Chapter 2

REFLECTION-REFRACTION AT POROUS-POROUS INTERFACE: EFFECT OF PORE CONNECTIONS

Biot’s theory is employed to study the reflection and refraction of plane harmonic waves at the welded interface between two dissimilar saturated poroelastic media. A pore alignment parameter is defined to classify the effects of connection between the interstices of the two media. Effects of pore alignment on the amplitude ratios and energy ratios have been calculated numerically for a particular model. Amplitude and energy ratios do not change significantly as we move from partial alignment to full alignment of pores. However, the effects on amplitudes and energies are quite significant for the values of the pore alignment parameter approaching zero. For extreme values of the pore alignment parameter, the amplitude and energy ratios have been plotted against the angle of incidence.

2.1 Elastodynamics of Fluid-saturated Porous Media

Following Biot (1956, 1962a,b), a set of differential equations governs the particle motion in an isotropic porous solid frame saturated by a non-viscous fluid. These equations, in the absence of body forces, are given by

\[
\tau_{ij,j} = \rho \ddot{u}_i + \rho_f \ddot{w}_i, \\
(-p_f)_i = \rho_f \ddot{u}_i + q \ddot{w}_i,
\]

where, \(u_i\) are displacement components of solid particles. The components \(w_i \left[= f(U_i - u_i)\right]\) represent the flow of fluid relative to the solid measured per unit area of the bulk.
medium, when \( f \) denotes porosity of solid matrix and \((U_1, U_2, U_3)\) is displacement of fluid particles. Comma before an index implies partial space derivative and dot over a variable denotes partial time derivative. Repeated index implies summation. Tensor \( \tau_{ij} \) and scalar \( p_f \) represent stress distribution and fluid pressure, respectively.

In isotropic porous medium, constitutive relations for stress components in porous aggregate \( (\tau_{ij}) \) and fluid pressure \( (p_f) \) are expressed as

\[
\tau_{ij} = 2\mu e_{ij} + \left[ (\lambda + \alpha^2 M) e + \alpha M \xi \right] \delta_{ij}, \quad (i,j = 1,2,3)
\]

\[
-p_f = M [\alpha e + \xi]
\]

where \( e = \nabla \cdot u \), \( \xi = \nabla \cdot w \) and

\[
e_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i}).
\]

\( \lambda, \mu \) are Lame’s constants for the solid;
\( \rho, \rho_f \) are mass densities of the bulk material and fluid respectively;
\( m \) is Biot’s parameter, which depends upon porosity \( f \); and
\( \alpha, M \) are elastic constant related to the coefficients of jacketed compressibility and unjacketed compressibility.

In terms of the displacement vectors \( \mathbf{u} \) and \( \mathbf{w} \), in a homogeneous poroelastic solid, the equations of motion (2.1) are expressed as follows.

\[
\mu \nabla^2 \mathbf{u} + (\lambda + \mu + \alpha^2 M) \nabla (\nabla \cdot \mathbf{u}) + \alpha M \nabla (\nabla \cdot \mathbf{w}) = \partial^2 (\rho \mathbf{u} + \rho_f \mathbf{w}) / \partial t^2,
\]

\[
\nabla [\alpha M (\nabla \cdot \mathbf{u}) + M (\nabla \cdot \mathbf{w})] = \partial^2 (\rho_f \mathbf{u} + m \mathbf{w}) / \partial t^2.
\]

2.2 General Solution: Harmonic Waves

Through the usual Helmholtz resolution of a vector, the displacement vectors in the two-phase homogeneous isotropic medium are written as

\[
\mathbf{u} = \nabla \phi + \nabla \times \mathbf{S}, \quad \nabla \cdot \mathbf{S} = 0;
\]

\[
\mathbf{w} = \nabla \psi + \nabla \times \mathbf{F}, \quad \nabla \cdot \mathbf{F} = 0.
\]
In terms of displacement potentials \((\phi, \psi; \mathbf{S}, \mathbf{F})\), the equations (2.4) are expressed as follows.

\[
\nabla (H \nabla^2 \phi - \rho \ddot{\phi}) + \nabla (\alpha M \nabla^2 \psi - \rho_f \ddot{\psi}) + (\mu \nabla^2 \nabla \times \mathbf{S} - \rho \nabla \times \ddot{\mathbf{S}}) = \rho_f \nabla \times \ddot{\mathbf{F}},
\]
\[
\nabla (\alpha M \nabla^2 \phi - \rho_f \ddot{\phi}) + \nabla (M \nabla^2 \psi - m \ddot{\psi}) = \rho_f \nabla \times \ddot{\mathbf{S}} + m \nabla \times \ddot{\mathbf{F}}, \tag{2.6}
\]

where \(H = \lambda + 2\mu + \alpha^2 M\). Applying divergence to system (2.6), we get a system of equations in dilatations \(e = \nabla . u \) \((= \nabla^2 \phi)\) and \(\epsilon = \nabla . w \) \((= \nabla^2 \psi)\) as follows.

\[
(H \nabla^2 e - \rho \ddot{e}) + (\alpha M \nabla^2 \epsilon - \rho_f \ddot{\epsilon}) = 0,
\]
\[
(\alpha M \nabla^2 e - \rho_f \ddot{e}) + (M \nabla^2 \epsilon - m \ddot{\epsilon}) = 0. \tag{2.7}
\]

