Chapter 3
Minimum Variance Unbiased Estimation and
Bayesian Estimation in Weibull Distribution Under
Progressive Type II Censored Data with Binomial
Removals

3.1 Introduction

The Weibull distribution is named after the Swedish physicist, Waloddi
Weibull who (1939 a,b) used it to represent the distribution of the breaking strength of
materials. In the Russian statistical literature this distribution is often referred to as the
Weibull-Gnedenko distribution. The Weibull distribution is one of the most
popular and widely used models of failure time in life testing and reliability
theory. The Weibull distribution has been shown to be useful for modeling and
analysis of life time data in medical, biological and engineering sciences.
Applications of the Weibull distribution in various fields are given in Zaharim et
al. (2008), Green et al. (1994) etc. Lieblein and Zelen (1956), Berretoni (1964), and
Nelson (1972) used it to describe the life length of ball bearings, electron tubes,
manufactured items, and electrical insulation, respectively. Pike (1966) and Peto and
Lee (1972) have given a theoretical motivation for its consideration in representing
time to appearance of tumor or until death in animals which were subjected to
carcinogenic insults over time (the multi-hit theory). Inference for Weibull distribution
based on progressively Type II censored data were discussed by many authors.
Sultan et al. (2007) have obtained approximate best linear unbiased estimates of the
location and scale parameters. They have also derived approximate maximum
likelihood estimates of these parameters. Tse and Yuen (1998) have obtained a
formula for expected experiment times for Weibull model under progressive
censoring with random removals. Tse and Yang (2003) gave reliability sampling
plan for Weibull distribution based on progressively Type II censored data with
binomial removals. The problem of interval estimation for parameters was considered
by Tse and Xiang (2003). They have considered seven different confidence interval
estimation procedures. Sarhan and Ruzaizaa (2010) have obtained maximum
likelihood estimates of the parameters. They also derived point and interval estimates of the parameters.

In this Chapter we assume that the lifetime has Weibull distribution with scale parameter $\alpha$ and known shape parameter $\beta$, and removal distribution is binomial. The inferences are based on progressive Type II censoring data with binomial removals. In Section 3.2 the conditional likelihood function is given. In Section 3.3 the UMVUE of parameter of $\alpha$ and its functions are derived. In Section 3.4 the UMVUE of $P(X < Y)$ is derived. In Section 3.5 the $(1-\alpha)100\%$ confidence interval for $p$th quantile is obtained. In Section 3.6 Bayesian estimators of the parameters, the reliability and hazard function are derived under various loss functions. The results of this chapter have been published in “Far East Journal of Theoretical Statistics” Vol. 34 (2), 2011, 109-122.

3.2 The model

Let the failure time distribution be Weibull with probability density function,

$$f(x) = \alpha \beta x^{\beta - 1} \exp\left[-\alpha x^\beta\right], \quad \alpha > 0, \ \beta > 0, \ x \geq 0. \quad (3.2.1)$$

where $\alpha$ and $\beta$ are scale and shape parameters respectively.

The cumulative distribution function is given by,

$$F(x) = 1 - \exp\left[-\alpha x^\beta\right], \quad x \geq 0. \quad (3.2.2)$$

The survival function is given by,

$$S(x) = \exp\left[-\alpha x^\beta\right], \quad x \geq 0 \quad (3.2.3)$$

The density given in (3.2.1) can be written as,

$$f(x) = \frac{a(x)[h(\alpha)]^d(x)}{g(\alpha)} \quad (3.2.4)$$
where \( a(x) = \beta x^{\beta - 1} \), \( h(\alpha) = \exp(-\alpha) \), \( d(x) = x^\beta \) and \( g(\alpha) = \frac{1}{\alpha} \) (3.2.5)

such that \( a(x) > 0 \) and \( g(\alpha) = \int_{x>0} a(x)[h(\alpha)]d(x)dx \).

The likelihood function is given by,

\[
L(\alpha; x/R = r) = c(\alpha \beta)^m \left( \prod_{i=1}^{m} x_i \right)^{\beta - 1} \exp\left\{ -\alpha \sum_{i=1}^{m} (1 + r_i)x_i^\beta \right\}
\]

Suppose that an individual unit being removed from the life test is independent of others but with the same probability \( p \). The joint probability mass function of \( r_1, r_2, \ldots, r_{m-1} \) is,

\[
P(R = r) = \frac{(n-m)!p^{\sum_{i=1}^{m-1} r_i}(1-p)^{(n-m)-(\sum_{i=1}^{m-1} r_i)}}{(n-m-\sum_{i=1}^{m-1} r_i)! \prod_{i=1}^{m-1} r_i!}
\]

We can write the joint likelihood function as,

\[
L(\alpha, p; x, R) = c(\alpha \beta)^m \left( \prod_{i=1}^{m} x_i \right)^{\beta - 1} \exp\left\{ -\alpha \sum_{i=1}^{m} (1 + r_i)x_i^\beta \right\}
\]

\[
\times (n-m)!p^{\sum_{i=1}^{m-1} r_i}(1-p)^{(n-m)-(\sum_{i=1}^{m-1} r_i)}
\]

\[
\left( n-m-\sum_{i=1}^{m-1} r_i \right)! \prod_{i=1}^{m-1} r_i!
\]

