CHAPTER 4

COINCIDENCES AND FIXED POINTS
THEOREMS

4.1  INTRODUCTION

Cauchy reciprocal continuity and Cauchy compatibility for a hybrid pair of single-valued and multivalued mappings have been introduced. A general principle has been given (Theorem 4.3.1) stating that nonvacuously Cauchy compatible and Cauchy reciprocal continuous hybrid pair, on a metric space, has a coincidence. A pair of continuous and commuting self-mappings on a closed interval $[0,1]$ has a common fixed point and continuous commuting mappings on a complete metric space need not have a coincidence. The results of Singh and Mishra (2002) have been generalized. In section 4.4, Menger (1951) introduced the notion of a probabilistic metric space as a generalization of a metric space. The development of fixed point theory in PM-spaces is due to Schweizer and Sklar (1960) and (1983). The notions of improving commutative maps have been extended to PM-spaces. Singh and Pant (1986) extended the notion of weak commutativity, Mishra (1991) extended the notion of compatibility and Cirić and Milovan Aranjdeelouc (2000) extended the notion of pointwise $R$-weak commutativity to PM-spaces. In section 4.4, the results of Suneel Kumar and Pant (2008) have been generalized. In Section 4.7, the concepts of Cauchy compatible mappings of type $(A)$ and Cauchy compatible mappings of type $(P)$ have been introduced and compared with Cauchy compatible mappings, which extend and improve some results of Pathak et al (1995) and Jungck et al (1993).
4.2 PRELIMINARIES

Definition 4.2.1. The mappings $T : X \to CL(X)$ and $f : X \to X$ are Cauchy reciprocal continuous on $X$ (respectively at $t \in X$) if and only if $fT x \in CL(X)$ for each $x \in X$ (respectively $fT(x) \in CL(X)$) and $\lim_{n \to \infty} fT x_n = fM$, $\lim_{n \to \infty} T f x_n = T l$, whenever $\{x_n\}$ is a Cauchy sequence in $X$ such that $\lim_{n \to \infty} T x_n = M \in CL(X)$ and $\lim_{n \to \infty} f x_n = t \in M$.

Definition 4.2.2. The mappings $T : X \to CL(X)$ and $f : X \to X$ are Cauchy compatible if and only if $fT x \in CL(X)$ for each $x \in X$ and $\lim_{n \to \infty} H(f x_n, T x_n) = 0$, whenever $\{x_n\}$ is a Cauchy sequence in $X$ such that $\lim_{n \to \infty} T x_n = M \in CL(X)$ and $\lim_{n \to \infty} f x_n = t \in M$.

Definition 4.2.3. (Singh and Mishra (2002)) Two mappings $T : X \to CL(X)$ and $f : X \to X$ are IT-commuting at a point $v \in X$ if $T f v \subseteq fT v$. Further $T$ and $f$ are IT-commuting on $X$ if they are IT-commuting at each point $v \in X$.

Definition 4.2.4. (Schweizer and Sklar (1983))A mapping $F : R \to R^+$ is called a distribution function if it is non-decreasing and left continuous with $\inf_{t \in R} F(t) = 0$ and $\sup_{t \in R} F(t) = 1$. In this section, $\mathcal{F}$, the set of all distribution functions and it denotes the specific distribution function defined by

$$H(t) = \begin{cases} 0 & t \leq 0 \\ 1 & t > 0 \end{cases}$$

Definition 4.2.5. (Schweizer and Sklar (1983))A PM-space is an ordered pair $(X, E)$, where $X$ is a nonempty set of elements and $E$ is a mapping from $X \times X$ to $\mathcal{F}$, the collection of all distribution functions. The value of $F$ at $(u, v)$ \in $X \times X$, represented by $F_{u,v}$ is assumed to satisfy the following conditions

(i) $F_{u,v}(t) = 1$ for all $t > 0$ if and only if $u = v$;

(ii) $F_{u,v}(0) = 0$;

(iii) $F_{u,v}(t) = F_{v,u}(t)$;
(iv) if $F_{u,v}(t)=1$ and $F_{v,w}(s)=1$, then $F_{u,w}(t+s)=1$ for all $u, v, w \in X$ and $t, s \geq 0$.

**Definition 4.2.6. (Schweizer and Sklar (1983))** A mapping $\Delta:[0,1] \times [0,1] \to [0,1]$ is called a triangular norm (briefly, $t$-norm) if the following conditions are satisfied.

(i) $\Delta(a, 1) = a$ for all $a \in [0,1]$;

(ii) $\Delta(a, b) = \Delta(b, a)$;

(iii) $\Delta(c, d) \geq \Delta(a, b)$ for $c \geq a, d \geq b$;

(iv) $\Delta(\Delta(a, b), c) = \Delta(a, \Delta(b, c))$ for all $a, b, c, d \in [0,1]$.

**Definition 4.2.7. (Schweizer and Sklar (1983)) (Four basic t-norms)**

(i) The minimum t-norm, denoted by $\Delta_M$, is defined by $\Delta_M(x, y) = \min(x, y)$;

(ii) The product t-norm, denoted by $\Delta_P$, is defined by $\Delta_P(x, y) = x \cdot y$;

(iii) The Lukasiewicz t-norm, denoted by $\Delta_L$, is defined by $\Delta_L(x, y) = \max(x + y - 1, 0)$;

(iv) The weakest t-norm, denoted by $\Delta_D$, is defined by

$$\Delta_D(x, y) = \begin{cases} 
\min(x, y) & \text{if } \max(x, y) = 1; \\
0, & \text{otherwise.}
\end{cases}$$

Then we have the inequalities $\Delta_D < \Delta_L < \Delta_P < \Delta_M$.

