Chapter 8
Minimum Variance Unbiased Estimation in
Two Parameter Non Regular Family of Distributions
Under Progressive Type II Censored Data

8.1 Introduction
In this chapter we consider the problem of UMVU estimation for two
parameter non regular family of distributions when progressive Type II censored
sample is available.

Let the life time distribution be given by the following p.d.f.,

\[ f(x; \theta_1, \theta_2) = \begin{cases} q(\theta_1, \theta_2)h(x), & a < \theta_1 \leq x \leq \theta_2 < b \\ 0, & \text{otherwise} \end{cases} \] \quad (8.1.1)

Here, \(-\infty \leq a < b \leq \infty\) are known, \(h(x)\) is an absolutely continuous function;
and \(q(\theta_1, \theta_2)\) is everywhere differentiable. The density in (8.1.1) was considered by
Hogg and Craig (1956) and discussed in Huzurbazar (1976). For the density given in
(8.1.1) the UMVU estimation was considered by Bar-Lev and Boukai (1985) and
considered the problem of UMVU estimation for two parameter non regular families
of distributions when complete and Type II censored sample is available.

The norming constant \(q(\theta_1, \theta_2)\) is defined as

\[ q(\theta_1, \theta_2) = \left[ \int_{\theta_1}^{\theta_2} h(x) \, dx \right]^{-1} \] \quad (8.1.2)
The cumulative distribution function is given by,

\[
F(x) = \begin{cases} 
0, & x < \theta_1 \\
\frac{q(\theta_1, \theta_2)}{q(\theta_1, x)}, & a < \theta_1 \leq x \leq \theta_2 < b \\
1, & \theta_2 < x 
\end{cases}
\]  

(8.1.3)

The survival function is given by,

\[
R(x) = \begin{cases} 
1, & x < \theta_1 \\
1 - \frac{q(\theta_1, \theta_2)}{q(x, \theta_2)}, & a < \theta_1 \leq x \leq \theta_2 < b \\
0, & \theta_2 < x 
\end{cases}
\]  

(8.1.4)

8.2 The Likelihood function

Let \((X_1, R_1), (X_2, R_2), \ldots, (X_m, R_m)\), denote a progressively Type II censored sample, where \(X_i = X_{i,m,n}\), for \(i = 1, 2, \ldots, m\) and \(X_1 < X_2 < \cdots < X_m\).

The conditional likelihood function can be written as, see Cohen (1963),

\[
L(\theta; \underline{x} / R = r) = c \prod_{i=1}^{m} f(x_i) \cdot [S(x_i)]^{r_i} 
\]  

(8.2.1)

where \(c = n(n-r_1-1)(n-r_1-r_2-2)\cdots(n-r_1-r_2-\cdots r_{m-1}-m+1)\),

and \(0 \leq r_i \leq (n-m-r_1-r_2-\cdots r_{i-1}) \) for \(i = 1, 2, \cdots m-1\).

Substituting (8.1.1) and (8.1.4) in (8.2.1) we get,

\[
L(\theta; \underline{x} / R = r) = c \prod_{i=1}^{m} \left[ q(\theta_1, \theta_2) h(x_i) \right] \left[ \frac{q(\theta_1, \theta_2)}{q(x_i, \theta_2)} \right]^{r_i} I(\theta_1 < x_i) I(x_i < \theta_2) 
\]
and \( I(A) \) denotes the indicator function of set \( A \). It is clear from (8.2.2) that \( (X_1, X_2, \ldots, X_m) \) is complete sufficient for \( q(\theta_1, \theta_2) \). Hence first we obtain unbiased estimator of \( \frac{1}{q(\theta_1, \theta_2)} \) based on \( X_1 \) and \( X_r \), say \( T_{1,r}(X_1, X_r) \). Then we construct a linear function of \( T_{1,r}(X_1, X_r), r = 2, 3, \ldots, m \) such that it becomes unbiased for \( \frac{1}{q(\theta_1, \theta_2)} \). Then by Lehmann-Scheffe theorem it gives unique UMVUE of \( \frac{1}{q(\theta_1, \theta_2)} \).