3.3 Unbiased estimation

Let \( Y_i = X_i^\beta \), \( i = 1, 2, \ldots, m \). (3.3.1)

Then \( Y_1 < Y_2 < \cdots < Y_m \) is progressive Type II censored sample from exponential distribution with mean \( \frac{1}{\alpha} \).
Consider the following transformation,
\[ Z_1 = nY_1 \]
\[ Z_i = (n-i+1-r_1-r_2-\cdots-r_{i-1})(Y_i-Y_{i-1}), \quad i = 2, 3, \ldots, m. \] (3.3.2)

It can be seen that,
\[ \sum_{i=1}^{m} Z_i = \sum_{i=1}^{m} (1+r_i)Y_i. \]

The joint density of \( Z_1, Z_2, \ldots, Z_m \),
\[ f(z, \alpha / R=r) = \alpha^m \exp\left\{-\alpha \sum_{i=1}^{m} z_i \right\} \] (3.3.3)

The variables \( Z_1, Z_2, \ldots, Z_m \) defined in (3.3.2) are all independent and identically distributed with exponential distribution.

Suppose \( T = \sum_{i=1}^{m} Z_i. \)

Then \( T = \sum_{i=1}^{m} (1+r_i)X_i^\beta \) (3.3.4)

Since (3.3.3) is a member of exponential family of distributions, \( T \) is a complete sufficient statistic for \( \alpha \). The distribution of \( T \) is gamma with parameters \( \alpha \) and \( m \), which is again a member of exponential family of distributions. The p.d.f. of \( T \) is given by,
\[ f(t, \alpha) = \frac{B(t, m) [h(\alpha)]^t}{[g(\alpha)]^m} \] (3.3.5)

where \( B(t, m) = \frac{m^{-1}}{m} \), \( h(\alpha) = \exp(-\alpha) \), and \( g(\alpha) = \frac{1}{\alpha} \).

Using (2.4.5) and (2.4.6), we get the minimum variance unbiased estimators of some important parametric functions as given below.

(i) The UMVUE of \( \exp[-k\alpha] \) is
\[ H_{k,m} = \left[ 1 - \frac{k}{\sum_{i=1}^{m} (1+r_i)x_i^\beta} \right]^{m-1}, \quad \sum_{i=1}^{m} (1+r_i)x_i^\beta > k \] (3.3.6)
(ii) Using (2.4.9) the UMVUE of the variance of $H_{k,m}$, is given by,

$$\tilde{\text{Var}}[H_{k,m}] = \left[1 - \frac{k}{\sum_{i=1}^{m} (1 + r_i)x_i^\beta}\right]^{2m-2} - \left[1 - \frac{2k}{\sum_{i=1}^{m} (1 + r_i)x_i^\beta}\right]^{m-1}$$

(iii) The UMVUE of $\frac{1}{\alpha}$ is given by,

$$G_{k,m} = \frac{\sqrt{m}}{m+k} \left[\sum_{i=1}^{m} (1 + r_i)x_i^\beta\right]^k$$  \hspace{1cm} (3.3.7)

(iv) The UMVUE of the variance of $G_{k,m}$ is given by,

$$\tilde{\text{Var}}[G_{k,m}] = \left[\frac{\sum_{i=1}^{m} (1 + r_i)x_i^\beta}{m+k}\right]^{2k} \left[\left(\frac{\sqrt{m}}{m+k}\right)^2 - \left(\frac{\sqrt{m}}{m+2k}\right)^2\right]$$

(v) Using (2.4.11) the UMVUE of density $f(x)$ is given by,

$$\phi_{x,m} = \left(\frac{\beta x^{\beta-1} (m-1)}{\sum_{i=1}^{m} (1 + r_i)x_i^\beta}\right) \left(1 - \frac{x^\beta}{\sum_{i=1}^{m} (1 + r_i)x_i^\beta}\right)^{m-2}$$  \hspace{1cm} (3.3.8)

$$x^\beta < \sum_{i=1}^{m} (1 + r_i)x_i^\beta, \quad m > 1$$

(vi) The UMVUE of variance of $\phi_{x,m}, \quad m > 2$ is given by
\[ \bar{\text{Var}}[\phi_{x,m}] = \begin{cases} \left( \frac{\beta x^{\beta-1} (m-1)}{t} \right)^2 \left[ 1 - \frac{x^\beta}{t} \right]^{2m-4} - \left( \frac{\beta x^{\beta-1} (m-1)}{t} \right), & t > 2x^\beta \\ \times \left( 1 - \frac{x^\beta}{t} \right)^{m-2} \left( \frac{\beta x^{\beta-1} (m-2)}{t - x^\beta} \right) \left( 1 - \frac{x^\beta}{t - x^\beta} \right)^{m-3}, & x^\beta < t \leq 2x^\beta \\ 0, & \text{otherwise} \end{cases} \]

where \( t = \sum_{i=1}^{m} (1 + r_i) x_i^\beta \) \hspace{1cm} (3.3.9)