**Definition 4.2.8. (Menger (1951))** A menger space is a triplet $(X, F, \Delta)$, where $(X, F)$ is a PM-space and $t$-norm $\Delta$ is such that the inequality $F_{u,v}(t + s) \geq \Delta(F_{u,v}(t), F_{v,w}(s))$ holds for all $u, v, w \in X$, and $t, s \geq 0$. Every metric space $(X, d)$ is a PM-space by taking $F : X \times X \to \exists$ by $F_{u,v}(t) = H(t - d(u, v))$ for all $u, v \in X$. 
Definition 4.2.9. (Singh and Pant (1986)) Two self mappings \( A \) and \( S \) of a PM-space \((X, F)\) are said to be weakly commuting if \( F_{ASz, SAz}(t) \geq F_{Az, Sz}(t) \) for each \( z \in X \) and \( t > 0 \).

Every pair of commuting self-mappings is weakly commuting, but the converse need not be true.

Two new concepts Cauchy compatible mappings and Cauchy reciprocal continuous in PM-spaces have been introduced:

Definition 4.2.10. Let \( A \) and \( S \) be self-mappings of a PM-space \((X, F)\). \( A \) and \( S \) are called Cauchy compatible mappings if and only if \( F_{ASu_n, SAu_n}(t) \to 1 \) for all \( t > 0 \), whenever \( \{u_n\} \) is a Cauchy sequence in \((X, d)\) such that \( Au_n \to z, S u_n \to z \) for some \( z \in X \).

Definition 4.2.11. Let \( A \) and \( S \) be self-mappings of a PM-space \((X, F)\). Then \( A \) and \( S \) are said to be Cauchy reciprocal continuous if \( ASu_n \to Az \) and \( SAu_n \to Sz \), whenever \( \{u_n\} \) is a Cauchy sequence in \( X \) such that \( Au_n \to z, Su_n \to z \) for some \( z \in X \).

Continuous mappings are Cauchy reciprocal continuous on \((X, d)\). But the converse may not be true.

Definition 4.2.12. (Ciric and Milovanovic Arandjeiovic (2000)) Two self-mappings \( S \) and \( T \) of a PM-space \((X, F)\) are called pointwise \( R \)-weakly commuting on \( X \) if given \( x \in X \), there exists an \( R > 0 \) such that \( F_{ASx, SAx}(t) \geq F_{Ax, Sz}(t/R) \) for \( t > 0 \).

Every pair of weakly commuting mappings is pointwise \( R \)-weakly commuting with \( R=1 \). \( R \)-weakly commuting maps can have a common fixed point without being pointwise weakly commuting. Compatible maps are necessarily pointwise \( R \)-weakly commuting.

Example 4.2.13. Define \( \varphi(t_1,t_2,t_3,t_4) = at_1 + bt_2 + ct_3 + dt_4 = 0 \), where \( a,b,c,d \in \mathbb{R} \) with \( a + b + c + d = 0, a > 0, a + c > 0, a + b > 0 \) and \( a + d > 0 \). Then \( \varphi \in \Phi \)
Example 4.2.14. Let $X = [0, \infty)$ with the usual metric. Define $T : X \to CL(X)$ and $f : X \to X$ by

$$T_x = \begin{cases} \left[\frac{1}{2}, x + 1\right] & \text{if } x > 0 \\ \{0\} & \text{if } x = 0 \end{cases}$$

and

$$f_x = \begin{cases} 2x + 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Let $\{x_n\} = \{0,0,0,...\}$ be a Cauchy sequence in $X$.

Then $\lim_{n \to \infty} T x_n = \{0\} \in CL(X), \lim_{n \to \infty} f x_n = 0, \lim_{n \to \infty} T f x_n = \{0\} = f\{0\} = \{0\}$ and $\lim_{n \to \infty} f T x_n = 0 = f(0) = 0$. Hence $A$ and $S$ are Cauchy reciprocal continuous at 0.

Example 4.2.15. Let $X = [0, \infty)$ with the usual metric. Define $T : X \to CL(X)$ and $f : X \to X$ by

$$T_x = \begin{cases} [0,x] & \text{if } x < 2 \\ [4,x+2] & \text{if } x \geq 2. \end{cases}$$

and

$$f_x = \begin{cases} x & \text{if } x < 2 \\ 4 & \text{if } x \geq 2. \end{cases}$$

Then $T$ and $f$ are Cauchy compatible for $x < 2$, but not for $x \geq 2$.

Let $\{x_n\} = \{2 + n\}$ be a sequence in $X$. Then $\lim_{n \to \infty} T x_n = [4, \infty), \lim_{n \to \infty} f x_n = 4$, $\lim_{n \to \infty} T f x_n = T(4) = [4,6], \text{and } \lim_{n \to \infty} f T x_n = f([4,\infty)) = \{4\}$. Therefore $H(T f x_n, f T x_n) \neq 0$. Hence $T$ and $f$ are not are Cauchy compatible.

The two new concepts of Cauchy compatible of type $(I')$ and Cauchy compatible of type $(A)$ have been introduced.
Definition 4.2.16. Let $S$ and $T$ be self-mappings of a metric space $(X, d)$. The mappings $S$ and $T$ are said to be Cauchy compatible of type $(P)$ if
\[ \lim_{n \to \infty} d(SSx_n, TTx_n) = 0 \]
whenever \( \{x_n\} \) is a Cauchy sequence in $X$ such that
\[ \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = t \]
for some $t$ in $X$.