Similarly, applying curl to system (2.6), we get a system of equations in rotations \(E = \nabla \times u \) \((= \nabla \times \nabla \times \mathbf{S})\) and \(G = \nabla \times w \) \((= \nabla \times \nabla \times \mathbf{F})\) as follows.

\[
(\mu \nabla^2 E - \rho \ddot{E}) = \rho_f \ddot{G}, \quad 0 = \rho_f \ddot{E} + m \ddot{G}. \tag{2.8}
\]

For time harmonic \((\sim e^{-i\omega t})\) potentials \((\phi, \psi, \mathbf{S}, \mathbf{F})\) to represent harmonic waves of angular frequency \(\omega\), the equations (2.7) - (2.8) transform to

\[
(H \nabla^2 + \rho \omega^2) e + (\alpha M \nabla^2 + \rho_f \omega^2) \epsilon = 0,
\]
\[
(\alpha M \nabla^2 + \rho_f \omega^2) e + (M \nabla^2 + m \omega^2) \epsilon = 0; \tag{2.9}
\]
\[
(\mu \nabla^2 - \rho \omega^2) E - \rho_f \omega^2 G = 0, \quad \rho_f E + m G = 0. \tag{2.10}
\]

The equations (2.9) are solved into a relation,

\[
(M \nabla^2 + m \omega^2) \psi = -(\alpha M \nabla^2 + \rho_f \omega^2) \phi, \tag{2.11}
\]

and a fourth-order partial differential equation given by

\[
[A \nabla^4 + \omega^2 B \nabla^2 + \omega^4 C] \phi = 0, \tag{2.12}
\]

where,

\[
A = (\lambda + 2\mu) M, \quad B = \rho M + m H - 2 \rho_f \alpha M, \quad C = \rho m - \rho_f^2; \quad H = \lambda + 2\mu + \alpha^2 M. \tag{2.13}
\]
The differential equation (2.12) is decomposed to satisfy two Helmholtz equations, given by

\[(\nabla^2 + \delta_i^2)\phi_i = 0, \quad (i = 1, 2), \quad (2.14)\]

where \(\delta_1^2\) and \(\delta_2^2\) are the roots of the quadratic equation \(AX^2 + BX + C = 0\) in unknown \(X\). This defines the propagation of two dilatational waves in saturated porous materials. Hence the general solution of differential equation (2.12) is given by

\[\phi = \phi_1 + \phi_2, \quad (2.15)\]

and, in terms of potentials \(\phi_i\), the relation (2.11) is expressed as

\[\psi = \mu_1\phi_1 + \mu_2\phi_2 \quad (2.16)\]

for some constants \(\mu_1\) and \(\mu_2\).

Similar to scalar potentials, for time harmonic \((\sim e^{-i\omega t})\) vector potentials \((F, G)\), the equations (2.10) are solved into

\[(\nabla^2 + \delta_3^2)E = 0, \quad \delta_3^2 = \omega^2\rho_3/\mu, \quad \rho_3 = \rho - \rho_f^2/m; \quad G = \mu_3E, \quad \mu_3 = -\rho_f/m. \quad (2.17)\]

This defines the propagation of lone shear wave in porous media with velocity \(\beta\).

Finally, we have a system of three wave equations, with particle motion given by

\[u = \nabla\phi_1 + \nabla\phi_2 + \nabla \times E \quad (2.18)\]

Potentials \(\phi_1\), \(\phi_2\) and \(E\) satisfy the wave equations, i.e.,

\[\nabla^2\phi_j = -\omega^2 \frac{1}{v_j^2}\phi_j, \quad (j = 1, 2); \quad \nabla^2E = -\omega^2 \frac{1}{v_3^2}E, \quad (2.19)\]

where

\[v_j = \omega/\delta_j, \quad (j = 1, 2, 3). \quad (2.20)\]

The velocities of propagation, \(v_j\), \((j = 1, 2, 3)\), of three waves are given by

\[v_j^2 = (\lambda + 2\mu)/\rho_j, \quad (j = 1, 2); \quad v_3^2 = \mu/\rho_3, \quad (2.21)\]
Figure 2.1: Geometry of the Medium
where the mass densities $\rho_j$ are defined as follows.

$$\rho_j = [B + (-1)^j \sqrt{(B^2 - 4AC)}/2M, \ (j = 1, 2), \ (2.22)$$

which ensures that $v_1 > v_2$ and hence the potential $\phi_1$ represents the fast $P$ (or $P_f$) wave and the potential $\phi_2$ represents the slow $P$ (or $P_s$) wave. The wave corresponding to the vector potential $E$, propagating with velocity $v_3$, is the lone shear (or $S$) wave. In terms of these densities, we can write

$$\mu_j = (\rho_f \alpha - \rho + \rho_j)/(\rho_f - m\alpha), \ (j = 1, 2). \ (2.23)$$

### 2.3 Motion in a Plane

In Cartesian coordinate system $(x, y, z)$, the plane $z = 0$ serves as the surface of porous solid half space occupying the region $z > 0$, as shown in Fig. 2.1. Isotropy in the medium provides a facility to study the wave motion confined to a plane without losing any information. Hence, the wave motion is studied in $x$-$z$ plane and all the quantities are independent of $y$-coordinate. We have $u = (u_x, 0, u_z)$ and $w = (w_x, 0, w_z)$ and these displacement components are expressed as follows.