(vii) Considering fixed \( x \), the UMVUE of reliability function \( R(x) = P(X > x) \), \( x \geq 0 \) is given by,

\[ \bar{R}(x) = \int_{X > x} \phi_{y,m} \, dy. \]

Using equation (3.3.8)

\[ \bar{R}(x) = \left( 1 - \frac{m}{\sum_{i=1}^{m} (1 + r_i) x_i^\beta} \right)^{m-1}, \quad x^\beta < \sum_{i=1}^{m} (1 + r_i) x_i^\beta \] \hspace{1cm} (3.3.10)

and

\[ \bar{\text{Var}}[\bar{R}(x)] = \begin{cases} \left[ 1 - \frac{x^\beta}{t} \right]^{2m-2} - \left[ 1 - \frac{2x^\beta}{t} \right]^{m-1}, & 0 < 2x^\beta < t \\ \left[ 1 - \frac{x^\beta}{t} \right]^{2m-2}, & x^\beta < t < 2x^\beta \\ 0, & \text{otherwise} \end{cases} \]
(viii) For fixed \( x \), the UMVUE of c.d.f. given in (3.2.2) is,

\[
\tilde{F}(x) = \begin{cases} 
0, & x < 0 \\
1 - \left(1 - \frac{x^\beta}{\sum_{i=1}^{m} (1+r_i)x_i^\beta}\right)^{m-1}, & x^\beta < \sum_{i=1}^{m} (1+r_i)x_i^\beta \\
1, & \text{otherwise}
\end{cases}
\]

Special Cases:

(a) Substituting \( k = 1 \) in (3.3.6) we get the UMVUE of \( \exp[-\alpha] \) as,

\[
H_{1,m} = \left[1 - \frac{1}{\sum_{i=1}^{m} (1+r_i)x_i^\beta}\right]^{m-1}, \quad \sum_{i=1}^{m} (1+r_i)x_i^\beta > 1
\]

(b) Substituting \( k = -1 \) in (3.3.6) we get the UMVUE of \( \exp[\alpha] \) as,

\[
H_{-1,m} = \left[1 + \frac{1}{\sum_{i=1}^{m} (1+r_i)x_i^\beta}\right]^{m-1}, \quad \sum_{i=1}^{m} (1+r_i)x_i^\beta > 0
\]

(c) Substituting \( k = 1 \) in (3.3.7) we get the UMVUE of \( \left(\frac{1}{\alpha}\right) \) as,

\[
G_{1,m} = \left[\frac{\sum_{i=1}^{m} (1+r_i)x_i^\beta}{m}\right]
\]

(d) Substituting \( k = -1 \) in (3.3.7) we get the UMVUE of \( \alpha \) as,

\[
G_{-1,m} = \frac{(m-1)}{\left[\sum_{i=1}^{m} (1+r_i)x_i^\beta\right]} \quad \text{(3.3.11)}
\]
(e) In case of Type II censored sample that is when \( r_i = 0, \ i = 1, \cdots, m-1 \) and \( r_m = n - m \) the UMVUE of \( \alpha \) can be obtained from (3.3.11) as,

\[
\alpha_m = \frac{m}{\sum_{i=1}^{m} x_i^\beta + (n-m)x_m^\beta}.
\]

In case of complete sample that is when \( r_i = 0, \ i = 1, \cdots, m \) and \( m = n \), the UMVUE of \( \alpha \) reduces to

\[
\frac{(n-1)}{\sum_{i=1}^{n} x_i^\beta}.
\]

(f) The \( p^{th} \) quantile \((0 < p < 1)\) of Weibull distribution is,

\[
\xi_p = \left( \frac{1}{\alpha} \right)^{\frac{1}{\beta}} \left[ \log \left( \frac{1}{(1-p)} \right) \right]^{\frac{1}{\beta}}
\]

(3.3.12)

Substituting \( k = \frac{1}{\beta} \) in (3.3.7) we get UMVUE of \( \xi_p \) as,

\[
\xi_p = \left( \frac{1}{m+1} \right)^{\frac{m}{\beta}} \left[ \sum_{i=1}^{m} (1+r_i)x_i^\beta \right]^{\frac{1}{\beta}} \left[ \log \left( \frac{1}{1-p} \right) \right]^{\frac{1}{\beta}}
\]

(3.3.7)

Since the joint density \( P(R = r) \) is independent of \( \alpha \) one gets the same estimators of \( p \) and its various functions as given in Chapter 2.