Definition 4.2.17. Let $f$ and $g$ be self-mappings of a metric space $(X, d)$. Then $f$ and $g$ are said to be Cauchy compatible of type $(A)$, if $f$ and $g$ satisfy the two conditions:
\[ \lim_{n \to \infty} d(fg x_n, g^2 x_n) = 0 \]
and
\[ \lim_{n \to \infty} d(fg x_n, f^2 x_n) = 0 \]
whenever \( \{x_n\} \) is a Cauchy sequence in $X$ such that
\[ \lim_{n \to \infty} fx_n = \lim_{n \to \infty} g x_n = t \]
for some $t$ in $X$.

### 4.3 COINCIDENCES AND FIXED POINTS OF CAUCHY RECIPROCAL CONTINUOUS AND CAUCHY COMPATIBLE HYBRID MAPPINGS

In this section, $C(T, f)$ stands for the collection of coincidence points of $T$ and $f$; that is, $C(T, f) = \{v : fv \in T v\}$. In this section some common fixed point theorems have been established using Cauchy reciprocal continuous and Cauchy compatible hybrid mappings, which generalizes Singh and Mishra(2002). The following theorem shows that Cauchy reciprocal continuous and nonvacuously Cauchy compatible imply reciprocal continuous and nonvacuously compatible.

**Theorem 4.3.1.** Let $(X, d)$ be a metric space and the mappings $T : X \to CL(X)$ and $f : X \to X$. If $T$ and $f$ are Cauchy reciprocal continuous and nonvacuously Cauchy compatible on $X$, then $C(T, f)$ is nonempty. Further, $T$ and $f$ have a common fixed point $ft$, provided $ft = ft$ for some $t \in C(T, f)$.

**Proof.** Since $T$ and $f$ are nonvacuously Cauchy compatible, there exists a Cauchy sequence $\{x_n\} \in X$ such that $\{fx_n\}$ and $\{Tx_n\}$ both converge, respectively, to $t \in X$ and $M \in CL(X)$, such that $t \in M$ and $\lim_{n \to \infty} H(Tfx_n, ftx_n) = 0$. 


The Cauchy reciprocal continuity of \( T \) and \( f \) gives \( H(Tt, fM) = 0 \) and \( Tt = fM \).

Now, \( t \in M \) implies that \( ft \in fM \).

Therefore \( ft \in Tt \), and \( C(T, f) \) is nonempty.

Note that \( ft = fft \) implies that \( ft \in fft = Tft \).

The following theorem is a generalization of Theorem 3.1 of Singh and Mishra (2002) in the setting of metric spaces.

Let \( \psi \) denote the family of mappings, \( \phi \) from the set \( \mathbb{R}^+ \) of nonnegative reals to itself such that \( \phi(t) < t \) for all \( t > 0 \).

**Theorem 4.3.2.** Let \((X, d)\) be a metric space and \( S, T : X \to CL(X) \) and \( f, g : X \to X \) such that

1. \( S(X) \subset g(X) \) and the pair \((S, f)\) is Cauchy reciprocal continuous and non-vacuously Cauchy compatible.

2. If there exists a \( \phi \in \psi \) such that \( H(Sx, Ty) \leq \phi(M(x, y)) \) for \( x, y \in X \), where

\[
M(x, y) = \max\{d(fx, gy), d(fx, Sy), d(gy, Ty), d(fx, Ty), d(gy, Sx)\},
\]

then \( C(S, f) \) and \( C(T, g) \) are nonempty. Further,

1a) \( S \) and \( f \) have a common fixed point \( ft \), provided \( fft = ft \) for some \( t \in C(S, f) \);

1b) \( T \) and \( g \) have a common fixed point \( gu \), provided \( ggu = gu \) and \( T, g \) are IT-commuting at \( u \in C(T, g) \);

1c) \( S, T, f \) and \( g \) have a common fixed point, provided conditions (1a) and (1b) given above are both true.
Proof.: By Theorem 4.3.1,

Condition (1) implies that \( C(S, f) \) is nonempty;
that is, \( ft \in S t \) for some \( t \in X \).

Since \( S(X) \subseteq g(X) \), there is a point \( u \in X \) such that \( ft = gu \in St \).

By condition (2),
\[
d(gu, Tu) \leq H(St, Tu) \\
\leq \phi(\max\{d(ft, gu), d(ft, St), d(gu, Tu), d(ft, Tu), d(gu, St)\}) \leq d(gu, Tu) < d(gu, Tu) \\
gu \notin Tu
\]

So \( gu \in Tu \), and \( C(T, g) \) is nonempty.

Conditions (Ia) and (Ib) may be shown as in the last part of the proof of Theorem 4.3.1.

Now, (Ic) Since \( S(X) \subseteq g(X) \), there is a point \( u \in X \) such that \( ft = gu \in St \).
Therefore \( ft = gu = ggu = f ft \), and \( f \) and \( g \) have a common fixed point. \( \square \)

The following corollary generalizes Corollary 3.3 of Singh and Mishra (2002).

**Corollary 4.3.3.** Let \((X, d)\) be a complete metric space and \( S, T : X \rightarrow CL(X) \), \( f, g : X \rightarrow X \), such that

1. \( S(X) \subseteq g(X), T(X) \subseteq f(X) \), and the pair \((S, f)\) is Cauchy compatible and
   Cauchy reciprocal continuous.