8.3 Unbiased estimation

We notice the following simple results,

\[
\frac{\partial q(\theta_1, \theta_2)}{\partial \theta_1} = q^2(\theta_1, \theta_2)h(\theta_1) \quad (8.3.1)
\]

\[
\frac{\partial q(\theta_1, \theta_2)}{\partial \theta_2} = -q^2(\theta_1, \theta_2)h(\theta_2) \quad (8.3.2)
\]

The marginal density of \( X_r \) and \( X_s, r < s \) have been derived in Kamps and Cramer (2001) and given by ,
\[ f_{X_r,X_s}(x_r,x_s) = c_{s-1} \left( \sum_{i=r+1}^{s} a_{i}^{(r)}(s) \left( \frac{1-F(x_s)}{1-F(x_r)} \right)^{\gamma_i} \right) \]

\[ \times \left( \sum_{i=1}^{s} a_{i}^{(r)}(1-F(x_r))^{\gamma_i} \right) \left[ \frac{f(x_r)}{1-F(x_r)} \right] \left[ \frac{f(x_s)}{1-F(x_s)} \right] \]  

(8.3.3)

\[ x_r \leq x_s, \quad 1 \leq r < s \leq n \]

where \( a_{i}^{(r)}(s) = \prod_{j=r+1}^{s} \frac{1}{\gamma_j - \gamma_i}, \quad r+1 \leq i \leq s, \quad j \neq i \)

\[ \gamma_j = n - j + 1 - \sum_{i=1}^{j-1} R_i \]

\[ c_{s-1} = \prod_{j=1}^{s-1} \gamma_j \] and the empty product \( \prod \phi \) is defined to be 1.

Now substituting \( r = 1 \) and \( s = r \) in (8.3.3) we get,

\[ f_{X_1,X_r}(x_1,x_r) = c_{r-1} \left( \sum_{i=2}^{r} a_{i}^{(1)}(1)[1-F(x_r)]^{\gamma_i-1} \right) \]

\[ \times \left[ 1-F(x_1) \right]^{\gamma_1-\gamma_i-1} f(x_1)f(x_r), \quad x_1 < x_r, \quad 1 < r \leq n \]  

(8.3.4)

Using (8.1.1) and (8.1.4) in (8.3.10) we get the joint density of \((X_1, X_r)\) is given by,

\[ f_{X_1,X_r}(x_1,x_r) = c_{r-1} \left( \sum_{i=2}^{r} a_{i}^{(1)}(1) \left[ \frac{q(\theta_1, \theta_2)}{q(x_r, \theta_2)} \right]^{\gamma_i-1} \right) \left[ \frac{q(\theta_1, \theta_2)}{q(x_1, \theta_2)} \right]^{\gamma_1-\gamma_i-1} \]
The main result of this chapter is given by following theorem.

**Theorem 8.3.1**: The UMVU estimator of U-estimable parametric function

\[
\left[ \frac{1}{q(\theta_1, \theta_2)} \right]
\]

based on progressively Type –II censored sample from the p.d.f. given in

(8.1.1) is of the form,

\[
\tilde{H}_m(X_1, X_2, \ldots, X_m) = \frac{\sum_{r=1}^{m} R_r T_{1,r} + T_{1,m}}{\sum_{r=1}^{m} R_r + 1}
\]  

(8.3.6)

Proof: Using the joint density of \((X_1, X_r)\) given in (8.3.4), we have,

\[
E\left\{ \frac{1}{q(x_1, x_r)} \right\} = c_{r-1} \left[ q(\theta_1, \theta_2) \right]^{\gamma_1} \sum_{i=2}^{r} a_i^{(1)}(r) \frac{\theta_2 \theta_2}{\theta_1 x_1} \left[ \frac{1}{q(x_1, x_r)} \right] \times
\]

\[
\left[ \frac{1}{q(x_r, \theta_2)} \right]^{\gamma_i-1} \left[ \frac{1}{q(x_1, \theta_2)} \right]^{\gamma_1-\gamma_i-1} h(x_1) h(x_r) dx_r dx_1
\]  

(8.3.7)

We use the following identity,

\[
\frac{1}{q(x_1, x_r)} = \frac{1}{q(x_1, \theta_2)} - \frac{1}{q(x_r, \theta_2)}
\]

in (8.3.7). Hence

\[
= c_{r-1} \left[ q(\theta_1, \theta_2) \right]^{\gamma_1} \sum_{i=2}^{r} a_i^{(1)}(r) \frac{\theta_2 \theta_2}{\theta_1 x_1} \left[ \frac{1}{q(x_1, \theta_2)} - \frac{1}{q(x_r, \theta_2)} \right]
\]
\[
E\left(\frac{1}{q(x_1, x_r)}\right) = c_{r-1}\left[\frac{1}{q(\theta_1, \theta_2)}\right]^{\gamma_1} \sum_{i=2}^{r} a_i^{(1)}(r) \frac{1}{\gamma_i(\gamma_i + 1)} \left[\frac{1}{q(x_1, \theta_2)}\right]^{\gamma_i} h(x_1)dx_1
\] (8.3.10)