$$u_x = \phi_{1,x} + \phi_{2,x} + \phi_{3,z}; \quad w_x = \mu_1 \phi_{1,x} + \mu_2 \phi_{2,x} + \mu_3 \phi_{3,z}; \quad (2.24)$$

$$u_z = \phi_{1,z} + \phi_{2,z} - \phi_{3,x}; \quad w_z = \mu_1 \phi_{1,z} + \mu_2 \phi_{2,z} - \mu_3 \phi_{3,x};$$

where $\phi_3$ is the component of vector potential $E$ in negative $y$-direction. Hence, the displacement potentials $\phi_j, \ (j = 1, 2, 3)$, represent the propagation of $P_f$, $P_s$ and $SV$ waves with velocities $v_j, \ (j = 1, 2, 3)$, respectively. Note that, in $x$-$z$ plane, $(u_x, u_z)$ are the components of displacement of solid particles and $(w_x, w_z)$ are the components of displacement of fluid particles relative to solid frame.
2.4 Formulation of the Problem

Consider two dissimilar saturated non-dissipative isotropic porous solids having a common boundary, as shown in Fig. 2.1. In the Cartesian coordinate system \((x, y, z)\), let the plane \(z = 0\) define this common boundary which is separating the two dissimilar porous media (say, \(M\) and \(M'\)). A wave travels through the medium \(M\) (i.e., \(z > 0\)) with velocity \(V_0\) and incident at the interface making angle \(\theta_0\) to the \(z\)-axis. For two-dimensional motion in \(x\)-\(z\) plane, the row matrix \((\sin\theta_0, 0, \cos\theta_0)\) represents the phase direction of the incident wave. The incident angle may vary from 0 to \(\pi/2\). Such an incidence results in the three waves reflected back into the medium \(M\) and three waves refracted to the continuing medium \(M'\). The directions of reflected waves in \(x\)-\(z\) plane of medium \(M\) are identified with angles \(\theta_j\), \((j = 1, 2, 3)\) and those refracted to the \(x\)-\(z\) plane of medium \(M'\) are identified with angles \(\theta'_j\), \((j = 1, 2, 3)\). Generalising this identification, the primed notations denote the quantities in medium \(M'\) corresponding to the quantities denoted without primes in medium \(M\).

2.5 Boundary Conditions

Following Deresiewicz and Skalak (1963), nonalignment of a portion of the pores can produce an interfacial flow area which is smaller than that in either medium adjacent to the interface. If we define \(f_0 = \min(f, f')\) and \(\zeta f_0\) \((0 \leq \zeta \leq 1)\) as the interfacial flow area in the interface element of unit area then \(\zeta\) is assumed as pore alignment parameter. \(\zeta = 1\) implies that pores of two media are completely connected at the interface and \(\zeta = 0\) corresponds to the case when there is no connection between the interstices of two media. The effect of nonalignment of the portion of pores might be accomplished physically by inserting a porous membrane between the two poroelastic media with fully aligned pores. Flow through such an interface would result in a pressure drop across the interface. As the pores in each of the individual poroelastic media are assumed to be interconnected, we assume that even the alignment of a small portion of the pores would result in the large reduction in the pressure drop, at the interface. Therefore, with the assumed consistency between the pressure drop
and normal component of filtration velocity, we choose to write the continuity requirement, regarding the pressure drop, as

\[ p_f' - p_f = [(1 - \zeta) / \zeta] \dot{w}_n. \]

In general, the boundary conditions appropriate for the interface between two different saturated poroelastic media are given by

\[ \begin{align*}
&i) \quad \tau_{zz}' = \tau_{zz}, \quad ii) \quad u_z' = u_z, \quad iii) \quad \tau_{xx}' = \tau_{xx}, \quad iv) \quad u_x' = u_x, \\
&v) \quad p_f' - p_f = [(1 - \zeta) / \zeta] \dot{w}_z, \quad vi) \quad w_z' = w_z.
\end{align*} \]

(quantities with prime correspond to the medium \( M' \))

The boundary conditions \( i) \) and \( iii) \) represent the continuities of normal and tangential stresses between the two media. Similarly, the conditions \( ii) \) and \( iv) \) ensure the continuities of normal and tangential components of displacements of solid particles in two porous media. The relation \( vi) \) is the continuity of discharge of the pore-fluids filling the pores on either side of the interface \( z = 0 \). The boundary condition \( v) \) deals with pressure drop and normal component of filtration velocity at the interface. These conditions ensure the conservation of energy at the interface between two dissimilar porous solids, for any value of \( \zeta \in [0, 1] \).

