2.0 Introduction

In certain applications involving discrete data, we come across data having frequency of an observation ‘zero’ significantly higher than the one predicted by the assumed model. The problem of high proportion of zeros has been an interest in data analysis and modeling. There are many situations in the medical field, engineering applications, manufacturing, economics, public health, road safety epidemiology and in other areas leading to similar situations. In highly automated production process, occurrence of defects is assumed to be Poisson. However, we get no defectives in many samples. This leads to excess number of zeros. Models having more number of zeros significantly are known as zero-inflated models.

In the literature, numbers of researchers have worked on family of zero-inflated power series distributions. Gupta et al. (1995) have studied the
structural properties and point estimation of parameters of Zero-Inflated Modified Power Series distributions and in particular for zero-inflated Poisson distribution. Murat and Szynal (1998) have studied the class of inflated modified power series distributions where inflation occurs at any of the support points. Moments, factorial moments, central moments, the maximum likelihood estimators and variance-covariance matrix of the estimators are obtained. Murat and Szynal (1998) extended the results of Gupta et al. (1995) to the discrete distributions inflated at any point \( s \).

Zero-inflated power series distribution contains two parameters. The first parameter indicates inflation (\( \pi \)) of zero and the other parameter (\( \theta \)) is that of power series distribution. Literature survey reveals that many researchers devoted to the inflation parameter of the model. In the present study, we focus on the referential aspect of the basic parameter of the model. In this chapter, we provide maximum likelihood parameters, Fisher information and asymptotic tests for testing the parameter of the Power Series distribution. Additionally, asymptotic confidence interval for the parameter is provided.

The rest of the chapter is organized as follows. In section 2.1, we report estimation of both the parameters of ZIPSD and corresponding asymptotic variances using full likelihood approach and conditional likelihood approach. In section 2.2, we provide three asymptotic tests for testing the parameter of PSD and asymptotic confidence intervals. In section 2.3, we provide Modified Zero-Inflated Power Series Distribution and inference related to the model. In the section 2.4, we provide three asymptotic tests for testing the parameter of MPSD and asymptotic confidence intervals. We introduce the Zero-Inflated Truncated Power Series Distribution in section 2.5.

2.1 Estimation of the Parameters Using Full Likelihood Function

Suppose a random sample \( X_1, X_2, \ldots, X_n \) of size \( n \) from ZIPSD defined in (1.2.2) is available. Then the likelihood function is given by
\[ L(\theta, \pi ; \mathbf{x}) = \prod_{i=1}^{n} \left( 1 - \pi + \frac{\pi b_0}{f(\theta)} \right)^{1-a_i} \left( \frac{\pi b x_i \theta^a}{f(\theta)} \right)^{a_i}, \quad \theta, \pi > 0 \]

where \( a_i = 0 \) if \( x_i = 0 \) and \( a_i = 1 \) if \( x_i = 1, 2, 3, \ldots \) \( \quad \ldots (2.1.1) \)

Maximum likelihood estimators of \( \theta \) and \( \pi \) can be obtained by maximizing \( \log L(\theta, \pi ; \mathbf{x}) \) with respect to \( \theta \) and \( \pi \) respectively, where

\[ \log L(\theta, \pi ; \mathbf{x}) = n_0 \log \left( 1 - \pi + \frac{\pi b_0}{f(\theta)} \right) + \sum_{i=1}^{n} a_i \log \pi + \sum_{i=1}^{n} a_i \log bx_i + \sum_{i=1}^{n} a_i x_i \log \theta - \sum_{i=1}^{n} a_i \log f(\theta) \]

\( \quad \ldots (2.1.2) \)

We note that \( \left( \sum_{i=1}^{n} a_i, \sum_{i=1}^{n} a_i x_i \right)' \) is jointly sufficient for \( (\pi, \theta)' \). Let \( n_0 \) be the number of zeros in the sample. Differentiating log likelihood function with respect to \( \pi \) and \( \theta \) we get

\[ \frac{\partial \log L}{\partial \pi} = \frac{n_0 \left( -1 + \frac{b_0}{f(\theta)} \right) + \sum_{i=1}^{n} a_i }{1 - \pi + \frac{b_0}{f(\theta)}} \frac{\pi}{\pi} \]

\( \quad \ldots (2.1.3) \)

and

\[ \frac{\partial \log L}{\partial \theta} = \frac{n_0 \left( - \frac{\pi b_0 f'(\theta)}{f(\theta)^2} \right) + \sum_{i=1}^{n} a_i x_i }{1 - \pi + \frac{b_0}{f(\theta)}} \frac{\theta}{\theta} - \frac{\sum_{i=1}^{n} a_i f'(\theta) }{f(\theta)}, \text{ respectively.} \quad \ldots (2.1.4) \]

Equating Eq. (2.1.3) and Eq. (2.1.4) to zero and solving for \( \pi \) and \( \theta \) we get

\[ \hat{\pi} = \frac{(n - n_0) f(\hat{\theta})}{n f(\hat{\theta}) - b_0} \]

\( \quad \ldots (2.1.5) \)

and
\[
\sum_{i=1}^{n} a_{i} x_{i} = \frac{n_{0} \hat{\pi} b_{0} f'(\hat{\theta})}{f(\hat{\theta})^{2} \left(1 - \hat{\pi} + \frac{b_{0}}{f(\hat{\theta})} \right)} + \frac{\sum_{i=1}^{n} a_{i} f'(\hat{\theta})}{f(\hat{\theta})}, \quad \ldots (2.1.6)
\]

where \( \hat{\pi} \) and \( \hat{\theta} \) are the mles of \( \pi \) and \( \theta \) respectively.

Substituting \( \hat{\pi} = \frac{n-n_{0}}{n(f(\hat{\theta}) - b_{0})} \) in Eq. (2.1.6) we get

\[
\sum_{i=1}^{n} a_{i} x_{i} = \frac{n_{0} \left( \frac{(n-n_{0}) f(\hat{\theta})}{n(f(\hat{\theta})-b_{0})} \right) b_{0} f'(\hat{\theta})}{f(\hat{\theta})^{2} \left(1 - \frac{(n-n_{0}) f(\hat{\theta})}{n(f(\hat{\theta})-b_{0})} + \frac{(n-n_{0}) f(\hat{\theta})}{n(f(\hat{\theta})-b_{0})} \right)} + \frac{\sum_{i=1}^{n} a_{i} f'(\hat{\theta})}{f(\hat{\theta})} + \frac{(n-n_{0}) f'(\hat{\theta})}{f(\hat{\theta})} \]

\[
\sum_{i=1}^{n} a_{i} x_{i} = \frac{n_{0} \left( \frac{(n-n_{0}) f(\hat{\theta})}{n(f(\hat{\theta})-b_{0})} \right) b_{0} f'(\hat{\theta})}{f(\hat{\theta})^{2} \left(1 - \frac{(n-n_{0}) f(\hat{\theta})}{n(f(\hat{\theta})-b_{0})} + \frac{(n-n_{0}) f(\hat{\theta})}{n(f(\hat{\theta})-b_{0})} \right)} + \frac{(n-n_{0}) f'(\hat{\theta})}{f(\hat{\theta})} \]

\[
\sum_{i=1}^{n} a_{i} x_{i} = \frac{(n-n_{0}) f'(\hat{\theta})}{f(\hat{\theta})} \left(1 + \frac{b_{0}}{f(\hat{\theta})-b_{0}} \right)
\]

\[
\sum_{i=1}^{n} a_{i} x_{i} = \frac{\hat{\theta} f'(\hat{\theta})}{f(\hat{\theta})} \left(1 + \frac{b_{0}}{f(\hat{\theta})-b_{0}} \right)
\]

\[
\bar{x} = \frac{\hat{\theta} f'(\hat{\theta})}{f(\hat{\theta})-b_{0}}, \quad \ldots (2.1.7)
\]

which is a non-linear equation in \( \theta \). Using Newton-Raphson method first we find \( \hat{\theta} \). Substituting this value of \( \hat{\theta} \) in Eq. (2.1.5), we obtain \( \hat{x} \). Let us denote the Fisher information matrix of \( \hat{\varphi} = (\pi, \theta)' \) as \( I(\hat{\varphi}) = (I_{ij}(\hat{\varphi})) ; i,j = 1,2 \). Therefore,
\[
I(\delta) = \begin{pmatrix}
- \frac{\partial^2 \log L}{\partial \pi^2} & - \frac{\partial^2 \log L}{\partial \pi \partial \theta} \\
- \frac{\partial^2 \log L}{\partial \pi \partial \theta} & - \frac{\partial^2 \log L}{\partial \theta^2}
\end{pmatrix}
= \begin{pmatrix}
I_{11} & I_{12} \\
I_{21} & I_{22}
\end{pmatrix}.
\]