2. If there exists a \( q \in (0, 1) \) such that \( H(Sx, Ty) \leq q(M(x, y)) \) for \( x, y \in X \), where
   \[
   M(x, y) = \max\{d(fx, gy), d(fx, Sx), d(gy, Ty), [d(fx, Ty) + d(gy, Sx)]/2\}
   \]

then \( C(S, f)\) and \( C(T, g) \) are nonempty. Further, conclusions (Ia), (Ib) and (Ic) are also true.
**Corollary 4.3.4.** Let \( f, g, S, T \) be self-mappings of a complete metric space \((X, d)\) such that \( S(X) \subset g(X), \ T(X) \subset f(X) \) and the pair \((S, f)\) is Cauchy compatible and Cauchy reciprocal continuous. If there exists a \( q \in (0, 1) \) such that \( d(Sx, Ty) \leq q(M(x, y)) \), where \( M(x, y) = \max\{d(fx, gy), d(fx, Sy), d(gy, Tx), d(fx, Ty), d(gy, Sx)\} \) for all \( x, y \in X \), then

(Iia) \( S \) and \( f \) have a common fixed point in \( X \);

(Iib) \( T \) and \( g \) have a coincidence at \( x = a \in X \);

(Iic) \( f, g, S \) and \( T \) have a common fixed point provided that \( T \) and \( g \) are weakly compatible.

The following results are proved in a compact metric space. Corollary 4.3.3 is being applied to obtain a coincidence theorem for the hybrid pair of single-valued and multivalued mappings. In this theorem continuous mappings are being used instead of reciprocal continuous mappings.

**Theorem 4.3.5.** Let \( f, g \) be continuous self mappings of a compact metric space \((X, d)\) and let \( S, T : X \rightarrow \text{CB}(X) \) be continuous such that \( S(X) \subset g(X) \), \( T(X) \subset f(X) \), and the pair \((S, f)\) is Cauchy compatible. If \( H(Sx, Ty) < M(x, y) \) where \( M(x, y) = \max\{d(fx, gy), d(fx, Sy), d(gy, Tx), [d(fx, Ty) + d(gy, Sx)]/2\} \neq 0 \), then \( C(S, f) \) and \( C(T, g) \) are nonempty. Further, (Ia)-(Ic) are also true.

**Proof.** To use Corollary 4.3.3, it has to be shown that \( C(S, f) \) and \( C(T, g) \) are nonempty.

We claim that \( M(x, y) = 0 \) for some \( x, y \in X \).

Otherwise the function \( w(x, y) = H(Sx, Ty)/M(x, y) \) is continuous and satisfies \( w(x, y) < 1 \) for \((x, y) \in X \times X\).
Since $X \times X$ is compact, there exist $v, z \in X$ such that $w(x, y) \leq w(v, z) = q < 1$ for all $x, y \in X$.

Also $H(Sx, Ty) \leq q(M(x, y))$ for $x, y \in X$ and some $q \in (0, 1)$.

By Corollary 4.3.3 there exist $u, t \in X$ such that $ft \in Sf$ and $ft = gy \in Ty$.

We have $M(t, u) = 0$, contradicting the condition $M(t,u) > 0$

Therefore $M(x, y) = 0$ for some $x, y \in X$.

Consequently, $fx = gy$, $fx \in Sx$, $gy \in Ty$.

\[ \square \]

**Corollary 4.3.6.** Let $f, g, S$ and $T$ be continuous self maps of a compact metric space $(X, d)$ such that $S(X) \subseteq g(x)$, $T(X) \subseteq f(x)$, and the pair $\{S, f\}$ is Cauchy compatible. If $d(Sx, Ty) < M(x, y)$ when $M(x, y) > 0$, then (IIa)-(IIc) are true.

### 4.4 A COMMON FIXED POINT THEOREM IN PROBABILISTIC METRIC SPACES USING CAUCHY RECIPROCAL CONTINUITY AND CAUCHY COMPATIBILITY

In this section the results of Suneel Kumar and Pant (2008) have been generalized in the setting of probabilistic metric spaces.

**Lemma 4.4.1. (Singh and Pant (1984))** Let $\{u_n\}$ be a sequence in a Menger space $(X, F, \Delta_M)$. If there exists a constant $h \in (0,1)$ such that $F_{u_n,u_{n+1}}(ht) \geq F_{u_{n-1},u_n}(t)$, for $n=1,2,3,...$, then $\{u_n\}$ is a Cauchy sequence in $X$.

Mihet (2005) established a fixed point theorem concerning probabilistic contractions satisfying an implicit relation. Popa (2002) used the family $F_4$ of implicit real functions to find the fixed points of two pairs of semi compatible mappings in a $d$-compatible topological space. We shall denote by $F_4$ the family of all real continuous functions $F : (\mathbb{R}^+)^4 \rightarrow \mathbb{R}$ satisfying the following properties:

- $(F_h)$ There exists an $h \geq 1$ such that, for every $u \geq 0$, $v \geq 0$ with $F(u,v,u,v) \geq 0$ or $F(u,v,v,u) \geq 0$ we have $u \geq v,$
\((f_u)F(u, u, 0, 0) < 0\) for all \(u > 0\).

Also, denote by \(\Phi\) the class of all real continuous functions \(\varphi: (\mathbb{R}^+) \to \mathbb{R}\) non-decreasing in the first argument and satisfying the following conditions:

\((H_1)\) for \(u, v \geq 0\), \(\varphi(u, v, u, v) \geq 0\) or \(\varphi(u, v, v, u) \geq 0\) implies that \(u \geq v\).

\((H_1)\) \(\varphi(u, u, 1, 1) < 0\) for all \(u \geq 1\).

Before proving the main results, the following lemma which generalizes Sunnel Kumar and Pant of (2008) will be proved.

**Lemma 4.4.2.** Let \((X, F, \Delta_M)\) be a complete Menger space. Further, let \((A, S)\) and \((B, T)\) be pointwise \(R\)-weakly commuting pairs of self mappings of \(X\) satisfying

\[A(X) \subseteq T(X), B(X) \subseteq S(X),\]

\[(4.4)\]

\[\varphi(F_{Au,Bv}(ht), F_{Su,Tv}(t), F_{Au,Su}(t), F_{Be,Tv}(ht)) \geq 0,\]

\[(4.5)\]

and

\[\varphi(F_{Au,Bv}(ht), F_{Su,Tv}(t), F_{Au,Su}(ht), F_{Be,Tv}(t)) \geq 0,\]

\[(4.6)\]

for all \(u, v \in X, t > 0, h \in (0, 1)\), and for some \(\varphi \in \Phi\). Then the continuity of one of the mappings in the Cauchy compatible pair \((A, S)\) or \((B, T)\) on \((X, F, \Delta_M)\) implies their Cauchy reciprocal continuity.