Making the transformation \( u = \frac{1}{q(x_1, \theta_2)} \) in (8.3.10) we have

\[
E\left(\frac{1}{q(x_1, x_r)}\right) = c_{r-1}\left[\frac{1}{q(\theta_1, \theta_2)}\right]^{\gamma_1} \left[\frac{1}{q(\theta_1, \theta_2)}\right]^{\gamma_1} \sum_{i=2}^{r} a_i^{(1)}(r) \frac{1}{\gamma_i(\gamma_i + 1)}
\]

That is
The unbiased estimator of \( \left[ \frac{1}{q(\theta_1, \theta_2)} \right] \) based on \( X_1 \) and \( X_r \) is

\[
T_{1, r} (X_1, X_r) = \frac{\left( \gamma_1 + 1 \right)}{c_{r-1} \left( \sum_{i=2}^{r} a_i^{(1)}(r) \frac{1}{\gamma_i (\gamma_i + 1)} \right) q(X_1, X_r)}
\]  

(8.3.12)

Using (8.3.12) in (8.3.6) we get the UMVUE of \( \left[ \frac{1}{q(\theta_1, \theta_2)} \right] \).

8.4 Examples

Using the estimation procedure given in this chapter we find the UMVUE of \( \left[ \frac{1}{q(\theta_1, \theta_2)} \right] \).

Example 8.4.1

Let \( f(x, \theta_1, \theta_2) = \frac{1}{\theta_2 - \theta_1}, \quad \theta_1 < x < \theta_2 \) \n
(8.4.1)

Suppose \( (X_1, R_1), (X_2, R_2), \ldots, (X_m, R_m) \) be a progressive Type II censored sample available from two parameter uniform distribution on the interval \( (\theta_1, \theta_2) \).

Here \( q(\theta_1, \theta_2) = \frac{1}{\theta_2 - \theta_1} \) and \( h(x) = 1 \).
We have 
\[
H_m(X_1, X_2, \ldots, X_m) = \frac{\sum_{r=1}^{m} R_r T_{1, r} + T_{1, m}}{\sum_{r=1}^{m} R_r + 1}
\]  
(8.4.2)

where

\[
T_{1, r}(x_1, \ldots, x_r) = \frac{(\gamma_1 + 1)(x_r - x_1)}{c_{r-1} \left( \sum_{i=2}^{r} a_i^{(1)}(r) \frac{1}{\gamma_i(\gamma_i + 1)} \right)}
\]  
(8.4.3)

In case of complete sample it can be shown that, see Bairamov and Tanil (2007),

\[
\left( \sum_{i=2}^{n} a_i^{(1)}(n) \right) = \frac{(n-1)}{n!} \text{ and } c_{n-1} = n!
\]

Under complete sample case the UMVUE of \[\frac{1}{q(\theta_1, \theta_2)}\] is

\[
\hat{H}_n(X_1, X_n) = \frac{(n+1)}{(n-1)}(X_n - X_1)
\]  
(8.4.4)

Result (8.4.4) is the UMVUE of range based on complete sample and it agrees with Patel and Bhatt (1991)

**Example 8.4.2**

Consider two parameter truncated exponential distribution with p.d.f.

\[
f(x, \theta_1, \theta_2) = \frac{\alpha \exp(-\alpha x)}{\exp(-\alpha \theta_1) - \exp(-\alpha \theta_2)}, \quad \theta_1 < x < \theta_2
\]  
(8.4.5)

Here \( \alpha \) is known and

\[
q(\theta_1, \theta_2) = \frac{1}{\exp(-\alpha \theta_1) - \exp(-\alpha \theta_2)}, \quad h(x) = \alpha \exp(-\alpha x).
\]

The UMVUE of \[\frac{1}{q(\theta_1, \theta_2)}\] is given by
\[ \tilde{H}_m(X_1, X_2, \ldots, X_m) = \frac{\sum_{r=1}^{m} R_r T_{1,r} + T_{1,m}}{\sum_{r=1}^{m} R_r + 1} \]

where \( T_{1,r}(X_1, X_r) = \frac{(\gamma_1 + 1) \left( \exp(-\alpha_1 X_1) - \exp(-\alpha X_r) \right)}{c_{r-1} \left( \sum_{i=2}^{r} a_i^{(1)}(r) \frac{1}{\gamma_i(\gamma_i + 1)} \right)} \)