Let
\[
I^{-1}(\delta) = \frac{1}{I_{11}I_{22} - I_{12}I_{21}} \begin{pmatrix}
I_{22} & -I_{12} \\
-I_{21} & I_{11}
\end{pmatrix} = \begin{pmatrix}
I_{11} & I_{12} \\
I_{21} & I_{22}
\end{pmatrix}.
\]

In the following we obtain the elements \(I_{ij}, i, j = 1, 2\).

Differentiating Eq. (2.1.3) with respect to \(\pi\), we get
\[
\frac{\partial^2 \log L}{\partial \pi^2} = -\frac{n_0 \left(1 + \frac{b_0}{f(\theta)}\right)^2}{\sum_{i=1}^{n} a_i} \left(1 - \pi + \pi \frac{b_0}{f(\theta)}\right)^2 \frac{\pi^2}{n_0}.
\]

Therefore,
\[
- \frac{\partial^2 \log L}{\partial \pi^2} = \frac{E(n_0) \left(1 + \frac{b_0}{f(\theta)}\right)^2 \left(1 - \pi + \pi \frac{b_0}{f(\theta)}\right)^2}{\sum_{i=1}^{n} a_i} + \frac{E(n_0) \left(1 + \frac{b_0}{f(\theta)}\right)^2 \left(1 - \pi + \pi \frac{b_0}{f(\theta)}\right)^2}{\pi^2},
\]

\[
= \frac{n_0 \left(1 + \frac{b_0}{f(\theta)}\right)^2 \left(1 + \frac{b_0}{f(\theta)}\right)^2}{\sum_{i=1}^{n} a_i} + \frac{n \pi \left(1 - \frac{b_0}{f(\theta)}\right)^2}{\pi^2},
\]

\[
= \frac{n \left(1 + \frac{b_0}{f(\theta)}\right)}{\pi^2} + \frac{n \left(1 - \frac{b_0}{f(\theta)}\right)}{\pi^2},
\]

\[
= \frac{n(f(\theta) - b_0)}{\pi(f(\theta) - \pi f(\theta) + \pi b_0)}.
\]

Therefore, \(I_{11} = \frac{n(f(\theta) - b_0)}{\pi(f(\theta) - \pi f(\theta) + \pi b_0)}\).

Further differentiating Eq. (2.1.4) with respect to \(\theta\), we get
\[
\frac{\partial^2 \log L}{\partial \theta^2} = -n_0 \pi b_0 \left\{ \left( 1 - \pi + \pi \frac{b_0}{f(\theta)} \right) \left( \frac{f(\theta)^2 f''(\theta) - 2 f'(\theta)^2 f(\theta)}{f(\theta)^4} \right) \right. \\
- \frac{f'(\theta)^2}{f(\theta)^2} \left( -\pi \frac{b_0}{f(\theta)} \right) \right\} \\
\left( 1 - \pi + \pi \frac{b_0}{f(\theta)} \right)^2 \\
- \frac{\sum_{i=1}^n a_i x_i}{\theta^2} - \frac{\sum_{i=1}^n a_i (f(\theta)f''(\theta) - f'(\theta)^2)}{f(\theta)^2}.
\]

Therefore,

\[
- E \left( \frac{\partial^2 \log L}{\partial \theta^2} \right) = \\
E(n_0) \pi b_0 \left\{ \left( 1 - \pi + \pi \frac{b_0}{f(\theta)} \right) \left( \frac{f(\theta)^2 f''(\theta) - 2 f'(\theta)^2 f(\theta)}{f(\theta)^4} \right) \right. \\
- \frac{f'(\theta)^2}{f(\theta)^2} \left( -\pi \frac{b_0}{f(\theta)} \right) \right\} \\
\left( 1 - \pi + \pi \frac{b_0}{f(\theta)} \right)^2 \\
+ \frac{\sum_{i=1}^n a_i X_i}{\theta^2} + \frac{\sum_{i=1}^n a_i (f(\theta)f''(\theta) - f'(\theta)^2)}{f(\theta)^2},
\]

\[
n_0 \pi b_0 \left\{ \left( 1 - \pi + \pi \frac{b_0}{f(\theta)} \right) \left( \frac{f(\theta)^2 f''(\theta) - 2 f'(\theta)^2 f(\theta)}{f(\theta)^4} \right) \right. \\
- \frac{f'(\theta)^2}{f(\theta)^2} \left( -\pi \frac{b_0}{f(\theta)} \right) \right\} \\
\left( 1 - \pi + \pi \frac{b_0}{f(\theta)} \right)^2 \\
+ \frac{n \pi \theta f'(\theta)}{f(\theta) \theta^2} + \left\{ \frac{n \pi \left( 1 - \frac{b_0}{f(\theta)} \right)}{f(\theta)} \left( \frac{f(\theta)f''(\theta) - f'(\theta)^2}{f(\theta)^2} \right) \right\},
\]

\[
n_0 b_0 \left\{ \left( 1 - \pi + \pi \frac{b_0}{f(\theta)} \right) \left( \frac{f(\theta)^2 f''(\theta) - 2 f'(\theta)^2 f(\theta)}{f(\theta)^4} \right) \right. \\
+ \frac{f'(\theta)^2}{f(\theta)^2} \left( \frac{\pi b_0}{f(\theta)^2} \right) \right\} \\
\left( 1 - \pi + \pi \frac{b_0}{f(\theta)} \right)^2 \\
+ \frac{n \pi f'(\theta)}{f(\theta) \theta} + \left\{ \frac{n \pi \theta f'(\theta)}{f(\theta)} \frac{f(\theta)f''(\theta) - f'(\theta)^2}{f(\theta)^2} \right\},
\]
\[
I_{22} = \left[ n\pi b_0 \left( \frac{f(\theta)^2 f''(\theta) - 2f'(\theta)^2 f(\theta)}{f(\theta)^4} \right) + \frac{\pi b_0 f'(\theta)^2}{f(\theta)^4 \left( 1 - \pi + \frac{b_0}{f(\theta)} \right)} \right] \\
+ \frac{n\pi f'(\theta)}{\theta f(\theta)} + \frac{n\pi \left( 1 - \frac{b_0}{f(\theta)} \right) \left( f(\theta)f''(\theta) - f'(\theta)^2 \right)}{f(\theta)^2} \right].
\]

Hence,

\[
\frac{\partial^2 \log L}{\partial \pi \partial \theta} = \left[ n_0 \left( 1 - \pi + \frac{b_0}{f(\theta)} \right) \left( \frac{b_0 f'(\theta)}{f(\theta)^2} \right) - \left( -1 + \frac{b_0}{f(\theta)} \right) \left( \pi b_0 f'(\theta) \right) \right] \\
\times \left( 1 - \pi + \frac{b_0}{f(\theta)} \right)^{-2},
\]

