attraction of a stable law with index \( \alpha, 0 < \alpha \leq 2 \), denoted by \( DNA(\alpha) \), \( 0 < \alpha \leq 2 \).

**Definition 1.1.15.** A d.f. \( F \) is said to belong to the domain of partial attraction of a semi-stable law with index \( \alpha, 0 < \alpha \leq 2 \), denoted by \( DPA(\alpha) \), \( 0 < \alpha \leq 2 \), if there exists a sequence \( (n_k) \) satisfying,
1. \( n_k < n_{k+1}, \ k \geq 1 \)

2. \( \frac{n_{k+1}}{n_k} \to \rho \) as \( k \to \infty \), \( 1 \leq \rho < \infty \).

and sequences \( (A_{n_k}) \) of real constants and \( (B_{n_k}) \) of positive constants \( (B_{n_k} \to \infty \) as \( k \to \infty \)) such that,

\[
\frac{S_{n_k} - A_{n_k}}{B_{n_k}} \overset{d}{\to} Y^*_\alpha,
\]

where \( Y^*_\alpha \sim G^*_\alpha \) and \( G^*_\alpha \) is semi-stable. When \( \alpha = 2 \), the d.f. \( F \) is said to belong to domain of partial attraction of normal distribution, denoted by \( DPA(2) \). Here \( B_n \) is a solution of the equation

\[
n(1 - F(B_n) + F(-B_n)) \simeq 1
\]

and

\[
A_n = \begin{cases} 
0 & \text{if } 0 < \alpha < 1 \\
\frac{n}{E(X_1)} & 1 < \alpha < 2.
\end{cases}
\]

**Definition 1.1.16.** A r.v. \( Y \) is said to be geometrically strictly stable (G.S.S.) if there exists \( a_n > 0 \), such that,

\[
Y \overset{d}{=} \left( \frac{1}{a_n} \right)^{\frac{n}{a_n}} \sum_{j=1}^{N^*(\frac{1}{a_n})} Y_j, \ n \geq 2,
\]

where \( \{Y_j, \ j \geq 1\} \) is a sequence of i.i.d. r.v. having the same distribution as that of \( Y \) and is independent of \( N \left( \frac{1}{n} \right) \), which is a geometric r.v. with parameter \( \frac{1}{n} \). Then
$f(t)$, the characteristic function of $Y$ has the representation,

$$f(t) = (1 + \psi(t))^{-1},$$

where $\exp\{-\psi(t)\}$ is the characteristic function of some strictly stable r.v.

### 1.2 Known Results

**Result 1.2.1. Allan Gut (1986)**

Let $\{n_k\}$ be a strictly increasing sub-sequence of the positive integers such that,

$$\liminf_{k \to \infty} \frac{n_k}{n_{k-1}} > 0$$

and let

$$\epsilon^* = \inf \left\{ \epsilon > 0 : \sum_{k=3}^{\infty} (\log n_k)^{-\epsilon^2} < \infty \right\}.$$

Suppose that $E(X_1) = 0$ and $E(X_1^2) = \sigma^2 < \infty$. Then,

$$\limsup_{k \to \infty} \frac{S_{n_k}}{\sqrt{n_k \log \log n_k}} = \sigma \epsilon^* \text{ a.s.}$$

and

$$\liminf_{k \to \infty} \frac{S_{n_k}}{\sqrt{n_k \log \log n_k}} = -\sigma \epsilon^* \text{ a.s.}$$

In particular, if $\epsilon^* = 0$, then

$$\frac{S_{n_k}}{\sqrt{n_k \log \log n_k}} \xrightarrow{a.s.} 0 \text{ as } k \to \infty.$$ 

For the converse, suppose that for $\epsilon^* > 0$, if

$$P \left( \limsup_{k \to \infty} \frac{|S_{n_k}|}{\sqrt{n_k \log \log n_k}} < \infty \right) > 0,$$
then, \( E(X_1^2) < \infty \) and \( E(X_1) = 0 \).