**Proof.** Assume that \(A\) and that \(S\) are Cauchy compatible and \(S\) is continuous.

It can be proved that \(A\) and \(S\) are Cauchy reciprocal continuous.

Let \(\{u_n\}\) be a Cauchy sequence such that \(Au_n \to z\) and

\[Su_n \to z\]

for some \(z \in X\) as \(n \to \infty\).

Since \(S\) is continuous, \(SAu_n \to Sz\) and \(SSu_n \to Sz\) as \(n \to \infty\).
Since \((A, S)\) is Cauchy compatible, \(F_{ASu_n, S Au_n}(t) \to 1.\)

This implies that \(F_{ASu_n, S z}(t) \to 1\), that is \(ASu_n \to S z\) as \(n \to \infty\).

By equation (4.4) for each \(n\), there exists a sequence \(\{v_n\} \in X\) such that \(ASu_n = T v_n\).

Thus, we have \(SSu_n \to S z; S Au_n \to S z; ASu_n \to S z;\)

and \(T v_n \to S z\) as \(n \to \infty\), whenever \(ASu_n = T v_n\).

It has to be shown that \(Bv_n \to S z\) as \(n \to \infty\).

Suppose not. Then by equation (4.5),

\[
\varphi(F_{ASu_n, Bv_n}(ht), F_{SSu_n, T v_n}(t), F_{ASu_n, SSu_n}(t), F_{Bv_n, T v_n}(ht)) \geq 0.
\]

Taking the limit as \(n \to \infty\), we get

\[
\varphi(F_{Sz, Bv_n}(ht), F_{Sz, Sz}(t), F_{Sz, Sz}(t), F_{Bv_n, Sz}(ht)) \geq 0;
\]

that is, \(\varphi(F_{Bv_n, Sz}(ht), 1, 1, F_{Bv_n, Sz}(ht)) \geq 0.\)

Using condition \((h_1)\), \(F_{Bv_n, Sz}(ht) \geq 1\) for all \(t > 0.\)

Hence \(F_{Bv_n, Sz}(ht) = 1\) and thus \(Bv_n \to S z.\)

By equation (4.5), \(\varphi(F_{Az, Bv_n}(ht), F_{Sz, T v_n}(t)), F_{Az, Sz}(t)), F_{Bv_n, T v_n}(ht)) \geq 0.\)

Taking the limit as \(n \to \infty\), we get

\[
\varphi(F_{Az, Sz}(ht), 1, F_{Az, Sz}(t), 1) \geq 0.
\]

Since \(\varphi\) is non-decreasing in the first argument,

\[
\varphi(F_{Az, Sz}(t), 1, F_{Az, Sz}(t), 1) \geq 0.
\]

Using condition \((h_1)\),

\(F_{Az, Sz}(t) \geq 1\) for all \(t > 0\) and therefore \(F_{Az, Sz}(t) = 1.\)

Hence \(A z = S z.\)

Since \(S Au_n \to S z\) and \(ASu_n \to S z = A z\) as \(n \to \infty,\)

\(A\) and \(S\) are Cauchy reciprocal continuous on \(X.\)

Assume that \(B\) and that \(T\) Cauchy compatible and that \(T\) is continuous.

It can be proved that \(B\) and \(T\) are Cauchy reciprocal continuous.
Let \( \{u_n\} \) be a Cauchy sequence such that \( B u_n \to z \) and

\[ T u_n \to z \quad \text{for some } z \in X \quad \text{as } n \to \infty. \]

Since \( T \) is continuous, \( TB u_n \to T \cdot z \) and \( TT u_n \to S \cdot z \) as \( n \to \infty \).

Since \((B, T)\) is Cauchy compatible, \( F_{BT u_n, T B u_n}\) \( (l) \to 1 \).

This implies that \( F_{BT u_n, T z}\) \( (l) \to 1 \); that is, \( B T u_n \to T \cdot z \) as \( n \to \infty \).

By equation (4.4), for each \( n \), there exists a sequence \( \{t_n\} \in X \) such that \( B T u_n = S t_n \).

Thus we have \( TT u_n \to T \cdot z \), \( B T u_n \to T \cdot z \), \( T B u_n \to T \cdot z \)

and \( S t_n \to T \cdot z \) as \( n \to \infty \) whenever \( B T u_n = S t_n \).

It will now be shown that \( A t_n \to T \cdot z \) as \( n \to \infty \).

Suppose not. Then by equation (4.5),

\[ \varphi(F_{BT u_n, A t_n}(h l), F_{TT u_n, S t_n}(l), F_{B T u_n, T T u_n}(l), F_{A t_n, S t_n}(h l)) \geq 0. \]

Taking the limit as \( n \to \infty \), we get

\[ \varphi(F_{T z, A t_n}(h l), F_{T z, T z}(l), F_{T z, T z}(l), F_{A t_n, T z}(h l)) \geq 0; \]

that is, \( \varphi(F_{A t_n, T z}(h l), 1, 1, F_{A t_n, T z}(h l)) \geq 0. \)

Using condition \((l_1), F_{A t_n, T z}(h l) \geq 1 \) for all \( l > 0 \).

Hence \( F_{A t_n, T z}(h l) = 1 \), and thus \( A t_n \to T \cdot z \).