Sharma (2008) modified these conditions further by taking into account the continuity equation for fluid-flow in pores at the boundary. In the present geometry of the media, the energy balance is maintained through the equation

\[ \tau_{zz}(\dot{u}_z + \ddot{w}_z) + \tau_{xx}\dot{u}_x - (p_f + \tau_{zz})\dot{w}_z = \tau_{zz}'(\dot{u}_z' + \ddot{w}_z') + \tau_{xx}'\dot{u}_x' - (p_f' + \tau_{zz}')\dot{w}_z' \]

(2.26)

Let us start with the simplest case of welded contact interface. Assuming that the surface pores of both the media are fully connected, following Deresiewicz and Skalak (1963), we should have

\[ \begin{align*}
&i) \quad \tau_{zz} = \tau_{zz}', \quad ii) \quad \dot{u}_z + \ddot{w}_z = \dot{u}_z' + \ddot{w}_z'; \quad iii) \quad \tau_{xx} = \tau_{xx}', \\
&iv) \quad \dot{u}_x = \dot{u}_x'; \quad v) \quad p_f = p_f'; \quad vi) \quad \dot{w}_z = \dot{w}_z'.
\end{align*} \]

(2.27)
Fully connected pores at the interface, as used by Deresiewicz and Skalak (1963), may not be a realistic situation. For example, the surface pores at the common (plane) boundary of two porous media may not connect fully, even when they have same porosity. Hence, an important role is played by the parameter $\zeta$ to represent the effective connections between the surface-pores of two media at the interface. Let this parameter, $\zeta$, be defined as

$$\zeta = \frac{\min(f, f')}{\max(f, f')} \nu,$$  \hspace{1cm} (2.28)

where, $f$ and $f'$ are porosities of the two porous media. The value of $\zeta$ can be zero when, either $\nu = 0$ or one of the porosities is zero. That means no connection between pores at the interface. The value 1 of $\zeta$ means fully connected pores and can be achieved only when $f = f'$ and $\nu = 1$. So, $\nu$ acts as a likelihood parameter that may be defined as the probability that the surface pores of two porous solids of same porosity are fully connected. Moreover, instead of pore pressure, the fluid discharge out of the pores may be depending upon the differential pressure (difference between normal stress in porous matrix and fluid pressure) existing in the porous aggregate. Hence, the boundary conditions $v)$ and $vi)$, then, may be written as

$$v) \quad \zeta(p_f + \tau_{zz}) = Z(1 - \zeta)\dot{w}_z; \quad vi) \quad \zeta(p'_f + \tau'_{zz}) = Z(1 - \zeta)\dot{w}_z.$$  \hspace{1cm} (2.29)

where, $Z$ (a constant) is assumed to be a non-zero, finite value for surface flow impedance (Denneman, et al., 2002), when the pores are partially connected (i.e., $0 < \zeta < 1$). These boundary conditions do ensure the conservation of energy at the interface.

2.6 Reflection and Refraction

We consider only two-dimensional problem in the $x$-$z$ plane. The incident wave is assumed to originate in medium $M$ and become incident at the interface $z=0$, making an angle $\theta_0$ with the $z$-axis. It results in three reflected waves ($P_f$, $P_s$ and SV) in medium $M$ and three waves ($P_f'$, $P_s'$ and $SV'$) transmitted to medium $M'$, as shown in Fig. 2.1.
From equations (2.8), displacement potentials for reflected waves are written as:

\[ \phi_j = A_j \exp[i\delta_j(x\sin\theta_j - z\cos\theta_j) - i\omega t]; \quad (j = 1, 2, 3) \]  

(2.30)

where \( \phi_3 = (-E)_y \) and arbitrary constants \( A_1, A_2, A_3 \) denote the amplitudes of reflected \( P_f, P_s \) and SV waves respectively.

Similarly the corresponding potentials for waves transmitted to medium \( M' \) are written as

\[ \phi'_j = B_j \exp[i\delta'_j(x\sin\theta'_j + z\cos\theta'_j) - i\omega t], \quad (j = 1, 2, 3) \]  

(2.31)

with the corresponding quantities defined for the medium \( M' \).

Displacement potentials for the incident wave are as follows:

(i) for incident \( P_f \)

\[ \phi_1 = A_0 \exp[i\delta_1(x\sin\theta_0 + z\cos\theta_0) - i\omega t], \quad \phi_2 = 0, \quad \phi_3 = 0. \]  

(2.32)

(ii) for incident \( P_s \) wave

\[ \phi_1 = 0, \quad \phi_2 = A_0 \exp[i\delta_2(x\sin\theta_0 + z\cos\theta_0) - i\omega t], \quad \phi_3 = 0. \]  

(2.33)

(iii) for incident SV wave

\[ \phi_1 = 0, \quad \phi_2 = 0, \quad \phi_3 = A_0 \exp[i\delta_3(x\sin\theta_0 + z\cos\theta_0) - i\omega t]. \]  

(2.34)

Corresponding to the potentials given by (2.30) - (2.34), the boundary conditions (2.27) - (2.29) are satisfied for all values of \( x \) if and only if

(i) \( \delta_j \sin\theta_j = \delta_0 \sin\theta_0 = \delta'_j \sin\theta'_j, \quad (j = 1, 2, 3), \)  

(2.35)

where \( \delta_0 = \delta_j, \quad (j = 1, 2, 3) \), as the incident wave is \( P_f, P_s \) and SV respectively.

(ii) \[ \sum_{i=1}^{6} a_{ij} Z_j = b_i, \quad (i = 1, 2, \ldots, 6), \]  

(2.36)

where \( Z_1 = B_1/A_0, \ Z_2 = B_2/A_0, \ Z_3 = B_3/A_0, \ Z_4 = A_1/A_0, \ Z_5 = A_2/A_0 \) and \( Z_6 = A_3/A_0 \) represent the amplitude ratios for refracted \( P_f, P_s, SV \) and reflected \( P_f, P_s \) and SV waves.
respectively.