Therefore,

\[
- \frac{E(n_0)}{\frac{\partial^2 \log L}{\partial \pi \partial \theta}} = \left[ n_0 \left( 1 - \pi + \frac{b_0}{f(\theta)} \right) \left( \frac{b_0 f'(\theta)}{f(\theta)^2} \right) - \left( -1 + \frac{b_0}{f(\theta)} \right) \left( \pi b_0 f'(\theta) \right) \right] \\
\times \left( 1 - \pi + \frac{b_0}{f(\theta)} \right)^{-2},
\]
\[
\frac{n\left(1 - \pi + \pi \frac{b_0}{f(\theta)}\right)\left(1 - \pi + \pi \frac{b_0}{f(\theta)}\right) b_0 f'(\theta) - \left(-1 + \frac{b_0}{f(\theta)}\right)^2 \left(-1 + \frac{b_0}{f(\theta)}\right)\pi b_0 f'(\theta)}{\left(1 - \pi + \pi \frac{b_0}{f(\theta)}\right)^2},
\]

\[
= \frac{n b_0 f'(\theta)}{f(\theta)^2 \left(1 - \pi + \pi \frac{b_0}{f(\theta)}\right)}\left(1 - \pi + \pi \frac{b_0}{f(\theta)}\right) - \pi \left(-1 + \frac{b_0}{f(\theta)}\right),
\]

\[
= \frac{n b_0 f'(\theta)}{f(\theta)^2 \left(1 - \pi + \pi \frac{b_0}{f(\theta)}\right)}.
\]

Therefore, \(I_{12} = \frac{n b_0 f'(\theta)}{f(\theta)(f(\theta) - \pi f(\theta) + \pi b_0)}\) \quad \ldots(2.1.11)

Assuming that Cramer-Huzurbazar conditions for asymptotic normality for maximum likelihood estimator are satisfied, we have following theorem:

**Theorem (2.1):** Let \(X_1, X_2, \ldots, X_n\) be a random sample from ZIPSD with parameters \(\pi\) and \(\theta\). Then the maximum likelihood estimators are obtained by solving Eq. (2.1.5) and Eq. (2.1.7). These maximum likelihood estimators have asymptotic bivariate normal distribution with mean vector \((\hat{\pi}, \hat{\theta})'\) and dispersion matrix \(I^{-1}(\hat{\theta})\) for \(n\) sufficiently large. That is as \(n \to \infty\),

\(\left(\sqrt{n}(\hat{\pi} - \pi), \sqrt{n}(\hat{\theta} - \theta)\right) \to N_2(0, I^{-1}(\hat{\theta}))\), where \(I^{-1}(\hat{\theta})\) is as defined in Eq. (2.1.8).

In ZIPSD, if we condition on number of zeros in the model, conditional distribution depends only on the parameter \(\theta\). If our interest is to infer about \(\theta\), it looks better to adopt such an approach. In the following we discuss the same.

**Conditional Likelihood Function**

We observe that the conditional density of \(X_i\) given \(A_i = a_i\) is independent of inflation parameter \(\pi\), since
\[ P(X_i = x_i | A_i = a_i) = \frac{P(X_i = x_i | A_i = a_i)}{P(A_i = a_i)} = \left( \frac{\frac{\pi b_{x_i} \theta^{x_i}}{f(\theta)}}{\pi \left(1 - \frac{b_0}{f(\theta)}\right)} \right)^{a_i} = \left( \frac{b_{x_i} \theta^{x_i}}{f(\theta) - b_0} \right)^{a_i}, \quad \ldots(2.1.12) \]

where \( a_i = 0 \) if \( x_i = 0 \) and \( a_i = 1 \) if \( x_i = 1,2,3,\ldots \). The marginal distribution of \( A = \sum_{i=1}^{n} a_i \) is Binomial with parameter \( n \) and \( \pi \left(1 - \frac{b_0}{f(\theta)}\right) \). We note that the distribution of \( A \) contains both the parameters. Now the conditional log-likelihood function based on Eq. (2.1.12) is given by

\[ \log L'(\theta; x) = \sum_{i=1}^{n} a_i \log \left( \frac{b_{x_i} \theta^{x_i}}{f(\theta) - b_0} \right), \quad \theta > 0. \]

\[ = \sum_{i=1}^{n} \log \left( \frac{b_{x_i} \theta^{x_i}}{f(\theta) - b_0} \right) \]

\[ = \sum_{i=1}^{n} \log b_{x_i} - \sum_{i=1}^{n} x_i \log \theta - \sum_{i=1}^{n} \log (f(\theta) - b_0) \quad \ldots(2.1.13) \]

The mle \( \tilde{\theta} \) of \( \theta \) is the solution to an equation,

\[ \bar{x} = \frac{\tilde{\theta} f'((\tilde{\theta}))}{f(\tilde{\theta}) - b_0}, \quad \ldots(2.1.14) \]

where \( \bar{x} = \frac{\sum_{i=1}^{n} x_i}{n-n_0} \) is the mean of the positive observations only. We note that mle of \( \theta \) based on standard likelihood (Eq. 2.1.7) and based on conditional likelihood (Eq. 2.1.14) are the same. Also it coincides with the one given by Kale (1998). In the following we obtain asymptotic variance of \( \tilde{\theta} \).

Differentiating log likelihood with respect to \( \theta \), we have

\[ \frac{\partial \log L'}{\partial \theta} = \sum_{i=1}^{n} \frac{x_i}{\theta} - \sum_{i=1}^{n} \frac{f'(\theta)}{(f(\theta) - b_0)} \]
and
\[
\frac{\partial^2 \log L^*}{\partial \theta^2} = -\sum_{i=1}^{n} \frac{x_i}{\theta^2} - \sum_{i=1}^{n} \frac{(f(\theta) - b_0)f''(\theta) - f'(\theta)^2}{(f(\theta) - b_0)^2}.
\]
Thus,
\[
- E\left(\frac{\partial^2 \log L^*}{\partial \theta^2}\right) = E\left(\sum_{i=1}^{n} \frac{x_i}{\theta^2}\right) + E\left(\sum_{i=1}^{n} \frac{(f(\theta) - b_0)f''(\theta) - f'(\theta)^2}{(f(\theta) - b_0)^2}\right),
\]
\[
= \frac{(n-n_0)}{(f(\theta)-b_0)} \left( \frac{f'(\theta)}{\theta} + \frac{f''(\theta)(f(\theta)-b_0)-f'(\theta)^2}{(f(\theta)-b_0)} \right) \quad \text{...(2.1.15)}
\]

Therefore, asymptotic variance of \( \hat{\theta} \) is different than the asymptotic variance of estimate of \( \theta \) based on the standard likelihood approach. The same is given by
\[
AV_{\hat{\theta}}(\theta) = \left( \frac{(n-n_0)}{(f(\theta)-b_0)} \left( \frac{f'(\theta)}{\theta} + \frac{f''(\theta)(f(\theta)-b_0)-f'(\theta)^2}{(f(\theta)-b_0)} \right) \right)^{-1} \quad \text{...(2.1.16)}
\]

Assuming that Cramer-Huzurbazar conditions required for asymptotic normality for maximum likelihood estimators are satisfied, we have following theorem:

**Theorem (2.2):** Let \( X_1, X_2, \ldots, X_n \) be a random sample from ZIPSD with parameters \( \pi \) and \( \theta \). Then the mle of \( \theta \) is the solution of the Eq. (2.1.14) and has asymptotic normal distribution with mean \( \theta \) and variance \( AV_{\hat{\theta}}(\theta) \) for \( n \) sufficiently large. That is as \( n \to \infty \), \( \left( \sqrt{n}(\hat{\theta} - \theta) \right) \to N_1(0, AV_{\hat{\theta}}(\theta)) \).