**Result 1.2.2. Allan Gut (1986)**

Let \( \{n_k\} \) be a strictly increasing sub-sequence of the positive integers such that,

\[
\liminf_{k \to \infty} \frac{n_k}{n_{k-1}} > 0.
\]

Further, let \( E(X_1) = 0 \) and \( E(X_1^2) = \sigma^2 < \infty \). Then,

\[
\limsup_{k \to \infty} \frac{S_{n_k}}{\sqrt{n_k \log \log n_k}} = \sigma \sqrt{2}
\]

and

\[
\liminf_{k \to \infty} \frac{S_{n_k}}{\sqrt{n_k \log \log n_k}} = -\sigma \sqrt{2}.
\]

Conversely, if

\[
P\left( \limsup_{k \to \infty} \frac{S_{n_k}}{\sqrt{n_k \log \log n_k}} < \infty \right) > 0.
\]

Then, \( E(X_1) = 0 \) and \( E(X_1^2) = \sigma^2 < \infty \).

**Result 1.2.3. Barndorff-Nielson (1961)**

Let \( (X_n), n \geq 1 \) be a sequence of i.i.d. Uniform \((0, 1)\) r.v.s, defined on a probability space \((\Omega, \mathcal{F}, P)\) with common d.f. \( F \) and

\[
M_n = \max\{X_1, X_2, ..., X_n\}.
\]

Then

\[
\limsup_{n \to \infty} \frac{n(1 - M_n)}{\log \log n} = 1 \quad \text{a.s.}
\]
and

\[ \liminf_{n \to \infty} \frac{n(1 - M_n)}{\log \log n} = 0 \text{ a.s.} \]

**Result 1.2.4. Chover (1966)**

Let \((X_n), n \geq 1\) be a sequence of i.i.d. r.v.s with d.f. \(F\) which is distributed according to a symmetric stable law with exponent \(\alpha\) \((0 < \alpha < 2)\), that is,

\[ E(e^{itX_n}) = e^{-|t|^\alpha}. \]

Let,

\[ S_n = \sum_{i=1}^{n} X_i, \quad n \geq 1 \]

be the partial sum. Then, for \(0 < \alpha < 2\),

\[ \limsup_{n \to \infty} \left| \frac{S_n}{\frac{1}{n^{1/\alpha}}} \right|^{\frac{1}{\log \log n}} = e^\frac{1}{\alpha} \text{ a.s.} \]

**Result 1.2.5. de Haan and Hordijk (1972)**

Let \((X_n)\) be i.i.d. with a continuous d.f. \(F\). Assume that \(F\) has a positive derivative \(F'\) for large \(x\). Let

\[ f(x) = \frac{1 - F(x)}{F'(x)} \text{ and } g(x) = f(x) \log \log \left(\frac{1}{1 - F(x)}\right) \]

and \(B_n\) is such that

\[ F(B_n) \approx 1 - \frac{1}{n}, \quad n > 1. \]

Assume that,

\[ \lim_{t \to \infty} \frac{g(t)}{t} = \gamma, \quad (1.2.1) \]
for some $\gamma$, $0 \leq \gamma < \infty$. Then a.s.,

$$\limsup_{n \to \infty} \frac{M_{1,n}}{B_n} = e^\gamma \quad \text{and} \quad \liminf_{n \to \infty} \frac{M_{1,n}}{B_n} = 1.$$ 

If

$$\lim_{t \to \infty} g''(t) = 0, \quad (1.2.2)$$

then, a.s.,

$$\limsup_{n \to \infty} \frac{M_{1,n} - B_n}{f(B_n) \log \log n} = 1 \quad \text{and} \quad \liminf_{n \to \infty} \frac{M_{1,n} - B_n}{f(B_n) \log \log n} = 0.$$ 

**Result 1.2.6. Divanji and Vasudeva (1986)**

Let $F \in DPA(\alpha)$, $0 < \alpha < 2$. Then there exists a slowly varying (S.V.) function $L$ and a function $\theta$ bounded by two positive numbers $b_1, b_2$, $0 < b_1 \leq b_2 < \infty$, such that

$$\lim_{x \to \infty} \frac{x^\alpha (1 - F(x) + F(-x))}{L(x) \theta(x)} = 1.$$ 

**Result 1.2.7. Divanji and Vasudeva (1986)**

Let $B_n$ be the smallest root of the equation: $n(1 - F(x) + F(-x)) = 1$. Then,

$$B_n \asymp n^{\frac{1}{\alpha}} \ell(n) \eta_n,$$

where $\ell$ is a function S.V. at $\infty$ and $\eta$ is a function such that both $\eta$ and $\frac{1}{\eta}$ are bounded.