By equation (4.5), \( \varphi(F_{B z, A t_n}(h l), F_{T z, S t_n}(l), F_{B z, T z}(l), F_{A t_n, S t_n}(h l)) \geq 0. \)

Taking the limit as \( n \to \infty \), we get

\[ \varphi(F_{B z, T z}(h l), 1, F_{B z, T z}(l), 1) \geq 0. \]

Since \( \varphi \) is non-decreasing in the first argument,

\[ \varphi(F_{B z, T z}(l), 1, F_{A z, T z}(l), 1) \geq 0. \]

Using condition \((l_1), F_{A z, S z}(l) \geq 1 \) for all \( l > 0 \), and therefore \( F_{B z, T z}(l) = 1. \)

Hence \( A z = S z \).

Since \( T B u_n \to T \cdot z \), and \( B T u_n \to T \cdot z = B z \) as \( n \to \infty \),
$B$ and $T$ are Cauchy reciprocal continuous on $X$.

Our main result, Theorem 4.4.3, is a generalization of Theorem 4.1 of Suneel Kumar and Pant (2008).

**Theorem 4.4.3.** Let $(X, F, \Delta_M)$ be a complete Menger space. Further, let $(A, S)$ and $(B, T)$ be a pointwise $R$-weakly commuting pair of self mappings of $X$ satisfying equations (4.4), (4.5) and (4.6). If one of the mappings in the Cauchy compatible pair $(A, S)$ or $(B, T)$ is continuous, then $A, B, S$ and $T$ have a unique common fixed point in $X$.

**Proof.** Let $u_0 \in X$.

By equation (4.4), the sequences $\{u_n\}$ and $\{v_n\}$ in $X$ are defined such that for all $n=0,1,2,...$

$$v_{2n+1} = Au_{2n} = Tu_{2n+1},
\quad v_{2n+2} = Bu_{2n+1} = Su_{2n+2}.$$  \hspace{1cm} (4.7)

By equation (4.5),

$$\varphi(F_{Au_2n,Bu_{2n+1}}(ht), F_{Su_{2n},Tu_{2n+1}}(t), F_{Au_{2n},Su_{2n}}(t), F_{Bu_{2n+1},Tu_{2n+1}}(ht)) \geq 0;$$  

that is, $\varphi(F_{v_{2n+1},v_{2n+2}}(ht), F_{v_{2n+1},v_{2n+2}}(t), F_{v_{2n+2},v_{2n+2}}(ht), F_{v_{2n+2},v_{2n+2}}(ht)) \geq 0$

Using condition $(l_1)$,

$$F_{v_{2n+1},v_{2n+2}}(ht) \geq F_{v_{2n+1},v_{2n+2}}(t).$$  \hspace{1cm} (4.8)

Similarly, by equation (4.6) and condition $(l_1)$, we have

$$F_{v_{2n+2},v_{2n+3}}(ht) \geq F_{v_{2n+2},v_{2n+3}}(t).$$  \hspace{1cm} (4.9)

Thus, for any $n$ and $t$, we have

$$F_{v_n,v_{n+1}}(ht) \geq F_{v_{n-1},v_n}(t).$$

Hence, by Lemma 4.4.1, $\{v_n\}$ is a Cauchy sequence in $X$. 

Since $X$ is complete, $\{v_n\}$ converges to a point $z \in X$
Its subsequences $\{Au_{2n}\}, \{Bu_{2n+1}\}, \{Su_{2n}\}$ and $\{Tu_{2n+1}\}$
also converge to $z$.

Suppose that $(A, S)$ is a Cauchy compatible pair and that $S$ is continuous.
Then, by Lemma 4.4.2, $A$ and that $S$ are Cauchy reciprocally continuous,
and $ASu_{2n} \to Az$ and $SAu_{2n} \to Sz$.
The Cauchy compatibility of $A$ and $S$ implies that
$$F_{ASu_{2n}, SAu_{2n}}(t) \to 1;$$
that is, $F_{Az, Sz}(t) \to 1$ as $n \to \infty$.
Therefore $Az = Sz$.

Since $A(X) \subseteq T(X)$, there exists a point $p$ in $X$ such that $Az = Tp$.

By equation (4.5),
$$\varphi(F_{Az, Bp}(ht), F_{Sz, Tp}(t), F_{Az, Sz}(t), F_{Bp, Tp}(ht)) \geq 0;$$
that is, $\varphi(F_{Az, Bp}(ht), 1, 1, F_{Bp, Az}(ht)) \geq 0$.
Using condition $(I_1)$, $F_{Az, Bp}(ht) \geq 1$ for all $t > 0$, and therefore
$$F_{Az, Bp}(ht) = 1.$$

Hence, $Az = Bp$.

Therefore $Az = Sz = Bp = Tp$.

Since $A$ and $S$ are pointwise $R$-weakly commuting mappings,
there exists an $R > 0$ such that $F_{ASz, SAz}(t) \geq F_{Az, Sz}(t/R) = 1$;
that is, $ASz = SAz$ and $AAz = ASz = SAz = SSz$.

Similarly, since $B$ and $T$ are pointwise $R$-weakly commuting mappings,
we have $BBp = BTp = TBP = TTp$.

By equation (4.5),
$$\varphi(F_{AAz, Bp}(ht), F_{SAz, Tp}(t), F_{AAz, SAz}(t), F_{Bp, Tp}(ht)) \geq 0;$$
that is, \( \varphi(F_{AAz,Az}(ht), F_{AAz,Az}(t), 1, 1) \geq 0 \).

As \( \varphi \) is non-decreasing in the first argument,

\( \varphi(F_{AAz,Az}(t), F_{AAz,Az}(t), 1, 1) \geq 0 \).