In the following we present moment estimator of ZIPSD.

**Moment Estimator of ZIPSD**

We have,
\[
E(X) = \pi \left( \frac{\theta f'(\theta)}{f(\theta)} \right) = \pi \mu(\theta), \quad \text{say}
\]
\[
E(X^2) = \frac{\theta \pi}{f(\theta)} \left( \theta f''(\theta) + f'(\theta) \right)
\]
and
We obtain moment estimators of $\pi$ and $\theta$ by solving,

$$\bar{X} = \pi \mu(\theta) \quad \text{...(2.1.18)}$$

and

$$\sum_{i=1}^{n} x_i^2 = \frac{\theta \pi}{f(\theta)} \left( \theta f''(\theta) + f'(\theta) \right) \cdot \quad \text{...(2.1.19)}$$

We know (using Central Limit Theorem) that sample mean is consistent and asymptotically normal for the populations mean. Therefore we have the following Theorem.

**Theorem (2.3):** Let $X_1, X_2, \ldots, X_n$ be a random sample from ZIPSD with parameters $\pi$ and $\theta$. Then the moment estimator of $\pi$ and $\theta$ are obtained by solving Eq. (2.1.18) and Eq. (2.1.19). The moment estimator of $\theta$ has asymptotic normal distribution with mean $\pi \mu(\theta)$ and variance $\frac{\sigma^2(\pi, \theta)}{n}$, for $n$ sufficiently large. That is as $n \to \infty$, $\left(\sqrt{n}(\hat{\theta} - \theta)\right) \to N_1\left(0, \frac{\sigma^2(\pi, \theta)}{n}\right)$.

Thus we have three estimators for $\theta$, which have asymptotic normal distribution with the mean $\theta$, but different variances. In the section below, we develop asymptotic tests for $\theta$, using these three estimators.

### 2.2 Tests for the Parameter $\theta$ of ZIPS Distribution

**Test Based On $\hat{\theta}$**

Suppose we wish to test $H_0: \theta = \theta_0$ vs $H_1: \theta \neq \theta_0$, Let us assume that $\pi$ is known. Therefore under $H_0$, from Theorem (2.1) we have

$$\left(\hat{\theta} - \theta_0\right) \sim AN\left(0, AV_\theta(\pi, \theta_0)\right). \quad \text{...(2.2.1)}$$
Define a test statistic \( Z_1 = \frac{\hat{\theta} - \theta_0}{\sqrt{AV_\theta(\pi, \theta_0)}} \). Based on \( Z_1 \) we define the test \( \psi_1 \) which rejects \( H_0 \) at \( \alpha \) level of significance, if \( |Z_1| > z_{1-\alpha/2} \), where \( z_{1-\alpha/2} \) is the upper \( 100(\alpha/2) \)th percentile of Standard Normal Variate (SNV).

Let \( \Phi(.) \) be the cumulative distribution function of SNV. Then the power of the test \( \psi_1 \) is given by

\[
\beta_{\psi_1}(\pi, \theta) = P(|Z_1| > z_{1-\alpha/2}),
\]

\[
= 1 - P\left\{ \frac{\hat{\theta} - \theta_0}{\sqrt{AV_\theta(\pi, \theta_0)}} < z_{1-\alpha/2} \right\},
\]

\[
= 1 - P\left\{ -z_{1-\alpha/2} < \frac{\hat{\theta} - \theta_0}{\sqrt{AV_\theta(\pi, \theta_0)}} < z_{1-\alpha/2} \right\},
\]

\[
= 1 - P\left( -z_{1-\alpha/2} \sqrt{AV_\theta(\pi, \theta_0)} < \hat{\theta} - \theta_0 < z_{1-\alpha/2} \sqrt{AV_\theta(\pi, \theta_0)} \right),
\]

\[
= 1 - P\left( \theta_0 - z_{1-\alpha/2} \sqrt{AV_\theta(\pi, \theta_0)} < \hat{\theta} < \theta_0 + z_{1-\alpha/2} \sqrt{AV_\theta(\pi, \theta_0)} \right),
\]

\[
= 1 - P\left\{ \frac{\theta_0 - \theta - z_{1-\alpha/2} \sqrt{AV_\theta(\pi, \theta_0)}}{\sqrt{AV_\theta(\pi, \theta)}} < \frac{\hat{\theta} - \theta}{\sqrt{AV_\theta(\pi, \theta)}} < \frac{\theta_0 - \theta + z_{1-\alpha/2} \sqrt{AV_\theta(\pi, \theta_0)}}{\sqrt{AV_\theta(\pi, \theta)}} \right\},
\]

\[
= 1 - \Phi(B) + \Phi(A)
\]

\[\text{...(2.2.2)}\]

where

\[
A = \frac{\theta_0 - \theta - z_{1-\alpha/2} \sqrt{AV_\theta(\pi, \theta_0)}}{\sqrt{AV_\theta(\pi, \theta)}} \quad \text{and} \quad B = \frac{\theta_0 - \theta + z_{1-\alpha/2} \sqrt{AV_\theta(\pi, \theta_0)}}{\sqrt{AV_\theta(\pi, \theta)}}.
\]

However, in practice \( \pi \) is unknown. Hence, we modify the test statistic by replacing \( \pi \) by its mle \( \hat{\pi}_0 \), when \( H_0 \) is true. By doing so we define test
Statistic $Z'_1 = \frac{\hat{\theta} - \theta_0}{\sqrt{AV_{\hat{\theta}}(\hat{\pi}_0, \theta_0)}}$, where $\hat{\pi}_0 = \frac{(n - n_0)f(\theta_0)}{n(f(\theta_0) - b_0)}$. Based on $Z'_1$, we propose a test $\psi'_1$, which rejects $H_0$ at $\alpha$ level of significance, if $|Z'_1| > z_{1-\alpha/2}$.

The power of this test is given by

$$\beta_{\psi'_1}(\theta, \pi) = \sum_{k=0}^{n}(1 - \Phi(\hat{B}_k) + \Phi(\hat{A}_k))P(n_0 = k), \quad \ldots(2.2.3)$$

where

$$\hat{A}_k = \theta_0 - \theta - z_{1-\alpha/2} \sqrt{AV_{\hat{\theta}}(\hat{\pi}_0, \theta_0)}, \quad \hat{B}_k = \theta_0 - \theta + z_{1-\alpha/2} \sqrt{AV_{\hat{\theta}}(\hat{\pi}_0, \theta_0)},$$

$$P(n_0 = k) = \binom{n}{k} P_0^k (1 - P_0)^{n-k}, \quad \text{with} \quad P_0 = \left(1 - \pi + \pi \frac{b_0}{f(\theta)}\right). \quad \ldots(2.2.4)$$

Below, we develop test based on $\tilde{\theta}$, estimator based on conditional likelihood approach.

**Test Based On $\tilde{\theta}$**

Theorem (2.2) gives

$$\left(\tilde{\theta} - \theta_0\right) \sim AN(0, AV_{\hat{\theta}}(\theta_0)). \quad \ldots(2.2.5)$$

Hence, we define test statistic $Z_2 = \frac{\tilde{\theta} - \theta_0}{\sqrt{AV_{\hat{\theta}}(\theta_0)}}$. Test based on $Z_2$ is $\psi_2$ which rejects $H_0$ at $\alpha$ level of significance, if $|Z_2| > z_{1-\alpha/2}$.