### 3.4 UMVU estimator of \( P(X < Y) \)

In the following theorem, we derive the UMVUE of \( P(X < Y) \). Let \( m_1 \) units (out of \( n_1 \)) on \( X \) and \( m_2 \) units (out of \( n_2 \)) on \( Y \) are recorded which follow Weibull distribution, given in (3.2.1) with parameters \( \alpha_1 \) and \( \alpha_2 \) respectively. Let \( r_1, r_2, \cdots, r_{m_1} \) and \( s_1, s_2, \cdots, s_{m_2} \) be corresponding removals. We denote,

\[
t_1 = \sum_{i=1}^{m_1} (1+r_i)x_i^\beta
\]

(3.4.1)
\[ t_2 = \sum_{i=1}^{m_2} (1+s_i)y_i^\beta \]  

(3.4.2)

**Theorem 3.4.1:** Under progressive Type II censored data the UMVU estimator of 
\[ P = P(X < Y) \]  
for the density given in (3.2.1) is given by,

\[
\tilde{P} = \begin{cases} 
\sum_{j=0}^{m_2-1} (-1)^j \frac{(m_1-1)!}{(m_1+j-1)(m_2-j-1)!} \left( \frac{t_1}{t_2} \right)^j & t_1 < t_2 \\
1 - \sum_{i=0}^{m_1-1} (-1)^i \frac{(m_1-1)!}{(m_1-i-1)(m_2+i-1)!} \left( \frac{t_2}{t_1} \right)^i & t_1 \geq t_2 
\end{cases}
\]  

(3.4.3)

where \( t_1 \) and \( t_2 \) are given by (3.4.1) and (3.4.2), respectively.

Proof:
We have \[ \tilde{P} = \iint_G \phi_{x,m_1,y,m_2} \, dx \, dy \]  

(3.4.4)

where \( G = \{(x,y) : 0 < x < t_1^\beta, \ 0 < y < t_2^\beta, \ x < y \} \)

We consider the case \( t_1 < t_2 \).

\[ \tilde{P} = \int_0^{t_1^\beta} \int_x^{t_2^\beta} \phi_{x,m_1,y,m_2} \, dy \, dx \]  

(3.4.5)

Using (3.3.8) in (3.4.5) we get,
Consider $I = \frac{1}{x} \beta y^{\beta-1} \left( \frac{m_2 - 1}{t_2} \right) \left[ 1 - y^{\beta} \right]^{m_2-2} dy$.

We make the transformation $z = \frac{y^{\beta}}{t_2}$. Then

$$I = (m_2 - 1) \int \frac{1}{x^{\beta}} (1 - z)^{m_2-2} dz.$$ 

Using (3.4.7) in (3.4.6) we get,

$$\tilde{P} = \beta \left[ \frac{m_1 - 1}{t_1} \right] \int_0^1 \frac{1}{x^{\beta-1}} \left[ 1 - \frac{x^{\beta}}{t_1} \right]^{m_1-2} \left[ 1 - \frac{x^{\beta}}{t_2} \right]^{m_2-1} dx.$$ 

We make the transformation $u = \frac{x^{\beta}}{t_1}$. Then equation (3.4.8) becomes,
\[(m_1 - 1) \int_0^1 [1 - u]^{m_1 - 2} \left(1 - u \left(\frac{t_1}{t_2}\right)\right)^{m_2 - 1} \, du \]
\[= (m_1 - 1) \sum_{j=0}^{m_2 - 1} \binom{m_2 - 1}{j} (-1)^j \left(\frac{t_1}{t_2}\right)^j \int_0^1 (1 - u)^{m_1 - 2} u^j \, du \]

Hence
\[
\tilde{p} = \sum_{j=0}^{m_2 - 1} (-1)^j \binom{m_1 - 1}{j} \binom{m_2 - 1}{j} \left(\frac{t_1}{t_2}\right)^j , \quad t_1 < t_2
\]

Similarly for the case \(t_i \geq t_j\), it can be shown that
\[
\tilde{p} = 1 - \sum_{i=0}^{m_2 - 1} (-1)^i \binom{m_1 - 1}{i} \binom{m_2 - 1}{i} \left(\frac{t_2}{t_1}\right)^i , \quad t_1 \geq t_2
\]

3.5 Exact confidence interval for \(p^{th}\) quantile

In this Section we shall obtain the \((1 - \alpha) \times 100\%\) confidence interval for \(p^{th}\) quantile of Weibull distribution.

**Theorem 3.5.1:** The \((1 - \alpha) \times 100\%\) confidence interval for \(p^{th}\) quantile is given by,

\[
\left[ \frac{2}{\chi^2_{2m,1-\alpha/2}} \right]^{1/\beta} \left[ \frac{m+1}{m} \bar{\xi} \right] p , \quad \left[ \frac{2}{\chi^2_{2m,\alpha/2}} \right]^{1/\beta} \left[ \frac{m+1}{m} \bar{\xi} \right] p \]
\( (3.5.1) \)

**Proof:** Using (3.3.7) and (3.3.9) we have,
\[
T = \frac{1}{\alpha} \left( \frac{\xi p}{\xi p} \right)^{\beta} \left( \frac{m + \frac{1}{\beta}}{|m|} \right)^{\beta}
\]

Here \( T \) has gamma distribution with parameters \( \alpha \) and \( m \). Making the transformation \( Q = 2T\alpha \), it can be shown that \( Q \) has chi-square distribution with \( 2m \) degrees of freedom. Therefore,

\[
P \left[ \chi^2_{2m, \frac{\alpha}{2}} \leq Q \leq \chi^2_{2m, 1 - \frac{\alpha}{2}} \right] = 1 - \alpha
\]