Using condition (II1),

\( F_{AAz,Az}(t) \geq 1 \) for all \( t > 0 \).

This yields \( F_{AAz,Az}(t) = 1 \),

which implies that \( AAz = Az \) and \( Az = AAz = SAz \).

Therefore \( Az \) is a common fixed point of \( A \) and \( S \).

By equation (4.5), we have that \( Bp = Az \) is a common fixed point of \( B \) and \( T \).

Thus \( Az \) is a common fixed point of \( A, B, S \) and \( T \).

Suppose that \( Aq(\neq Az) \) is another common fixed point of \( A, B, S \) and \( T \).

By equation (4.5),

\( \varphi(F_{AAz,BAq}(ht), F_{SAz,TAq}(t), F_{AAz,SAz}(t), F_{BAq,TAq}(ht)) \geq 0; \)

that is, \( \varphi(F_{Az,Aq}(ht), F_{Az,Aq}(t), 1, 1) \geq 0 \).

Since \( \varphi \) is non-decreasing in the first argument, \( \varphi(F_{Az,Aq}(t), F_{Az,Aq}(t), 1, 1) \geq 0 \).

Using Condition (II1), we obtain \( F_{Az,Aq}(t) \geq 1 \) for all \( t > 0 \),

which gives \( F_{Az,Aq}(t) = 1 \),

which implies that \( Az = Aq \).

Therefore \( Az \) is the unique common fixed point of \( A, B, S \) and \( T \).

\( \square \)

**Corollary 4.4.4.** Let \( (X, F, \Delta_M) \) be a complete Menger space. Let \( A \) and \( B \) be self mappings of \( X \) satisfying

\[ \varphi(F_{Au,Bv}(ht), F_{u,v}(t), F_{Au,u}(t), F_{Be,v}(ht)) \geq 0 \]  \hfill (4.10)

and

\[ \varphi(F_{Au,Bv}(ht), F_{u,v}(t), F_{Au,u}(t), F_{Be,v}(ht)) \geq 0 \]  \hfill (4.11)
for all $u, v \in X$, $t > 0$, $h \in (0, 1)$ and, for some $\varphi \in \Phi$. If $A$ and $B$ are Cauchy reciprocally continuous mappings, then $A$ and $B$ have a unique common fixed point in $X$.

The following example shows that our main result is a nonempty generalization.

**Example 4.4.5.** Let $X = \mathbb{R}^+$ and let $F$ be a function defined by

$$F_{u,v}(t) = \begin{cases} t/(t + |u - v|) & \text{if } t > 0; \\ 0, & \text{if } t = 0. \end{cases}$$

Then $(X, F)$ is a probabilistic metric space. Let $A, B, S$ and $T$ be self mappings of $X$ defined by

$$A u = \begin{cases} 0 & \text{if } u = 0 \\ 1 & \text{if } u > 0, \end{cases}$$

$$B u = \begin{cases} 0 & \text{if } u = 0 \text{ or } u > 6 \\ 2 & \text{if } 0 < u \leq 6, \end{cases}$$

$$S u = \begin{cases} 0 & \text{if } u = 0 \\ 2 & \text{if } u > 0, \end{cases}$$

and

$$T u = \begin{cases} 0 & \text{if } u = 0 \\ 4 & \text{if } 0 < u \leq 6 \\ u - 6 & \text{if } u > 6. \end{cases}$$

Then $A, B, S$ and $T$ satisfy all of the conditions of Theorem 4.4.3 with $h \in (0, 1)$, and have a unique common fixed point at $u = 0$. Take the Cauchy sequence $u_n = 0, 0, 0, \ldots$. Then $\lim_{n \to \infty} A u_n = 0$, $\lim_{n \to \infty} S u_n = 0$, $\lim_{n \to \infty} A S u_n = 0$, and
\[ \lim_{n \to \infty} S \cdot A u_n = 0. \] Therefore \( A \) and \( S \) are Cauchy reciprocal continuous. Since 
\[ A S u_n = 0 \text{ and } S A u_n = 0, \quad F_{A S u_n, S A u_n(t)} \to 1 \text{ for all } t > 0. \] Therefore \( A \) and \( S \) are Cauchy compatible. However, \( A \) and \( S \) are not continuous, even at the common fixed point. The sequence \( u_n = 6 + 1/n, \ n \geq 1 \) is a Cauchy sequence in \( X \). The mappings \( B \) and \( T \) are non-compatible. Since \( B u_n \to 0, \ T u_n \to 0, \ TB u_n = 0 \) and \( B T u_n = 2 \), it follows that \( B \) and \( T \) are non-compatible.

### 4.5 COMMON FIXED POINT THEOREMS FOR DIFFERENT COMPATIBLE TYPE MAPPINGS

In this section, two new concepts Cauchy compatible mappings of type \((A)\) and Cauchy compatible mappings of type \((P)\) are introduced and compared with Cauchy compatible mappings. The purpose of this section is to prove some common fixed point theorems for different types of compatible mappings.

**Proposition 4.5.1.** Let \( S, T : (X, d) \to (X, d) \) be continuous mappings. Then \( S \) and \( T \) are Cauchy compatible if and only if they are Cauchy compatible of type \((P)\).

**Proof.** Let \( \{x_n\} \) be a Cauchy sequence in \( X \) such that
\[ \lim_{n \to \infty} S x_n = \lim_{n \to \infty} T x_n = z \text{ for some } z \in X. \]
Since \( S \) and \( T \) are continuous,
\[ \lim_{n \to \infty} SS x_n = \lim_{n \to \infty} S T x_n = S z \text{ and } \lim_{n \to \infty} TS x_n = \lim_{n \to \infty} TT x_n = T z. \]
Suppose that \( S \) and \( T \) are Cauchy compatible.