The coefficient $a_{ij}$ in equations (2.36) are as follows:

$$a_{11} = T'_1, \quad a_{12} = T'_2, \quad a_{13} = \mu' \delta_3 \sin 2\theta'_3,$$

$$a_{14} = -T_1, \quad a_{15} = -T_2, \quad a_{16} = \mu \delta_3 \sin 2\theta_3,$$

$$a_{21} = \mu' \delta_1 \sin 2\theta'_1, \quad a_{22} = \mu' \delta_2 \sin 2\theta'_2, \quad a_{23} = \mu' \delta_3 \cos 2\theta'_3,$$

$$a_{24} = \mu \delta_1 \sin 2\theta_1, \quad a_{25} = \mu \delta_2 \sin 2\theta_2, \quad a_{26} = \mu \delta_3 \cos 2\theta_3,$$

$$a_{31} = M'(\alpha' + \mu'_1) \delta_1^2 \zeta, \quad a_{32} = M'(\alpha' + \mu'_2) \delta_2^2 \zeta,$$

$$a_{33} = 0, \quad a_{34} = (1 - \zeta) \omega \delta_1 \mu \cos \theta_1 - M(\alpha + \mu_1) \delta_1^2 \zeta,$$

$$a_{35} = (1 - \zeta) \omega \delta_2 \mu \cos \theta_2 - M(\alpha + \mu_2) \delta_2^2 \zeta, \quad a_{36} = (1 - \zeta) \omega \delta_3 \alpha_0 \sin \theta_3,$$

$$a_{41} = \delta_1' \sin \theta'_1, \quad a_{42} = \delta_2' \sin \theta'_2, \quad a_{43} = \delta_3' \cos \theta'_3,$$

$$a_{44} = -\delta_1 \sin \theta_1, \quad a_{45} = -\delta_2 \sin \theta_2, \quad a_{46} = \delta_3 \cos \theta_3,$$

$$a_{51} = \delta_1' \sin \theta'_1, \quad a_{52} = \delta_2' \cos \theta'_2, \quad a_{53} = -\delta_3' \sin \theta'_3,$$

$$a_{54} = \delta_1 \cos \theta_1, \quad a_{55} = \delta_2 \cos \theta_2, \quad a_{56} = \delta_3 \sin \theta_3,$$

$$a_{61} = \mu' \delta_1' \cos \theta'_1, \quad a_{62} = \mu' \delta_2' \cos \theta'_2, \quad a_{63} = -\alpha' \delta_3' \sin \theta'_3,$$

$$a_{64} = \mu \delta_1 \cos \theta_1, a_{65} = \mu \delta_2 \cos \theta_2, a_{66} = \alpha_0 \delta_3 \sin \theta_3.$$

where

$$T_j = [2 \mu \sin^2 \theta_j - (H + \alpha M \mu_j)] \delta_j^2, \quad T'_j = [2 \mu' \sin^2 \theta'_j - (H' + \alpha' M' \mu'_j)] \delta'_j^2; \quad (j = 1, 2).$$

The constant term $b_i$ on the right side of equations (2.36) are given by

(i) for incident $P'_1$ wave

$$b_1 = -a_{14}, \quad b_2 = a_{24}, \quad b_3 = M(\alpha + \mu_1) \delta_1^2 \zeta + (1 - \zeta) \omega \delta_1 \mu \cos \theta_1,$$

$$b_4 = -a_{44}, \quad b_5 = a_{54}, \quad b_6 = a_{64}; \quad (2.37)$$

(ii) for incident $P'_2$ wave

$$b_1 = -a_{15}, \quad b_2 = a_{25}, \quad b_3 = M(\alpha + \mu_2) \delta_2^2 \zeta + (1 - \zeta) \omega \delta_2 \mu \cos \theta_2,$$

$$b_4 = -a_{45}, \quad b_5 = a_{55}, \quad b_6 = a_{65}; \quad (2.38)$$

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(iii) for incident SV wave

\[ b_1 = a_{16}, \quad b_2 = -a_{26}, \quad b_3 = -a_{36}, \]
\[ b_4 = a_{46}, \quad b_5 = -a_{56}, \quad b_6 = -a_{66}. \]  

(2.39)

2.7 Energy Ratios

The energy in a poroelastic body is calculated from the stresses and particle velocities of fluid and solid particles at the surface of the body. We now consider the distribution of energy between different reflected and transmitted waves at the surface element of unit area. Following Achenbach (1973), the scalar product of surface traction and particle velocity per unit area, denoted \( P^* \), represents the rate at which energy is communicated per unit area of the surface. If the outer normal on the surface element is \( \hat{n} \), we have

\[ P^* = \tau_{rm} n_m \dot{u}_r \]  

(2.40)

where \( \tau_{rm} \) is the stress tensor, \( n_m \) are the direction cosines of the unit normal \( \hat{n} \) and \( \dot{u}_r \) are components of particle velocity. The time average of \( P^* \) over a period, denoted by \( \langle P^* \rangle \), represents the average energy transmission per unit surface area per unit time. For fluid-saturated porous medium, taking into account the energy communicated to the fluid portion, we have the rate of energy transmission at \( z = 0 \), given by

\[ P^* = \tau_{zz} \partial u_z / \partial t + \tau_{xz} \partial u_x / \partial t + (-p_f) \partial w_z / \partial t. \]  

(2.41)