Arranging (3.5.2) we get,

\[
P \left[ \frac{2}{\chi^2_{2m, 1 - \frac{\alpha}{2}}} \leq \left( \frac{\xi p}{\xi p} \right)^{\beta} \left( \frac{m + \frac{1}{\beta}}{|m|} \right)^{\beta} \leq \frac{2}{\chi^2_{2m, \frac{\alpha}{2}}} \right] = 1 - \alpha
\]

### 3.6 Bayesian estimation

In this section, we use the Bayes procedure to derive the point estimates of the parameters based on progressively Type II censoring data with binomial removals. For this purpose we assume that the parameters \( \alpha \) and \( p \) behave as independent random variables. We also assumed that the random variable \( \alpha \) has gamma distribution as a priori distribution with known parameters \( \theta \) and \( \lambda \) while \( p \) has beta
prior distribution with known parameters $a$ and $b$. The prior distribution of $\alpha$ takes the following form,

$$g_1(\alpha) = \frac{\theta^\lambda}{\lambda} \exp(-\theta \alpha) \alpha^{\lambda-1} \quad \theta, \lambda > 0, \alpha > 0 \quad (3.6.1)$$

The prior distribution of $p$ is given by,

$$g_2(p) = \frac{1}{B(a, b)} p^{a-1} (1-p)^{b-1} \quad a, b > 0, 0 < p < 1 \quad (3.6.2)$$

The joint prior distribution of $\alpha$ and $p$ is given by,

$$g(\alpha, p) = g_1(\alpha) g_2(p) \quad (3.6.3)$$

Using (3.6.1) and (3.6.2) in (3.6.3) we get,

$$g(\alpha, p) = \frac{1}{B(a, b)} \theta^\lambda \exp(-\theta \alpha) \alpha^{\lambda-1} p^{a-1} (1-p)^{b-1} \quad (3.6.4)$$

Using (3.2.8) the joint posterior distribution of $(\alpha, p)$, is given by,

$$\pi(\alpha, p | x, r) = \frac{\alpha^{m+\lambda-1} \exp(-\alpha S_r) [S_r]^{m+\lambda} p^{a^* - 1} (1-p)^{b^* - 1}}{B(a^*, b^*) [m+\lambda]}$$

where $S_r = \left[ \theta + \sum_{i=1}^{m} (1+r_i)x_i \beta \right]$, $a^* = a + \sum_{i=1}^{m-1} r_i$, and

$$b^* = b + (m-1)(n-m) - \sum_{i=1}^{m-1} (m-i)r_i$$

The marginal posterior distribution of $\alpha$ is,
\[
\pi_1(\alpha / \bar{x}, r) = \frac{1}{\frac{\alpha^{m+\lambda-1}}{m+\lambda}} \exp(-\alpha S_r) [S_r]^{m+\lambda} \times \frac{1}{B\left(a^*, b^*\right)} p^{a^*-1} (1-p)^{b^*-1} dp
\]

That is,
\[
\pi_1(\alpha / \bar{x}, r) = \frac{\alpha^{m+\lambda-1}}{m+\lambda} \exp(-\alpha S_r) [S_r]^{m+\lambda} \tag{3.6.6}
\]

and the marginal posterior distribution of \( p \) is,
\[
\pi_2(p / \bar{x}, r) = \frac{p^{a^*-1} (1-p)^{b^*-1}}{B\left(a^*, b^*\right)} \int_0^\infty \frac{\alpha^{m+\lambda-1}}{m+\lambda} \exp(-\alpha S_r) [S_r]^{m+\lambda} d\alpha
\]

That is,
\[
\pi_2(p / \bar{x}, r) = \frac{1}{B(a^*, b^*)} p^{a^*-1} (1-p)^{b^*-1} \tag{3.6.7}
\]

Note that the posterior distribution of \( \alpha \) is gamma with parameters \( \theta \) and \( \lambda \) while the posterior distribution of \( p \) is beta with parameters \( a^* \) and \( b^* \).

### 3.6.1 Bayes estimator based on squared error loss function (SELF)

Under squared error loss function the usual estimator of parameters (or the given function of parameters) is the posterior mean. Therefore the Bayes estimators of the parameters \( \alpha \) and \( p \) say \( \hat{\alpha}_{\text{SELF}} \) and \( \hat{p}_{\text{SELF}} \) are given by
\[
\hat{\alpha}_{\text{SELF}} = \int_0^\infty \alpha \pi_1(\alpha / \bar{x}, r) d\alpha \tag{3.6.8}
\]

and
\[
\hat{p}_{\text{SELF}} = \int_0^1 p \pi_2(p / \bar{x}, r) dp \tag{3.6.9}
\]
We have

\[ \hat{P}_{SELF} = \frac{1}{B(a^*, b^*)} \int_0^1 p^{a^*} (1 - p)^{b^* - 1} \, dp \]