The power of the test $\psi_2$ is given by

$$\beta_{\psi_2}(\theta, \pi) = 1 - \Phi(B) + \Phi(A) \quad \ldots(2.2.6)$$

where,

$$A = \frac{\theta_0 - \theta - z_{1-\alpha/2} \sqrt{AV_{\hat{\theta}}(\theta_0)}}{\sqrt{AV_{\hat{\theta}}(\theta)}}, \quad B = \frac{\theta_0 - \theta + z_{1-\alpha/2} \sqrt{AV_{\hat{\theta}}(\theta_0)}}{\sqrt{AV_{\hat{\theta}}(\theta)}}.$$
Test Based On the Moment Estimator $\tilde{\theta}$ of $\theta$

It is clear that the problem of testing $H_0 : \theta = \theta_0$ vs $H_1 : \theta \neq \theta_0$ is equivalent to testing $H_0 : \mu(\theta) = \mu(\theta_0)$ vs $H_1 : \mu(\theta) \neq \mu(\theta_0)$, where $\mu(\theta) = \frac{\theta f'(\theta)}{f(\theta)}$. We have from Theorem (2.3), sample mean is consistent and asymptotically normal for the population mean.

That is $\bar{X} \sim AN \left( \pi \mu(\theta), \frac{\sigma^2(\pi, \theta)}{n} \right)$.

Therefore, under $H_0$, we have

$$
\sqrt{n} \left( \frac{\bar{X}}{\pi} - \mu(\theta_0) \right) \sim AN \left( 0, \frac{\sigma^2(\pi, \theta_0)}{\pi^2} \right).
$$

Define test statistic

$$
Z_3 = \frac{\sqrt{n} \left( \frac{\bar{X}}{\pi} - \mu(\theta_0) \right)}{\sigma^2(\pi, \theta_0)} \sim N(0,1), \text{ when } \pi \text{ is known.}
$$

The test $\psi_3$ rejects $H_0$ at $\alpha$ level of significance if $|Z_3| > z_{1-\alpha/2}$.

That is, reject $H_0$ if

$$
\frac{\sqrt{n} \left( \frac{\bar{X}}{\pi} - \mu(\theta_0) \right)}{\sqrt{\frac{\sigma^2(\pi, \theta_0)}{n}}} > z_{1-\alpha/2}.
$$

The power of the test $\psi_3$ is given by

$$
\beta_{\psi_3}(\theta, \pi) = 1 - \Phi(B') + \Phi(A') \quad \text{...(2.2.7)}
$$

where

$$
A' = \frac{\pi \left( \mu(\theta_0) - z_{1-\alpha/2} \sqrt{\frac{\sigma^2(\pi, \theta_0)}{n \pi^2}} \right) - \pi \mu(\theta)}{\sqrt{\frac{\sigma^2(\pi, \theta)}{n}}}
$$

and $B' = \frac{\pi \mu(\theta_0)}{\sqrt{\frac{\sigma^2(\pi, \theta)}{n}}}$.
\[
\pi \left( \mu(\theta_0) + z_{1-\alpha/2} \sqrt{\frac{\sigma^2(\pi, \theta_0)}{n \pi^2}} \right) - \pi \mu(\theta) \quad \text{and} \quad B' = \frac{z_{1-\alpha/2} \sqrt{\frac{\sigma^2(\pi, \theta)}{n \pi^2}}}{\sqrt{\frac{\sigma^2(\pi, \theta)}{n}}}.
\]

If \( \pi \) is unknown, we modify the test statistic by replacing \( \pi \) by its estimate \( \hat{\pi}_0 \) under \( H_0 \). By doing so we define test statistic

\[
Z'_3 = \frac{\sqrt{n} \left( \frac{\bar{X}}{\hat{\pi}_0} - \mu(\theta_0) \right)}{\sqrt{\frac{\sigma^2(\hat{\pi}_0, \theta_0)}{\hat{\pi}_0^2}}}.
\]

where \( \hat{\pi}_0 \) is given by \( \hat{\pi}_0 = \frac{\bar{X}}{\mu(\theta_0)} \).

Based on \( Z'_3 \), we propose test \( \psi'_3 \) which rejects \( H_0 \) at \( \alpha \) level of significance if

\[
|Z'_3| > Z_{1-\alpha/2}
\]

The power of the test is given by

\[
\beta_{\psi'_3}(\pi, \theta) = \sum_{k=0}^{n} (1 - \Phi(B''_k) + \Phi(A''_k))P(n_0 = k),
\]

where

\[
A''_k = \frac{\hat{\pi}_0 \left( \mu(\theta_0) - z_{1-\alpha/2} \sqrt{\frac{\sigma^2(\hat{\pi}_0, \theta_0)}{n \pi^2}} \right) - \pi \mu(\theta)}{\sqrt{\frac{\sigma^2(\pi, \theta)}{n}}},
\]

\[
B''_k = \frac{\hat{\pi}_0 \left( \mu(\theta_0) + z_{1-\alpha/2} \sqrt{\frac{\sigma^2(\hat{\pi}_0, \theta_0)}{n \pi^2}} \right) - \pi \mu(\theta)}{\sqrt{\frac{\sigma^2(\pi, \theta)}{n}}}
\]

and \( P(n_0 = k) = \binom{n}{k} P_0^k (1 - P_0)^{n-k} \), with \( P_0 = \left( 1 - \pi + \pi \frac{a_0}{f(\theta)} \right) \).
Using the tests developed above, we can define two sided asymptotic confidence intervals for \( \theta \), by inverting acceptance regions of the tests appropriately. Below we report the same.

**Asymptotic Confidence Interval for \( \theta \)**

Asymptotic confidence interval for \( \theta \) based on the test \( \psi_1 \) is given by

\[
\left( \hat{\theta} - z_{1-\alpha/2} \sqrt{AV_\theta(\hat{\theta}, \hat{\theta})}, \quad \hat{\theta} + z_{1-\alpha/2} \sqrt{AV_\theta(\hat{\theta}, \hat{\theta})} \right)
\]

where, \( AV_\theta(\hat{\theta}, \hat{\theta}) \) is an estimate of asymptotic variance of \( \hat{\theta} \) and asymptotic confidence interval for \( \theta \) based on the test \( \psi_2 \) is given by

\[
\left( \hat{\theta} - z_{1-\alpha/2} \sqrt{AV_\theta(\hat{\theta}, \hat{\theta})}, \quad \hat{\theta} + z_{1-\alpha/2} \sqrt{AV_\theta(\hat{\theta}, \hat{\theta})} \right)
\]

where \( AV_\theta(\hat{\theta}, \hat{\theta}) \) is an estimate of \( AV_\theta(\hat{\theta}) \), which is given in the Eq. (2.1.16).

Asymptotic confidence interval for \( \theta \) based on the test \( \psi_3 \) is given by

\[
\left( \frac{\bar{X}}{\hat{\pi}} - z_{1-\alpha/2} \sqrt{AV_\theta(\hat{\theta}, \hat{\theta})}, \quad \frac{\bar{X}}{\hat{\pi}} + z_{1-\alpha/2} \sqrt{AV_\theta(\hat{\theta}, \hat{\theta})} \right)
\]

where, \( AV_\theta(\hat{\theta}, \hat{\theta}) = \frac{\sqrt{n} \left( \frac{\bar{X}}{\pi} - \mu(\theta) \right)}{\sigma^2(\pi, \theta)} \).

In the following section we consider a zero-inflated model for a more general class of discrete distributions known as Modified Power Series Distributions.