With the help of the expression \( \langle R(f)R(g) \rangle = \frac{1}{2} \Re(f \overline{g}) \), for any two complex functions \( f \) and \( g \), we obtain the energy ratios giving the rate of average energy transmission of all the transmitted and reflected waves, to that of incident wave. These energy ratios, \( E_i \), \( (i = 1, 2, 3, ..., 6) \), for refracted \( P_f, P_s, SV \) waves and reflected \( P_f, P_s, SV \) waves, respectively, are expressed as

\[ E_i = \langle P_i^* \rangle / \langle P_0^* \rangle, \quad (i = 1, 2, ..., 6), \]  

(2.42)
where

\[
\begin{align*}
\langle P_1^* \rangle &= [\lambda' + 2\mu' + M'(\alpha' + \mu'_1)^2] |Z_1|^2 \Re(\cos\theta'_1)/v_1^3, \\
\langle P_2^* \rangle &= [\lambda' + 2\mu' + M'(\alpha' + \mu'_2)^2] |Z_2|^2 \Re(\cos\theta'_2)/v_2^3, \\
\langle P_3^* \rangle &= \mu'|Z_3|^2 \Re(\cos\theta'_3)/v_3^3, \\
\langle P_4^* \rangle &= [\lambda + 2\mu + M(\alpha + \mu_1)^2] |Z_4|^2 \Re(\cos\theta_1)/v_1^3, \\
\langle P_5^* \rangle &= [\lambda + 2\mu + M(\alpha + \mu_2)^2] |Z_5|^2 \Re(\cos\theta_2)/v_2^3, \\
\langle P_6^* \rangle &= \mu|Z_6|^2 \Re(\cos\theta_3)/v_3^3 \tag{2.43}
\end{align*}
\]

and

(i) for incident \( P_f \) wave

\[
\langle P_0^* \rangle = [\lambda + 2\mu + M(\alpha + \mu_1)^2] \cos\theta_0/v_1^3. \tag{2.45}
\]

(ii) for incident \( P_s \) wave

\[
\langle P_0^* \rangle = [\lambda + 2\mu + M(\alpha + \mu_2)^2] \cos\theta_0/v_2^3. \tag{2.46}
\]

(iii) for incident SV wave

\[
\langle P_0^* \rangle = \mu \cos\theta_0/v_3^3. \tag{2.47}
\]

Generalisation

All the expressions involving \( m \) derived in the preceding sections are applicable only to non-dissipative poroelastic solids. Taking into account the viscosity of the interstitial fluid, these expressions can be made applicable to a general saturated poroelastic solid by replacing Biot’s parameter \( m \) by \( (m - \eta/\omega\chi) \). \( \chi \) is permeability and \( \eta \) denotes the viscosity of the interstitial fluid. For higher frequencies, where the Poiseuille flow breaks down, a correction factor is applied to viscosity \( \eta \), replacing it by \( \eta F \), where \( F \) is a complex function of frequency \( \omega \) and is evaluated following Biot (1956).
2.8 Numerical Example

Since a large number of parameters enter into the final expressions, in order to study the dependence of amplitude and energy ratios on the angle of incidence of incident wave as well as pore-alignment parameter, we confine our numerical work to a particular model. We may mentions here that the aim of this study is to discuss the effects of pore alignment on the reflection and refraction. Therefore, for the sake of simplicity, poroelastic solid is assumed to be a non-dissipative one. Keeping in mind the availability of numerical data, we consider the model consisting of water-saturated sandstone in welded contact with water-saturated limestone. Following the experimental results given by Yew and Jogi (1976) and earlier data given by Fatt (1959), we choose the following values of relevant parameters:

(i) for water-saturated sandstone (medium $M$)

\[
\begin{align*}
\lambda &= 3.034\text{GPa}, \quad \mu = 9.22\text{GPa}, \quad M = 8.87\text{GPa} \quad \alpha = 0.3227 \\
\rho &= 2170\text{kg/m}^3, \quad \rho_f = 1000\text{kg/m}^3, \quad m = 3731\text{kg/m}^3, \quad \beta = 0.268.
\end{align*}
\]

(ii) for water-saturated limestone (medium $M'$)

\[
\begin{align*}
\lambda' &= 14.44\text{GPa}, \quad \mu' = 12.09\text{GPa}, \quad M' = 15.91\text{GPa} \quad \alpha' = 0.262 \\
\rho &= 2240\text{kg/m}^3, \quad \rho_f' = 1000\text{kg/m}^3, \quad m' = 6944\text{kg/m}^3, \quad \beta' = 0.144.
\end{align*}
\]

Corresponding to the above given values of parameters, the system of equations (2.36) is solved for $Z_i$ by Gauss elimination method using a computer program in FORTRAN-77. For fixed values of $\nu$ between 0 and 1, the angle of incidence is considered to be varying from normal incidence ($\theta_0 = 0$) to grazing incidence ($\theta_0 = 90^\circ$). Amplitude ratios $Z_i$ are found to depend upon the angle of incidence. The energy ratios are then calculated numerically using the relation (2.42) - (2.46). Variations in amplitude ratios ($Z_i$) and energy ratios ($E_i$) of all the reflected and refracted waves with incident direction are computed for three values of $\nu \in (0, 1)$. The statistical parameter $\nu$ represents the probability that the surface pores
Figure 2.2: Amplitude ratios of reflected and refracted waves at interface between water-saturated sandstone and water-saturated limestone for different pore-connections; incident $P_f$ wave
of two porous solids of same porosity are fully connected, i.e., the difference in porosities of two media are taken into account. With fixed porosities \( f \) and \( f' \) in present study, an increase in \( \nu \) directly increases the extent of pore-connections at the interface. Amplitude variations are exhibited in Figs. 2.2 to 2.4 and energy variations are exhibited in Figs. 2.5 to 2.7. Detailed discussion of the plots in these figures is illustrated as follows.