Thus

\[ \hat{P}_{SELF} = \frac{a^*}{a^* + b^*} \quad (3.6.9) \]

\[ \hat{\alpha}_{SELF} = \frac{1}{m + \lambda} \int_0^\infty \alpha^{m + \lambda + 1} \exp(-\alpha S_r) [S_r]^{m + \lambda} \, d\alpha \]

\[ = \frac{(m + \lambda)}{S_r} \quad (3.6.10) \]

The Bayes risk associated with parameters \( \alpha \) and \( p \) is given by,

\[ \text{Risk}(\hat{\alpha}_{SELF}) = \int_0^\infty \alpha^2 \pi_1(\alpha / \bar{x}, r) - [\hat{\alpha}_{SELF}]^2 \]

\[ = \frac{1}{m + \lambda} \int_0^\infty \alpha^{m + \lambda + 1} \exp(-\alpha S_r) [S_r]^{m + \lambda} \, d\alpha - \left[ \frac{(m + \lambda)}{S_r} \right]^2 \quad (3.6.11) \]

\[ \text{Risk}(\hat{P}_{SELF}) = \int_0^1 p^2 \pi_2(p / \bar{x}, r) - [\hat{P}_{SELF}]^2 \]

\[ = \frac{1}{B(a^*, b^*)} \int_0^1 p^{a^* + 1}(1 - p)^{b^* - 1} \, dp - \left[ \frac{a^*}{a^* + b^*} \right]^2 \]

\[ = \frac{a^* b^*}{(a^* + b^* + 1)(a^* + b^*)^2} \quad (3.6.12) \]
The reliability function \( R(x) \) and hazard function \( h(x) \) at mission time \( x \) for Weibull distribution are \( R(x) = \exp\left(-\alpha x^\beta\right) \), \( x > 0 \) and \( h(x) = \alpha \beta x^{\beta-1} \), \( x > 0 \).

The Bayes estimator of reliability function and hazard function are given by,

\[
\hat{R}_{SELF}(x) = \frac{1}{m + \lambda} \int_0^\infty \alpha^{m+\lambda-1} \exp\left(-\alpha x^\beta\right) \exp(-\alpha S_r) \left[ S_r \right]^{m+\lambda} d\alpha
\]

\[
= \frac{\left[ S_r \right]^{m+\lambda}}{\left[ S_r + x^\beta \right]^{m+\lambda}}
\]

(3.6.13)

\[
\hat{h}_{SELF}(x) = \frac{1}{m + \lambda} \int_0^\infty \alpha^{m+\lambda-1} \left( \alpha \beta x^{\beta-1} \right) \exp(-\alpha S_r) \left[ S_r \right]^{m+\lambda} d\alpha
\]

\[
= \frac{(m + \lambda) \beta x^{\beta-1}}{S_r}
\]

(3.6.14)

### 3.6.2 Bayes estimator based on LINEX loss function (LL)

In the estimation of reliability function, use of symmetric loss function may be inappropriate as has been recognized by Canfield (1970). Overestimation of reliability function or average failure time is usually much more serious than underestimate of reliability function or mean failure time. Feynman (1987) remarks that in the disaster of a space shuttle, the management may have overestimated the average life or reliability of solid fuel rocket booster. Also an underestimate of the failure rate results in more serious consequences than an overestimation of failure rate. In this case an asymmetric loss function might be appropriate. Varian (1975) introduced LINEX (Linear-Exponential) loss function which is asymmetric and used by several authors. Under the assumption that the minimal loss occurs at \( \alpha^* = \alpha \) the LINEX loss function is defined as

\[
L_1(\Delta) \propto \exp(c\Delta) - c\Delta - 1; \quad c \neq 0
\]

(3.6.15)
where $\Delta = (\alpha^* - \alpha)$. $\alpha^*$ is an estimate of $\alpha$. The sign and magnitude of the shape parameter $c$ represents the direction and degree of asymmetry respectively. If $c > 0$ the overestimation is more serious than underestimation and vice-versa.

The Bayes estimator of $\alpha$ under LINEX loss function defined by (3.6.15) is given by,

$$\hat{\alpha}_{LL} = -\frac{1}{c} \log E_{\pi} \left[ \exp(-c\alpha) \right]$$

(3.6.16)

where $E_{\pi}[..]$ denote the expectation under posterior distribution, provided that it exists and finite. The problem of choosing the value of $c$ is discussed in Calabria and Pulcini (1996).