**Result 1.2.8. Divanji and Vasudeva (1986)**

Let $F \in DPA(\alpha)$, $0 < \alpha < 2$. Then,

$$\limsup_{n \to \infty} \left| \frac{S_n}{B_n} \right| \frac{1}{\log \log n} = e^{\frac{1}{\alpha}} \quad \text{a.s.}$$
Result 1.2.9. Hüsler (1985)

Let \( (X_n) \) be i.i.d. with a continuous d.f. \( F \). Assume that \( F \) has a positive derivative \( F' \) for large \( x \). Let

\[
f(x) = \frac{1 - F(x)}{F'(x)} \quad \text{and} \quad g(x) = f(x) \log \log \left( \frac{1}{1 - F(x)} \right)
\]

and \( B_n \) is such that,

\[
F(B_n) \approx 1 - \frac{1}{n}, \quad n > 1.
\]

If

\[
\liminf_{k \to \infty} \frac{n_{k-1}}{n_k} > 0
\]

and (1.2.1) holds, then a.s.,

\[
\limsup_{k \to \infty} \frac{M_{1,n_k}}{B_{n_k}} = e^\gamma \quad \text{and} \quad \liminf_{k \to \infty} \frac{M_{1,n_k}}{B_{n_k}} = 1.
\]

If

\[
\liminf_{k \to \infty} \frac{n_{k-1}}{n_k} > 0
\]

and (1.2.2) holds, then a.s.,

\[
\limsup_{k \to \infty} \frac{M_{1,n_k} - B_{n_k}}{f(B_{n_k}) \log \log n_k} = 1 \quad \text{and} \quad \liminf_{k \to \infty} \frac{M_{1,n_k} - B_{n_k}}{f(B_{n_k}) \log \log n_k} = 0.
\]

Result 1.2.10. Hüsler (1985)

Assume that

\[
\liminf_{k \to \infty} \frac{n_{k-1}}{n_k} = 0 \quad \text{and} \quad \limsup_{k \to \infty} \frac{n_{k-1}}{n_k} \neq 1
\]
hold and that
\[ \sum_k (\log n_k)^y \begin{cases} < \infty & \text{for all } y < -\gamma' \\ = \infty & \text{for all } y > -\gamma', \end{cases} \]
for some \( \gamma' \).

1. If (1.2.1) holds then, a.s.
\[ \limsup_{k \to \infty} \frac{M_{n_k}}{B_{n_k}} = e^{\gamma - \gamma'} \quad \text{and} \quad \liminf_{k \to \infty} \frac{M_{n_k}}{B_{n_k}} = 1. \]

2. If (1.2.2) holds then, a.s.
\[ \limsup_{k \to \infty} \frac{M_{n_k} - B_{n_k}}{f(B_{n_k}) \log \log n_k} = \gamma' \quad \text{and} \quad \liminf_{k \to \infty} \frac{M_{n_k} - B_{n_k}}{f(B_{n_k}) \log \log n_k} = 0. \]

Result 1.2.11. Kesten and Maller (2004)

Let \( F \in \text{DPA}(2) \). Then, there exists \( A_n^* \in \mathbb{R} \) and \( B_n^* \uparrow \infty \), such that,
\[ -1 = \liminf_{n \to \infty} \frac{\binom{r}{X} S_n - A_n^*}{B_n^*} < \limsup_{n \to \infty} \frac{\binom{r}{X} S_n - A_n^*}{B_n^*} = 1 \text{ a.s.} \]
and
\[ 0 < \limsup_{n \to \infty} \frac{|\binom{r}{X} S_n - A_n^*|}{B_n^*} < \infty \text{ a.s..} \]

Result 1.2.12. Kiefer (1971)