2.8.1 Amplitude ratios

(i) For incident \( P_f \) wave (Fig. 2.2)

It can be noted that increase of \( \nu \) increases the amplitude of reflected waves when incidence is near normal incidence. As the incidence shifts towards the grazing direction, the reflection gets weaker with increase of pore connections (or, \( \nu \)). The effect of \( \nu \) is much more significant on \( P_s \) wave. In general, reflected waves are more sensitive to a change in \( \nu \), when its values deviates from 0. That means, a little connection between pores may have a large effect on the amplitude of reflected waves. The connecting pores do not affect the reflected \( SV \) wave when incidence is near-normal or near-grazing direction.

Amplitude of refracted \( P_f \) wave increase with increase in \( \nu \). But this increase disappears as the incidence approaches to grazing direction. Effect of pore-connections at the interface is largest on the \( P_f \) and \( P_s \) waves when incidence is close to normal direction. For larger \( \nu > 0.5 \), any change in \( \nu \) may not affect the amplitudes of \( P_f \) and \( SV \) waves. In contrast to refracted \( P_f \) wave, the amplitudes of refracted \( P_s \) and \( SV \) waves are largest when \( \nu \) is mid-way, i.e., \( \nu = 0.5 \). For incidence near-normal or near-grazing direction, the connecting pores do not affect the refracted \( SV \) wave.

(ii) For incident \( P_s \) wave (Fig. 2.3)

Amplitude of all the reflected \( P_f \) wave may increases many fold with the improvement in pore-connections at the interface at all incidences, except near critical angle or grazing.
Figure 2.3: Amplitude ratios of reflected and refracted waves at interface between water-saturated sandstone and water-saturated limestone for different pore-connections; incident $P_s$ wave
Figure 2.4: Amplitude ratios of reflected and refracted waves at interface between water-saturated sandstone and water-saturated limestone for different pore-connections; incident SV wave
direction. However, maximum amplitude is obtained for $\nu = 0.5$ and further increase of $\nu$ slightly reduces the amplitude. Contrary to reflected $P_f$ wave, with the increase of $\nu$ from 0 to 0.5, amplitude of $P_s$ wave decreases mainly for incidence before critical angle for reflected $SV$ wave. Effect of $\nu$ on the amplitude of reflected $SV$ is much larger for incidence around its critical angle. At normal incidence, amplitude of reflected $SV$ may not be disturbed with the pore-connection at the interface.

In general, the amplitude of all the three refracted waves decrease with increase in $\nu$ upto 0.5. However, the amplitude of $P_s$ wave increases for increase of $\nu$ upto 1. For incidence at critical angle for refracted $P_f$ wave, the pore-connections may have no effect on two refracted $P$ waves. On the contrary, effect of $\nu$ is quite significant on refracted $SV$ wave, when incidence is near its critical angle. Amplitude of refracted $SV$ may not be sensitive to pore-connection at the interface when incidence is near-normal. But same is not the case with refracted $P_f$ or $P_s$ wave.

(iii) For incident $SV$ wave (Fig. 2.4)

Amplitudes of each of the three reflected waves increases with the increase of $\nu$ mainly before the critical angle for reflected $P_f$ wave. For $P_s$ and $SV$ waves, this increase may be observed even after the critical incidence but quite before grazing incidence. All the reflected waves are insensitive to pore-connections near normal or grazing incidence. Effect of $\nu$ is largest on any reflected wave when incidence is around the critical angles for reflected or refracted $P_f$ waves. However, at this critical incidence, no effect of $\nu$ is observed on reflected $P_f$ wave.

Increase of $\nu$ increases the amplitudes of refracted $P_f$ and $P_s$ waves but $\nu = 0.5$ yields maxima. On the contrary, refracted $SV$ wave loses its amplitude with increase of $\nu$ upto 0.5 and reverses a bit after that. Insensitivity of refracted waves to $\nu$ is observed at normal as well as grazing incidence. Sensitivity to $\nu$ is maximum for incidence around the critical angles for reflected / refracted $P_f$ waves. The peak of refracted $SV$ wave points to the
Figure 2.5: Energy ratios of reflected and refracted waves at interface between water-saturated sandstone and water-saturated limestone for different pore-connections; incident $P_f$ wave
incidence at critical angle for refracted SV wave.

2.8.2 Energy ratios

(i) For incident $P_f$ wave (Fig. 2.5)

The reflected waves get more energy when the pores at the interface are connected or these connections are improved. Such a bonus is lost when incidence approaches to the grazing direction. However, for reflected $P_f$ wave, the sensitivity to $\nu$ vanishes near $40^\circ$ incidence and reverses thereafter. At normal incidence, the strength of $P_s$ wave improves upto five-folds with the increase of $\nu$ from 0.01 to 0.99. However, the reflected SV wave not at all sensitive to the pore-connections at the interface. With critical angle for refracted $P_f$ wave near $55^\circ$, some peaks are observed in the energies of reflected $P_s$ and SV waves.