### 2.3 Modified Zero-Inflated Power Series Distribution

Zero-inflated model for a more general class of discrete distributions known as Modified Power Series Distributions (MPSD) is introduced by Gupta (1974). This class includes among others the generalized Poisson, generalized negative binomial distributions. The generalized Poisson distribution possesses properties of overdispersion and underdispersion which make it a very good descriptive model in the field of genetics, ecology and many others. Generalized negative binomial distribution has been found
useful in queuing theory, branching process, random walk and for a survey article on MPSD. Gupta et al. (1995) studied the Zero-Inflated Modified Power Series (ZIMPSD) distribution with the structural properties, maximum likelihood estimators and variance covariance matrix. They have also obtained the confidence interval for the parameters involved in the model. Murat and Szynal (1998) studied the class of inflated modified power series distributions where inflation occurs at any of the support points. Moments, factorial moments, central moments, the maximum likelihood estimators, variance-covariance matrix of the estimators are obtained. Murat and Szynal (1998) extended the results of Gupta et al. (1995) to the discrete distributions inflated at any point 's' in the support. Here, we provide the inference of the model using full likelihood approach and conditional likelihood approach.

**Modified Zero-Inflated Power Series Distribution**

The probability mass function of MZIPSD is given by,

\[
P(X = x) = \begin{cases} 
1 - \pi + \frac{\pi b_0}{f(\theta)}, & \text{for } x = 0, \\
\frac{\pi b_x (g(\theta))^x}{f(\theta)}, & \text{for } x = 1, 2, \ldots, \text{ and } 0 < \pi < 1
\end{cases}
\]

\[\text{...(2.3.1)}\]

**Estimation of the Parameters Using Full Likelihood Function**

Suppose a random sample \(X_1, X_2, \ldots, X_n\) of size \(n\) from MZIPSD is available. Then the log likelihood function is given by

\[
\log L(\theta, \pi ; x) = n_0 \log \left(1 - \pi + \frac{\pi b_0}{f(\theta)}\right) + \sum_{i=1}^{n} a_i \log \pi + \sum_{i=1}^{n} a_i \log bx_i + \sum_{i=1}^{n} a_i x_i \log (g(\theta)) - \sum_{i=1}^{n} a_i \log f(\theta)
\]

\[\text{...(2.3.2)}\]

Maximum likelihood estimators of \(\theta\) and \(\pi\) are obtained by solving the following two equations
\[ \hat{\pi} = \frac{(n - n_0) f(\hat{\theta})}{n \left(f(\hat{\theta}) - b_0\right)} \]  
\[ \sum_{i=1}^{n} a_i x_i \frac{n_0 \hat{\pi} b_0 f'(\hat{\theta})}{f(\hat{\theta})^2 \left(1 - \hat{\pi} + \hat{\pi} \frac{b_0}{f(\hat{\theta})}\right)} + \sum_{i=1}^{n} a_i f'(\hat{\theta}) \frac{\hat{\pi}}{f(\hat{\theta})}, \]

Substituting \( \hat{\pi} = \frac{(n - n_0) f(\hat{\theta})}{n \left(f(\hat{\theta}) - b_0\right)} \) in Eq. (2.3.4) we get

\[ \bar{x} = \frac{g(\hat{\theta}) f'(\hat{\theta})}{g'(\hat{\theta}) \left(f(\hat{\theta}) - b_0\right)}, \]

which is non-linear equation in \( \theta \). Using Newton–Raphson method first we find \( \hat{\theta} \), substituting this value of \( \hat{\theta} \) in Eq. (2.3.3) we find \( \hat{\pi} \). The Fisher information matrix of \( \delta = (\pi, \theta)' \) is given by

\[
I(\delta) = \begin{pmatrix}
-E \left( \frac{\partial^2 \log L}{\partial \pi^2} \right) & -E \left( \frac{\partial^2 \log L}{\partial \pi \partial \theta} \right) \\
-E \left( \frac{\partial^2 \log L}{\partial \pi \partial \theta} \right) & -E \left( \frac{\partial^2 \log L}{\partial \theta^2} \right)
\end{pmatrix} = \begin{pmatrix}
I_{11} & I_{12} \\
I_{21} & I_{22}
\end{pmatrix}
\]

where

\[
I_{11} = \frac{n \left(f(\theta) - b_0\right)}{\pi \left(f(\theta) - \pi f(\theta) + \pi b_0\right)}, \]

\[
I_{12} = \frac{n b_0 f'(\theta)}{f(\theta) \left(f(\theta) - \pi f(\theta) + \pi b_0\right)},
\]

and,
Assuming that conditions required for asymptotic normality for maximum likelihood estimators are satisfied, we have following theorem:

**Theorem (2.4):** Let $X_1, X_2, \ldots, X_n$ be a random sample from MZIPSD with parameters $\pi$ and $\theta$. Then the maximum likelihood estimator obtained by solving Eq. (2.3.3) and Eq. (2.3.5), have asymptotic bivariate normal distribution with mean vector $(\pi, \theta)'$ and dispersion matrix $I^{-1}(\delta)$ for $n$ sufficiently large. That is as $n \to \infty$, $(\sqrt{n}\hat{\pi} - \pi, \sqrt{n}(\hat{\theta} - \theta)) \to N_2(0, I^{-1}(\delta)))$.

In the following we present conditional likelihood approach and obtain mles for $\theta$.

**Conditional Likelihood Function Approach**

We observe that the conditional density of $X_i$ given $A_i = a_i$ is independent of inflation parameter $\pi$, since

$$P(X_i = x_i | A_i = a_i) = \left( \frac{b_{x_i} (g(\theta))^5}{f(\theta) - b_0} \right)^{a_i} \quad \text{ \ldots (2.3.9)}$$

Now the conditional log likelihood function is given by

$$\log L^*(\theta; X) = \sum_{i=1}^{n-a_0} \log b_{x_i} + \sum_{i=1}^{n-a_0} x_i \log(g(\theta)) - \sum_{i=1}^{n-a_0} \log(f(\theta) - b_0) \quad \text{ \ldots (2.3.10)}$$

The mle $\hat{\theta}$ of $\theta$ is the solution to an equation
where \( \bar{x} = \frac{\sum_{i=1}^{n-n_0} x_i}{n-n_0} \) is the mean of the positive observations only. We note that mle of \( \theta \) based on full likelihood (Eq. 2.3.5) and based on conditional likelihood (Eq. 2.3.11) are the same. Now,

\[
-\mathbb{E} \left( \frac{\partial^2 \log L^*}{\partial \theta^2} \right) = (n-n_0) \left\{ \frac{g'(\theta)f'(g(\theta))}{g'(\theta)(f(\theta)-b_0)} \left( \frac{g'(\theta)^2 - g(\theta)g''(\theta)}{g(\theta)^2} \right) + \frac{f''(\theta)(f(\theta)-b_0) - f'(\theta)^2}{(f(\theta)-b_0)^2} \right\}
\]

and \( AV_{\tilde{\theta}}(\theta) = \left( (n-n_0) \left\{ \frac{g'(\theta)f'(\theta)}{g'(\theta)(f(\theta)-b_0)} \left( \frac{g'(\theta)^2 - g(\theta)g''(\theta)}{g(\theta)^2} \right) + \frac{f''(\theta)(f(\theta)-b_0) - f'(\theta)^2}{(f(\theta)-b_0)^2} \right\} \right)^{-1} \)

Therefore, asymptotic variance of \( \tilde{\theta} \) is different from the asymptotic variance of estimate of \( \theta \) based on the full likelihood approach.