Let \((X_n), n \geq 1 \) be a sequence of i.i.d. r.v.s, defined on a probability space \((\Omega, \mathcal{F}, P)\) with common d.f. \( F \), Uniform over \((0,1)\) and \( M_{r,n} = r^{th} \) maxima of \( \{X_1, X_2, ..., X_n\} \), \( 1 \leq r \leq n \). Then
\[ \limsup_{n \to \infty} \frac{n(1 - M_{r,n})}{\log \log n} = 1 \text{ a.s.} \]
and
\[ \lim_{n \to \infty} \inf \left( n(1 - M_{r,n}) \right)^{\frac{1}{\log \log n}} = e^{-\frac{1}{r}} \text{ a.s.} \]


Let \( X_1, X_2, ..., X_n \) be i.i.d. r.v.s from Uniform \((0, 1)\). Identify \( X_{r,n} \) as lower \( r \)th order statistics of \( X_1, X_2, ..., X_n \). Then
\[ \limsup \frac{\log X_{r,n} + \log n}{\log \log n} = 0 \text{ a.s. and } \liminf \frac{\log X_{r,n} + \log n}{\log \log n} = -\frac{1}{r} \text{ a.s.} \]

Result 1.2.14. **Li and Tomkins (1991)**

Let \( X_1, X_2, ... \) be a sequence of i.i.d. r.v.s with d.f. \( F \) such that \( F(x) < 1 \) for all \( x \).

For \( n \geq 1 \), let \( a_n = F^{-1} \left( 1 - \frac{1}{n} \right) \). For an integer \( r \geq 1 \) and each \( n \geq r \), let \( X_{n-r+1,n} \) be the \( r \)th largest of \( X_1, X_2, ..., X_n \); let \( \alpha \geq -1 \). Then the following statements are equivalent:
\[ \sum_{n=r}^{\infty} n^\alpha P \left( \left| \frac{X_{n-r+1,n}}{a_n} - 1 \right| > \epsilon \right) < \infty \text{ for all } \epsilon > 0 \tag{1.2.3} \]
\[ \sum_{n=r}^{\infty} n^\alpha P (X_{n-r+1,n} > (1 + \epsilon)a_n) < \infty \text{ for all } \epsilon > 0 \tag{1.2.4} \]
\[ \sum_{n=r}^{\infty} n^{\alpha+r} (1 - F((1 + \epsilon)a_n))^r < \infty \text{ for all } \epsilon > 0 \tag{1.2.5} \]
\[ \int_1^{\infty} \left( \frac{1 - F(x)}{1 - F(\delta x)} \right)^{r-1} \frac{dF(x)}{(1 - F(\delta x))^{n+2}} < \infty \text{ for all } 0 < \delta < 1 \tag{1.2.6} \]

Result 1.2.15. **Li and Tomkins (1991)**

Let \( X_1, X_2, ... \) be a sequence of i.i.d. r.v.s with d.f. \( F \) such that \( F(x) < 1 \) for all \( x \).

For \( n \geq r \geq 1 \), define \( a_n \) and \( X_{n-r+1,n} \) as in the above result. Then, for every integer
\( r \geq 1 \) and number \( \alpha > -1 \), there exist a real sequence \((a_n)\) such that

\[
\sum_{n=r}^{\infty} n^\alpha P \left( \left| \frac{X_{n-r+1,n}}{a_n} - 1 \right| > \epsilon \right) < \infty \text{ for all } \epsilon > 0
\]

Result 1.2.16. Mikosch (1999): Karamata’s Theorem

Let \( L \) be a S.V. function, locally bounded in \([x_0, \infty)\) for some \( x_0 > 0 \). Then, as \( x \to \infty \),

1. for \( \alpha > -1 \),

\[
\int_{x_0}^{x} t^\alpha L(t) dt \sim (\alpha + 1)^{-1} x^{\alpha+1} L(x).
\]

2. for \( \alpha < -1 \),

\[
\int_{x}^{\infty} t^\alpha L(t) dt \sim -(\alpha + 1)^{-1} x^{\alpha+1} L(x).
\]

Result 1.2.17. Peter Hall (1979)