Reflected $P_f$ wave loses its major share in incident energy, when pores at the interface are connected. This loss is the maximum for normal incidence and compensates upto its critical angle. For post-critical incidence, the refracted $P_f$ wave becomes evanescent by losing its (vertical) energy flux and propagates along the interface. Energy shares of $P_s$ and SV waves increase with $\nu$ with maximum at $\nu = 0.5$. For further increase in $\nu$, the refracted SV wave weakens only slightly but loss of refracted SV wave is quite significant. Effect of $\nu$ of refracted $P_s$ and SV waves, reverses for post-critical incidence. No effect of $\nu$ is noticed near grazing incidence.

(ii) For incident $P_s$ wave (Fig. 2.6)

Reflected $P_f$ and SV waves increase their energy shares with the increase of $\nu$ upto 0.5. But, the energy of reflected $P_s$ wave decreases many-fold with the increase of $\nu \leq 0.5$. For instance, the strongest reflected wave for any incidence is $P_s$ wave. However, it weaken for incidence around critical angle for refracted SV wave, i.e., near $40^\circ$. For increase of $\nu$ beyond 0.5, behaviour reverses a little for each of the three reflected waves. At normal incidence,
Figure 2.6: Energy ratios of reflected and refracted waves at interface between water-saturated sandstone and water-saturated limestone for different pore-connections; incident $P_s$ wave
both the reflected $P$ waves are too sensitive to the pore-connections at the interface. But, reflected $SV$ is almost insensitive to any change in $\nu$. Critical angles for reflected $P_f$ and $SV$ waves are near $30^\circ$ and $48^\circ$ respectively.

Energy shares of all the three refracted waves are nearly compatible. After its critical angle near $25^\circ$, the refracted $P_f$ starts propagating along the interface. Similarly, the refracted $SV$ loses its vertical energy flux for incidence of $P_s$ wave beyond its critical angle near $42^\circ$. For incidence before this critical angle, $P_s$ wave increases its energy stake with the increase of $\nu$ upto 1. However, the increase of energy shares of refracted $P_f$ and $SV$ waves increase with $\nu$ only upto 0.5 and decrease a bit for increase of $\nu \in (0.5, 1)$. For post-critical incidence, the refracted $P_s$ gets its maximum energy share when $\nu = 0.5$. But, near grazing, refracted $P_s$ wave is strongest when pores are nearly disconnected. The strength is gained at the cost of reflected $P_s$ wave because, in case of post-critical incidence, the whole incident energy is shared between reflected and refracted $P_s$ waves.

(iii) For incident $SV$ wave (Fig. 2.7)

Each of the three reflected waves improve their energy share with an increase of $\nu$, for incidence upto critical angle for refracted $SV$ wave near $70^\circ$. However, except for $P_s$ wave, the other two reflected waves gains more when $\nu$ increases in $(0, 0.5)$. Critical angle for reflected $P_s$ wave is found near $42^\circ$. For incidence after this angle, reflected $P_f$ wave propagate along the interface with no vertical energy flux. $\nu$ may have no effect on reflected waves when $SV$ wave is incident normal to the interface or nearly parallel to the interface. Reflected $SV$ wave is strongest when incidence approaches to the grazing direction. For incidence beyond critical angle for refracted $SV$ wave, the reflected $SV$ wave gains strength at the cost of its refracted counterpart, but reflected $P_s$ wave loses a similar when incidence moves to grazing direction. Note that no reflected wave is observed for normal incidence of $SV$ wave and whole energy is refracted to the rarefied medium.

Critical angles for refracted $P_f$ and $SV$ waves are noted for incidence near $35^\circ$ and $73^\circ$.
Figure 2.7: Energy ratios of reflected and refracted waves at interface between water-saturated sandstone and water-saturated limestone for different pore-connections; incident SV wave
respectively. For incidence beyond these critical angles, only refracted wave is $P_s$ wave. At normal incidence, the whole incident energy refracted as $SV$ wave. Effect of $\nu \in (0, 0.5)$ is quite large on each of the three refracted waves. But $\nu \in (0.5, 1)$ affects only the refracted $P_s$ wave. The energies of refracted $P$ wave increase but energy of refracted $SV$ decreases with increase of $\nu \in (0, 0.5)$. Refracted $P_s$ wave, however, loses a significant part of its gain for increase of $\nu \in (0.5, 1)$. No refracted wave is noticed for incidence near grazing direction.

2.9 Concluding Remarks

It is concluded that pore connections has no effect at grazing incidence for all the waves. Also it does not affect the normal incidence of $SV$ waves. It is observed that amplitude and energy ratios change infinitesimally with the change in the value of $\nu$, particularly when pores at the interface connect only slightly. This implies that presence of pore-connections matters much more than the extent or magnitude of these connections. That means it may not be important whether the pores at the interface of two media are fully connected or partially connected. This behaviour is in accordance with the change in pressure drop for nonalignment of a portion of pores. From the numerical results and also from various energy-plots in Figs. 2.5 to 2.7, we see that $\sum_{i=1}^{6} E_i = 1$ for each angle of incidents of every incident wave. This implies that no energy is dissipated during transmission at the interface and incident energy is completely shared by the three reflected and three refracted waves.