Assuming that Cramer-Huzurbazar conditions required for asymptotic normality for mles are satisfied, we have following theorem:

**Theorem (2.5):** Let \( X_1, X_2, \ldots, X_n \) be a random sample from MZIPSD with parameters \( \pi \) and \( \theta \). Then the mle of \( \theta \) is solution to the Eq. (2.3.11) and has asymptotic normal distribution with mean \( \theta \) and variance \( AV_{\tilde{\theta}}(\theta) \) for \( n \) sufficiently large. That is as \( n \to \infty \), \( \sqrt{n}(\hat{\theta} - \theta) \to N_1(0, AV_{\tilde{\theta}}(\theta)) \).

In the following we present moment estimator of MZIPSD.

**Moment Estimator of MZIPSD**

We have,

\[
E(X) = \frac{\pi g(\theta)f'(g(\theta))}{f(\theta)} = \pi \mu(\theta)
\]
\[
E(X^2) = \frac{g(\theta) \pi}{f(\theta)} \left( g(\theta) f''(g(\theta)) + f'(g(\theta)) \right) \quad \text{and}
\]
\[
\text{Var}(X) = \frac{g(\theta) \pi}{f(\theta)} \left( g(\theta) f''(g(\theta)) + f'(g(\theta)) - \frac{\pi g(\theta) (f'(g(\theta)))^2}{f(\theta)} \right),
\]
\[= \sigma^2(\pi, \theta) \quad \text{say.}\]

Let,
\[
\bar{X} = \pi \mu(\theta)
\]
\[
\sum_{i=1}^{n} x_i^2 = \frac{\pi g(\theta)}{f(\theta)} \left( g(\theta) f''(g(\theta)) + f'(g(\theta)) \right),
\]
\[\quad \text{...(2.3.14)}\]

Solving Eq. (2.3.13) and Eq. (2.3.14) we get moment estimators of \(\pi\) and \(\theta\).

**Theorem (2.6):** Let \(X_1, X_2, \ldots, X_n\) be a random sample from MZIPSD with parameters \(\pi\) and \(\theta\). Then the moment estimator of \(\pi\) and \(\theta\) are obtained by solving in the Eq. (2.3.13) and Eq. (2.3.14). The moment estimator of \(\theta\) has asymptotic normal distribution with mean \(\pi \mu(\theta)\) and variance \(\frac{\sigma^2(\pi, \theta)}{n}\), for \(n\) sufficiently large. That is as \(n \to \infty\), \(\sqrt{n}(\hat{\theta} - \theta) \to N(0, \frac{\sigma^2(\pi, \theta)}{n})\).

### 2.4 Tests for the Parameter \(\theta\) of MZIPS Distribution

**Test Based On \(\hat{\theta}\)**

Suppose we wish to test \(H_0: \theta = \theta_0\) vs \(H_1: \theta \neq \theta_0\). Let us assume that \(\pi\) is known. Therefore, under \(H_0\), from Theorem (2.4) we have
\[
\left( \hat{\theta} - \theta_0 \right) \sim \text{AN}(0, AV_\theta(\pi, \theta_0)). \quad \text{...(2.4.1)}
\]

Define a test statistic to be \(Z_4 = \frac{\hat{\theta} - \theta_0}{\sqrt{AV_\theta(\pi, \theta_0)}}\). Based on \(Z_4\) we define the test \(\psi_4\) which rejects \(H_0\) at \(\alpha\) level of significance, if \(|Z_4| > z_{1-\alpha/2}\), where \(z_{1-\alpha/2}\) is the upper \(100(\alpha/2)^{th}\) percentile of \(\text{SNV}\).
Let $\Phi(.)$ be the cumulative distribution function of SNV. Then the power of the test $\psi_4$ is given by

$$
\beta_{\psi_4}(\pi, \theta) = 1 - \Phi(B) + \Phi(A),
$$

where

$$
A = \frac{\theta_0 - \theta - z_{1-\alpha/2} \sqrt{AV_0(\pi, \theta_0)}}{\sqrt{AV_0(\pi, \theta)}}, \\
B = \frac{\theta_0 - \theta + z_{1-\alpha/2} \sqrt{AV_0(\pi, \theta_0)}}{\sqrt{AV_0(\pi, \theta)}}.
$$

However, in practice $\pi$ is unknown. Hence we modify the test statistic by replacing $\pi$ by its maximum likelihood estimator ($\hat{\pi}_0$), when $H_0$ is true. By doing so, we define test $Z'_4 = \frac{\hat{\theta} - \theta_0}{\sqrt{AV_0(\hat{\pi}_0, \theta_0)}}$, where $\hat{\pi}_0 = \frac{(n - n_0)f(\theta_0)}{n(f(\theta_0) - b_0)}$.

Based on $Z'_4$, we propose a test $\psi'_4$ rejects $H_0$ at $\alpha$ level of significance, if $|Z'_4| > Z_{1-\alpha/2}$.

The power of this test is given by

$$
\beta_{\psi_4}(\pi, \theta) = \sum_{k=0}^{n} \left( 1 - \Phi(\hat{B}_k) + \Phi(\hat{A}_k) \right) P(n_0 = k), \quad \text{... (2.4.2)}
$$

where

$$
\hat{B}_k = \frac{\theta_0 - \theta - z_{1-\alpha/2} \sqrt{AV_0(\hat{\pi}_0, \theta_0)}}{\sqrt{AV_0(\pi, \theta)}}, \\
\hat{A}_k = \frac{\theta_0 - \theta - z_{1-\alpha/2} \sqrt{AV_0(\hat{\pi}_0, \theta_0)}}{\sqrt{AV_0(\pi, \theta)}},
$$

$$
P(n_0 = k) = \binom{n}{k} p_0^k (1 - p_0)^{n-k}, \text{with } p_0 = \left( 1 - \pi + \pi \frac{b_0}{f(\theta)} \right). \quad \text{... (2.4.3)}
$$

Below we develop test based on $\tilde{\theta}$, estimator based on conditional likelihood approach.

**Test Based On $\tilde{\theta}$**

Theorem (2.5) gives

$$
(\tilde{\theta} - \theta_0) \sim \text{AN}(0, AV_0(\theta_0)). \quad \text{... (2.4.4)}
$$
Hence, we define test statistic \( Z_5 = \frac{\tilde{\theta} - \theta_0}{\sqrt{AV_{\tilde{\theta}}(\theta_0)}} \). A test based on \( Z_5 \) which rejects \( H_0 \) at \( \alpha \) level of significance, if \( |Z_5| > z_{1-\alpha/2} \).

The power of the test \( \psi \) is given by
\[
\beta_{\psi}(\pi, \theta) = 1 - \Phi(B) + \Phi(A),
\]
where,
\[
A = \frac{\theta_0 - \theta - z_{1-\alpha/2} \sqrt{AV_{\tilde{\theta}}(\theta_0)}}{\sqrt{AV_{\tilde{\theta}}(\theta)}},
B = \frac{\theta_0 - \theta + z_{1-\alpha/2} \sqrt{AV_{\tilde{\theta}}(\theta_0)}}{\sqrt{AV_{\tilde{\theta}}(\theta)}}.
\]

**Test Based on the Moment Estimator \( \tilde{\theta} \) of \( \theta \)**

It is clear that the problem of testing \( H_0 : \theta = \theta_0 \) vs \( H_1 : \theta \neq \theta_0 \) is equivalent to testing \( H_0 : \mu(\theta) = \mu(\theta_0) \) vs \( H_1 : \mu(\theta) \neq \mu(\theta_0) \), where
\[
\mu(\theta) = \frac{g(\theta)f'(g(\theta))}{f(\theta)}.
\]
We have from Theorem (2.6), sample mean is consistent and asymptotically normal for the population mean.

That is \( \bar{X} \sim \mathcal{N} \left( \pi \mu(\theta), \frac{\sigma^2(\pi, \theta)}{n} \right) \).