Let \( X_1, X_2, \ldots, X_n \) be i.i.d. r.v.s with d.f. \( F \) such that

\[
1 - F(x) = e^{-x^r L(x)}, \ r > 0
\]

and \( L \) a S.V. function. Let \( X_{n-r+1,n} \) denotes the \( r^{th} \) maxima. Define

\[
U(y) = - \log(1 - F(y))
\]

and introduce \( V(.) \) as its inverse function. Then

\[
\limsup \frac{\gamma \log n}{\log \log n} \left( \frac{X_{n-r+1,n}}{V(\log n)} - 1 \right) = \frac{1}{r} \text{ a.s.}
\]

and

\[
\liminf \frac{\gamma \log n}{\log \log n} \left( \frac{X_{n-r+1,n}}{V(\log n)} - 1 \right) = 0 \text{ a.s.}
\]
Result 1.2.18. Schwabe and Gut (1994)

Let \((X_n)\) be i.i.d. with \(EX_1 = 0, V(X_1) = 1\) and let \(S_n = \sum_{j=1}^{n} X_j\). If

\[ \frac{n_{k+1}}{n_k} \to \infty, \text{ as } k \to \infty \]

then

\[ \limsup_{k \to \infty} \frac{S_{n_k}}{\sqrt{n_k \log k}} = \sqrt{2} \text{ a.s.} \]

Result 1.2.19. Seneta (1976)

Let \(L(\cdot)\) be a S.V. function and let \(y_n \to \infty, z_n \to \infty\), be any two real sequences. Then, for any \(\delta > 0\),

\[ \lim_{n \to \infty} z_n^{-\delta} \frac{L(y_nz_n)}{L(y_n)} = 0 \quad \text{and} \quad \lim_{n \to \infty} z_n^{-\delta} \frac{L(y_nz_n)}{L(y_n)} = \infty. \]

Result 1.2.20. Spitzer (1964): Borel-Cantelli Lemma

Let \((A_n)\) be a sequence of events in a probability space \((\Omega, \mathcal{F}, P)\)

1. If

\[ \sum_{n=1}^{\infty} P(A_n) < \infty, \text{ then } P(A_n \text{ i.o.}) = 0. \]

2. If

\[ \sum_{n=1}^{\infty} P(A_n) = \infty, \]

and if \(A_1, A_2, \ldots\), are mutually independent, then

\[ P(A_n \text{ i.o.}) = 1. \]
Result 1.2.21. Spitzer (1964): Extended Borel-Cantelli Lemma

Let \((E_n)\) be a sequence of events in a probability space \((\Omega, \mathcal{F}, P)\). If
\[
\sum_{n=1}^{\infty} P(E_n) = \infty
\]
and if
\[
\liminf \frac{\sum_{r=1}^{n} \sum_{s=1}^{n} P(E_r \cap E_s)}{(\sum_{r=1}^{n} P(E_r))^2} \leq C < \infty.
\]
Then
\[
P(E_n \text{ i.o.}) \geq \frac{1}{C}.
\]

Result 1.2.22. Vasudeva and Divanji (2006)

Let \((X_n)\) be a sequence of i.i.d. r.v.s with a common d.f. \(F\) and let
\[
S_n = \sum_{i=1}^{n} X_i \quad \text{and} \quad M_n = \max_{1 \leq j \leq n} X_j, \quad n \geq 1.
\]
Assume that there exists a sequence \((B_n)\) of positive constants such that, \(\left( \frac{S_n}{B_n} \right)\) converges to a stable law with index \(\alpha\), \(0 < \alpha < 2\). Let \((n_k)\) be any sub-sequence of positive integers such that,
\[
\frac{n_{k+1}}{k^\gamma n_k} \to \infty,
\]
as \(k \to \infty\) for some \(\gamma > 0\). Then,
\[
\limsup_{k \to \infty} \left| \frac{S_{n_k}}{B_{n_k}} \right|^\frac{1}{H\alpha} = e^\frac{1}{\alpha} \quad \text{a.s.}
\]

Result 1.2.23. Vasudeva and Divanji (2006)

Let \(F \in DA(\alpha), 0 < \alpha < 1\). Let \((n_k)\) be any sub-sequence of positive integers such
that
\[ \frac{n_{k+1}}{n_k} \to \infty \]
as \( k \to \infty \). Then,
\[ \limsup_{k \to \infty} \left( \frac{S_{n_k}}{B_{n_k}} \right)^{\frac{1}{\log k}} = e^{\frac{1}{\alpha}} \text{ a.s.} \]