The Bayes estimator of $\alpha$ under LINEX loss function is,

$$E[\exp(-\alpha c)] = \frac{1}{m+\lambda} \int_0^\infty \alpha^{m+\lambda-1} \exp(-\alpha c)\exp(-\alpha S_r) [S_r]^{m+\lambda} d\alpha$$

$$= \left[ 1 + \frac{c}{S_r} \right]^{-1(m+\lambda)}$$

(3.6.17)

Using (3.6.17) in (3.6.16) we get

$$\hat{\alpha}_{LL} = \frac{1}{c(m+\lambda)} \log \left[ 1 + \frac{c}{S_r} \right]$$

(3.6.18)

and that of $p$ is

$$\hat{p}_{LL} = -\frac{1}{c} \log E_{\pi} \left[ \exp(-pc) \right]$$

(3.6.19)

$$E[\exp(-pc)] = \frac{1}{B(a^*, b^*)} \int_0^1 \exp(-pc) p^{a^*-1}(1-p)^{b^*-1} dp$$
\[
1 + \sum_{k=1}^{\infty} \left( \prod_{j=0}^{k-1} \frac{a^* + j}{a^* + b^* + j} \right) \frac{(-c)^k}{k!} 
\] (3.6.20)

Using (3.6.20) in (3.6.19) we get,

\[
\hat{p}_{LL} = -\frac{1}{c} \log \left[ 1 + \sum_{k=1}^{\infty} \left( \prod_{j=0}^{k-1} \frac{a^* + j}{a^* + b^* + j} \right) \frac{(-c)^k}{k!} \right] 
\] (3.6.21)

The Bayes risk associated with parameters \( \alpha \) and \( p \) under LINEX loss function is given by,

\[
Risk(\hat{\alpha}_{LL}) = \log E_{\pi}[\exp(-c\alpha)] + cE_{\pi}(\alpha) 
\] (3.6.22)

\[
E(\alpha) = \frac{(m+\lambda)}{S_r} 
\]

\[
Risk(\hat{\alpha}_{LL}) = (m+\lambda) \log \left[ \frac{S_r}{c+S_r} \right] + \frac{c(m+\lambda)}{S_r} 
\] (3.6.23)

\[
Risk(\hat{p}_{LL}) = \log E_{\pi}[\exp(-c\alpha)] + cE_{\pi}(p) 
\] (3.6.24)

\[
E_{\pi}(p) = \frac{a^*}{a^* + b^*} 
\]

\[
Risk(\hat{p}_{LL}) = \log \left[ 1 + \sum_{k=1}^{\infty} \left( \prod_{j=0}^{k-1} \frac{a^* + j}{a^* + b^* + j} \right) \frac{(-c)^k}{k!} \right] + c \left( \frac{a^*}{a^* + b^*} \right) 
\] (3.6.25)

The Bayes estimator of reliability function and hazard function are given by,

\[
\hat{R}_{LL}(x) = -\frac{1}{c} \log \left[ \sum_{j=0}^{\infty} \frac{(-c)^j}{j!} \left( 1 + \frac{jx^\beta}{S_r} \right)^{-(m+\lambda)} \right] 
\] (3.6.26)
3.6.3 Bayes estimator based on General Entropy loss function (GELF)

In many practical situations, it appears to be more realistic to express the loss in terms of the ratio \( \frac{\alpha^*}{\alpha} \). In this case a useful asymmetric loss function is the General Entropy loss function (GELF) proposed by Calabria and Pulcini (1994). It is defined as

\[
L_2(\alpha^*, \alpha) \propto \left( \frac{\alpha^*/\alpha}{c} \right)^{c_1} - c_1 \log \left( \frac{\alpha^*/\alpha}{c} \right) - 1; \tag{3.6.28}
\]

whose minimum occurs at \( \alpha^* = \alpha \). The sign of the shape parameter \( c_1 \) reflects the deviation of the asymmetry. The magnitude of \( c_1 \) reflects the degree of asymmetry.

The Bayes estimator of \( \alpha \) under GELF defined in (3.6.28) is,

\[
\hat{\alpha}_{GELF} = \left[ E_\pi \left( \alpha^{-c_1} \right) \right]^{-\frac{1}{c_1}} \tag{3.6.29}
\]

The Bayes estimators of \( \alpha, p, \) reliability and hazard function based GELF are,

\[
E \left( \alpha^{-c_1} \right) = \frac{1}{m + \lambda} \int_0^{\infty} \alpha^{m + \lambda - c_1 - 1} \exp(-\alpha S_{r}) S_{r}^{m + \lambda} d\alpha
\]

\[
= \frac{(m + \lambda - c_1) S_{r}^{c_1}}{(m + \lambda)} \tag{3.6.30}
\]

\[
\hat{\alpha}_{GELF} = \left( \frac{m + \lambda - c_1}{m + \lambda} \right)^{-\frac{1}{c_1}} \frac{1}{S_{r}} \tag{3.6.31}
\]
\[
\hat{p}_{GELF} = \left[ \frac{1}{E\left( p^{-c_1} \right)} \right]^{-1} \left( \frac{1}{c_1} \right)
\]  
(3.6.32)

\[
E\left( p^{-c_1} \right) = \frac{1}{B(a^*, b^*)} \int_0^1 p^{a^* - c_1 - 1} (1 - p)^{b^* - 1} dp
\]

\[
= \frac{B(a^* - c_1, b^*)}{B(a^*, b^*)}
\]  
(3.6.33)