Therefore, under \( H_0 \), we have
\[
\sqrt{n} \left( \frac{\bar{X}}{\pi} - \mu(\theta_0) \right) \sim \mathcal{N} \left( 0, \frac{\sigma^2(\pi, \theta_0)}{\pi^2} \right).
\]

Define test statistic
\[
Z_6 = \frac{\sqrt{n} \left( \frac{\bar{X}}{\pi} - \mu(\theta_0) \right)}{\frac{\sigma^2(\pi, \theta_0)}{\pi^2}} \sim \mathcal{N} (0,1), \text{ when } \pi \text{ is known.}
\]

The test \( \psi \) rejects \( H_0 \) at \( \alpha \) level of significance if \( |Z_6| > z_{1-\alpha/2} \).

That is, reject \( H_0 \) if
\[
\left( \sqrt{n} \left( \frac{\bar{X}}{\pi} - \mu(\theta_0) \right) \right) > z_{1-\alpha/2}.
\]
The power of the test $\psi_6$ is given by

$$\beta_{\psi_6}(\pi, \theta) = 1 - \Phi(B') + \Phi(A'), \quad \ldots(2.4.6)$$

where $A' = \frac{\pi \left( \mu(\theta_0) - z_{1-\alpha/2} \sqrt{\frac{\sigma^2(\pi, \theta_0)}{n \pi^2}} \right) - \pi \mu(\theta)}{\sqrt{\frac{\sigma^2(\pi, \theta)}{n}}}$

and

$B' = \frac{\pi \left( \mu(\theta_0) + z_{1-\alpha/2} \sqrt{\frac{\sigma^2(\pi, \theta_0)}{n \pi^2}} \right) - \pi \mu(\theta)}{\sqrt{\frac{\sigma^2(\pi, \theta)}{n}}},$

If $\pi$ is unknown, we modify the test statistic by replacing $\pi$ by its estimate $(\hat{\pi}_0)$ under $H_0$. By doing so, we define test statistic

$$Z'_6 = \frac{\sqrt{n} \left( \frac{X}{\hat{\pi}_0} - \mu(\theta_0) \right)}{\sqrt{\frac{\sigma^2(\hat{\pi}_0, \theta_0)}{\hat{\pi}_0^2}}}, \quad \ldots(2.4.7)$$

where $\hat{\pi}_0$ is given by $\hat{\pi}_0 = \frac{X}{\mu(\theta_0)}$.

Based on $Z'_6$ we propose a test $\psi'_6$ which rejects $H_0$ at $\alpha$ level of significance if $|Z'_6| > z_{1-\alpha/2}$. The power of the test is given by

$$\beta_{\psi'_6}(\pi, \theta) = \sum_{k=0}^n (1 - \Phi(B_k'')) + \Phi(A_k'')P(n_0 = k), \quad \ldots(2.4.8)$$

where

$$A_k'' = \frac{\pi \left( \mu(\theta_0) - z_{1-\alpha/2} \sqrt{\frac{\sigma^2(\hat{\pi}_0, \theta_0)}{n \pi^2}} \right) - \pi \mu(\theta)}{\sqrt{\frac{\sigma^2(\pi, \theta)}{n}}}$$. 


\[ B_k^n = \frac{\hat{\mu}(\theta_0) + z_{1 - \alpha/2} \sqrt{\frac{\sigma^2(\hat{\mu}_0, \theta_0)}{n\pi^2}} - \pi \mu(\theta)}{\sqrt{\frac{\sigma^2(\pi, \theta)}{n}}} \]

and \( P(n_0 = k) = \left( \frac{n}{k} \right)^P \left( 1 - P_0 \right)^{n-k} \), with \( P_0 = \left( 1 - \pi + \pi \frac{a_0}{f(\theta)} \right) \).

Using the tests developed above, we can define two sided asymptotic confidence intervals for \( \theta \), by inverting acceptance regions of the tests appropriately. Below we report the same.

**Asymptotic Confidence Interval for the Parameter \( \theta \)**

Asymptotic confidence interval for \( \theta \) based on the test \( \psi_4 \) is given by

\[
\left( \hat{\theta} - z_{1 - \alpha/2} \sqrt{AV(\hat{\pi}, \hat{\theta})}, \hat{\theta} + z_{1 - \alpha/2} \sqrt{AV(\hat{\pi}, \hat{\theta})} \right) \quad \ldots(2.4.9)
\]

where, \( AV(\hat{\pi}, \hat{\theta}) \) is an estimate of asymptotic variance of \( \hat{\theta} \) and asymptotic confidence interval for \( \theta \) based on the test \( \psi_5 \) is given by

\[
\left( \tilde{\theta} - z_{1 - \alpha/2} \sqrt{AV(\tilde{\pi}, \tilde{\theta})}, \tilde{\theta} + z_{1 - \alpha/2} \sqrt{AV(\tilde{\pi}, \tilde{\theta})} \right) \quad \ldots(2.4.10)
\]

where \( AV(\tilde{\pi}, \tilde{\theta}) \) is an estimate of the asymptotic variance of \( \tilde{\theta} \) as given in the Eq. (2.3.12).

Asymptotic confidence interval for \( \theta \) based on the test \( \psi_6 \) is given by

\[
\left( \frac{X}{\hat{\pi}} - z_{1 - \alpha/2} \sqrt{AV(\hat{\pi}, \tilde{\theta})}, \frac{X}{\hat{\pi}} + z_{1 - \alpha/2} \sqrt{AV(\hat{\pi}, \tilde{\theta})} \right), \quad \ldots(2.4.11)
\]

where \( AV(\hat{\pi}, \tilde{\theta}) = \sqrt{n} \left( \frac{X}{\pi} - \mu(\theta) \right) \).

If the data set under study does not contain observations after some known point in the support, we have to modify ZIPSD accordingly in order to
get better inferential properties. Modifying ZIPSD in this way leads to a truncated ZIPSD. Below we discuss the same.

2.5 Zero-Inflated Truncated Power Series Distribution (ZITPSD)

Before we define truncated ZIPSD, we first consider the Truncated Power Series Distribution (TPSD) truncated at the support point 't' onwards, where 't' is known. That is, the data set under study does not contain any observation after the point 't', where 't' is the point in the support. Then the probability mass function of TPSD is given by

\[ P(X = x) = \frac{b_x \theta^x}{f(\theta)(1 - P(X > t))}, \quad \text{for } x = 0, 1, 2, \ldots, t \]

\[ = \frac{b_x \theta^x}{f(\theta) \sum_{y=0}^{t} b_y \theta^y} \]

\[ = \frac{b_x \theta^x}{\left(\sum_{y=0}^{t} b_y \theta^y\right)}, \quad \text{where } G(\theta) = \sum_{y=0}^{t} b_y \theta^y \]

It is clear that the truncated distribution is also a Power series distribution. Based on the same, we define ZITPSD as follows:

Let the probability mass function of a random variable X is given by

\[ P(X = x) = \begin{cases} 
1 - \pi + \pi \frac{b_0}{G(\theta)} & \text{for } x = 0 \\
\pi \frac{b_x \theta^x}{G(\theta)} & \text{for } x = 1, 2, 3, \ldots, t. 
\end{cases} \]

\[ \quad \text{...(2.5.1)} \]

where \( G(\theta) = \sum_{y=0}^{t} b_y \theta^y \)
We can see that, the probability mass function of ZITPSD is similar to the probability mass function of ZIPSD (Eq. 1.2.2). Therefore all the tests and confidence intervals developed for ZIPSD can be applied to the ZITPSD distributions easily. We omit the details for brevity.

In the following chapter we study inference for zero-inflated Poisson distribution and zero-inflated Truncated Poisson distribution using results reported in this chapter.