**Result 1.2.24. Vasudeva and Divanji (2006)**

Let \( F \in DA(H_{1,\gamma}), \gamma > 0 \). Let \((n_k)\) be an increasing sub-sequence of positive integers such that,
\[ \frac{n_{k+1}}{n_k} \to \infty \]
as \( k \to \infty \), then,
\[ \limsup_{k \to \infty} \left| \frac{M_{n_k}}{B_{n_k}} \right|^{\frac{1}{\log k}} = e^{\frac{1}{\alpha}} \text{ a.s.}, \]
where \( B_n \) is the smallest root of the equation
\[ n(1 - F(B_n)) \simeq 1. \]

**Result 1.2.25. Vasudeva and Savitha (1992)**

Let \( F \in DA(H_{1,\gamma}), \gamma > 0 \). Further let \((n_k)\) be a strictly increasing sub-sequence of positive integers such that,
\[ \limsup_{k \to \infty} \frac{n_k}{n_{k+1}} < 1 \]
and let \((t_k)\) be a random sub-sequence of integers such that, \( \frac{t_k}{n_k} \to 1 \) a.s.. Then,
\[ \limsup_{k \to \infty} \left| \frac{M_{t_k}}{B_{n_k}} \right|^{\frac{1}{\log \log n_k}} = e^{\frac{1}{\gamma}} \text{ a.s.} \]
where

\[ \epsilon^* = \inf \{ \epsilon > 0 : \sum_{k} (\log n_k)^{-\epsilon} < \infty \} \]

and

\[ \lim \inf_{k \to \infty} \left| \frac{M_{t_k}}{B_{n_k}^{\frac{1}{\log \log n_k}}} \right|^{\frac{1}{\log \log n_k}} = 1 \text{ a.s.} \]

Result 1.2.26. Vasudeva and Savitha (1992)

Let \( F \in DA(H_{1,\gamma}) \), \( \gamma > 0 \). Further let \((n_k)\) be a strictly increasing sub-sequence of positive integers such that,

\[ \lim \inf_{k \to \infty} \frac{n_k}{n_{k+1}} > 0 \]

and let \((t_k)\) be a random sub-sequence of integers such that, \( \frac{t_k}{n_k} \to 1 \) a.s.. Then,

\[ \lim \sup_{k \to \infty} \left| \frac{M_{t_k}}{B_{n_k}^{\frac{1}{\log \log n_k}}} \right|^{\frac{1}{\log \log n_k}} = e^\gamma \text{ a.s.} \]

and

\[ \lim \inf_{k \to \infty} \left| \frac{M_{t_k}}{B_{n_k}^{\frac{1}{\log \log n_k}}} \right|^{\frac{1}{\log \log n_k}} = 1 \text{ a.s.} \]

Result 1.2.27. Xuewen Lu and Yongcheng Qi (2006)

Let \( F \in DA(\alpha) \), \( 0 < \alpha < 2 \). Then, for the trimmed sum \( (^{(r)}S_n) \),

\[ \lim \sup_{n \to \infty} \left( \left| \frac{(^{(r)}S_n - A_n)}{B_n} \right|^{\frac{1}{\log \log n}} \right)^{\frac{1}{\log \log n}} = e^{\frac{1}{\alpha(r+1)}} \text{ a.s.}, \]

where \( A_n \in R \) and \( B_n > 0 \) satisfies

\[ 1 - F(B_n) + F(-B_n) \simeq \frac{1}{n}. \]
Result 1.2.28. Xuewen Lu and Yongcheng Qi (2006)

Let $F \in DA(2)$. Then, there exists $A_n \in R$ and $B_n \to \infty$ as $n \to \infty$ such that,

$$\limsup_{n \to \infty} \left( \frac{|(r)S_n - A_n|}{B_n} \right)^{\frac{1}{\log \log n}} = \beta \in \left[ 1, e^{\frac{1}{2(r+1)}} \right] \text{ a.s.}$$