Substituting (3.6.33) in (3.6.32) we get

\[
\hat{p}_{GELF} = \left[ \frac{B(a^* - c_1, b^*)}{B(a^*, b^*)} \right]^{-1} \left( \frac{1}{c_1} \right).
\]  
(3.6.34)

\[
\hat{R}_{GELF}^{(x)} = \left[ E\left( \exp\left[ -\alpha x^\beta \right] \right) \right]^{-1} \left( \frac{1}{c_1} \right)
\]  
(3.6.35)

and

\[
E\left( \exp\left[ \alpha c_1 x^\beta \right] \right) = \frac{1}{m + \lambda} \int_0^\infty \alpha^{m+\lambda-1} \exp\left( \alpha c_1 x^\beta \right) \exp(-\alpha S_r) [S_r]^{m+\lambda} d\alpha
\]

\[
= \left[ 1 - \frac{c_1 x^\beta}{S_r} \right]^{-(m+\lambda)}
\]

Thus,

\[
\hat{R}_{GELF}^{(x)} = \left[ 1 - \frac{c_1 x^\beta}{S_r} \right]^{\frac{m+\lambda}{c_1}}
\]  
(3.6.36)

Now

\[
\hat{h}_{GELF}^{(x)} = \left[ E\left( \alpha \beta x^\beta - 1 \right) \right]^{-1} \left( \frac{1}{c_1} \right)
\]  
(3.6.37)
\[ E\left(\alpha \beta x^{\beta-1}\right)^{-c_1} = \frac{1}{m + \lambda} \int_0^{\infty} \alpha^{m + \lambda - 1}\left(\alpha \beta x^{\beta-1}\right)^{-c_1} \exp(-\alpha S_r) [S_r]^{m + \lambda} d\alpha \]

\[ = \frac{m + \lambda - c_1}{m + \lambda} \left[ \frac{S_r}{\beta x^{\beta-1}} \right]^{c_1} \]  

(3.6.38)

Substituting (3.6.38) in (3.6.37) we get,

\[ h_{GELF}(x) = \left[ \frac{m + \lambda - c_1}{m + \lambda} \right]^{c_1} \left[ \frac{S_r}{\beta x^{\beta-1}} \right] \]  

(3.6.39)

The Bayes risk associated with parameters $\alpha$ based on General Entropy loss function is given by,

\[ \text{Risk}\left(\hat{\alpha}_{GELF}\right) = \log \left[ E\left(\alpha^{-c_1}\right)\right] + c_1 E(\log \alpha) \]  

(3.6.40)

Now $E(\log \alpha) = \frac{1}{m + \lambda} \int_0^{\infty} (\log \alpha) \alpha^{m + \lambda - 1} \exp(-\alpha S_r) [S_r]^{m + \lambda} d\alpha$

\[ = \left[ \frac{S_r}{m + \lambda} \right]^{m + \lambda} \int_0^{\infty} (\log \alpha) \alpha^{m + \lambda - 1} \exp(-\alpha S_r) d\alpha \]  

(3.6.41)

We use the following result given in Gradshteyn and Ryzhik (2007, page 573, Section 4.352),

\[ \int_0^{\infty} x^{\nu - 1} \exp(-\mu x) \log x dx = \frac{\nu}{\mu^\nu} [\psi(\nu) - \log \mu] \]  

to simplify (3.6.41).

Thus \[ E(\log \alpha) = \psi(m + \lambda) - \log S_r \]  

(3.6.42)
Using (3.6.30) and (3.6.42) in (3.6.40) the Bayes risk of $\alpha$ can be given as,
\[
Risk(\hat{\alpha}_{GELF}) = -\frac{1}{c_1} \log \left[ \frac{m+\lambda-c_1}{m+\lambda} \left( S_r \right)^{c_i} \right] + c_1 \left[ \psi(m+\lambda) - \log S_r \right]. \tag{3.6.43}
\]

The Bayes risk associated with parameters $\alpha$ based on General Entropy loss function is given by,
\[
Risk(\hat{p}_{GELF}) = \log \left[ E\left( p^{-c_1} \right) \right] + c_1 E(\log p) \tag{3.6.44}
\]
\[
E(\log p) = \frac{1}{B(a^*, b^*)} \int_0^1 (\log p) p^{a^* - 1} (1-p)^{b^* - 1} dp. \tag{3.6.45}
\]

We use the following result given in Gradshteyn and Ryzhik (2007, page 540, Section 4.253),
\[
\frac{1}{r^2} B\left( \frac{\mu}{r}, \nu \right) \left[ \psi\left( \frac{\mu}{r} \right) - \psi\left( \frac{\mu}{r} + \nu \right) \right], \quad \mu, \nu, r > 0
\]
to solve (3.6.45).

We have
\[
E(\log p) = \psi(a^*) - \psi(a^* + b^*) \tag{3.6.46}
\]

Using (3.6.33) and (3.4.46) in (3.6.44) we obtain the Bayes risk for $p$ as,
\[
Risk(\hat{p}_{GELF}) = \log \left[ \frac{B(a^*, b^*)}{B(a^* - c_1, b^*)} \right] + c_1 \left[ \psi(a^*) - \psi(a^* + b^*) \right]. \tag{3.6.47}
\]