Then we have \( \lim_{n \to \infty} d(ST x_n, TS x_n) = 0. \)

Now, since we have
\[
d(SS x_n, TT x_n) \leq d(SS x_n, ST x_n) + d(ST x_n, TT x_n)
\]
\[
\leq d(SS x_n, ST x_n) + d(ST x_n, TS x_n) + d(TS x_n, TT x_n),
\]
it follows that \( \lim_{n \to \infty} d(SSx_n, TTx_n) = 0 \),
and the mappings \( S \) and \( T \) are Cauchy compatible of type \( (P) \).

Conversely, suppose that \( S \) and \( T \) are Cauchy compatible mappings of type \( (P) \);
that is, \( \lim_{n \to \infty} d(SSx_n, TTx_n) = 0 \),
whenever \( \{x_n\} \) is Cauchy a sequence in \( X \) such that \( \lim_{n \to \infty} f x_n = \lim_{n \to \infty} g x_n = t \) for some \( t \) in \( X \).

We then have

\[
\begin{align*}
d(STx_n, TSx_n) &\leq d(STx_n, SSx_n) + d(SSx_n, TSx_n) \\
 &\leq d(STx_n, SSx_n) + d(SSx_n, TTx_n) + d(TTx_n, TSx_n)
\end{align*}
\]

Therefore, it follows that \( \lim_{n \to \infty} d(STx_n, TSx_n) = 0 \). \( \square \)

**Proposition 4.5.2.** Let \( S, T : (X, d) \to (X, d) \) be Cauchy compatible mappings of type \( (A) \). If one of \( S \) and \( T \) is continuous, then \( S \) and \( T \) are Cauchy compatible of type \( (P) \).

**Proof.** Let \( \{x_n\} \) be a Cauchy sequence in \( X \) such that

\[
\lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = z \text{ for some } z \in X.
\]

Suppose that \( S \) and \( T \) are Cauchy compatible mappings of type \( (A) \).

Assume, without loss of generality, that \( S \) is continuous.

We then have

\[
\begin{align*}
d(SSx_n, TTx_n) &\leq d(SSx_n, STx_n) + d(STx_n, TTx_n)
\end{align*}
\]

and so, since \( S \) and \( T \) are Cauchy compatible of type \( (A) \), we have

\[
\lim_{n \to \infty} d(SSx_n, TSx_n) = 0 \text{ and } \lim_{n \to \infty} d(STx_n, TTx_n) = 0.
\]

Therefore it follows that

\[
\lim_{n \to \infty} d(SSx_n, TTx_n) = 0 \quad \square
\]
Proposition 4.5.3. Let $S, T : (X, d) \to (X, d)$ be continuous mappings. Then

(1) $S$ and $T$ are Cauchy compatible if and only if they are Cauchy compatible of type (P).

(2) $S$ and $T$ are Cauchy compatible of type (A) if and only if they are Cauchy compatible of type (P).

Proposition 4.5.4. Let $S, T : (X, d) \to (X, d)$ be mappings. If $S$ and $T$ are Cauchy compatible of type (P) and $Sz = Tz$ for some $z \in X$, then $SSz = STz = TSz = TTz$.

Proof. Let $\{x_n\}$ be the Cauchy sequence in $X$ defined by $x_n = z, \quad n = 1, 2, \ldots$.

Then we have $Sx_n, Tx_n \to Sz$ as $n \to \infty$.

Since $S$ and $T$ are Cauchy compatible of type (P), we have

$$d(SSz, TTz) = \lim_{n \to \infty} d(SSx_n, TSx_n) = 0,$$

Therefore, $SSz = TTz$.

But $Sz = Tz$ implies that $SSz = STz = TSz = TTz$. \qed

Proposition 4.5.5. Let $S, T : (X, d) \to (X, d)$ be mappings. Let $S$ and $T$ be Cauchy compatible mappings of type (P) and let $Sx_n, Tx_n \to z$ as $n \to \infty$ for some $z \in X$. Then we have the following:

(1) $\lim_{n \to \infty} TTx_n = Sz$ if $S$ is continuous at $z$.

(2) $\lim_{n \to \infty} SSx_n = Tz$ if $T$ is continuous at $z$.

(3) $STz = TSz$ and $Sz = Tz$ if $S$ and $T$ are continuous at $z$.

Proof. (1) Suppose that $S$ is continuous at $z$.

Since $\lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = z$ for some $z \in X$, it follows that
SSx_n \to S z \text{ as } n \to \infty.

Again, since S and T are Cauchy compatible of type (P),

\lim_{n \to \infty} d(TT x_n, SS x_n) = 0 \text{ and }

so, since we have

\[ d(TT x_n, S z) \leq d(TT x_n, SS x_n) + d(SS x_n, S z). \]

it follows that \( TT x_n \to S z \) as \( n \to \infty. \)

(2) Suppose that \( T \) is continuous at \( z \).

Since \( \lim_{n \to \infty} S x_n = \lim_{n \to \infty} T x_n = z \) for some \( z \in X \), it follows that

\( TT x_n \to T z \) as \( n \to \infty. \)

Again, since S and T are Cauchy compatible of type (P),

we have \( \lim_{n \to \infty} d(TT x_n, SS x_n) = 0 \) and

so, since we have

\[ d(SS x_n, T z) \leq d(TT x_n, SS x_n) + d(TT x_n, T z). \]

it follows that \( SS x_n \to T z \) as \( n \to \infty. \)

(3) Suppose that S and T are continuous at \( z \).

Since \( T x_n \to z \) as \( n \to \infty \) and

\( S \) is continuous at \( z \), by (1), \( TT x_n \to S z \) as \( n \to \infty. \)

On the other hand, since \( T x_n \to z \) as \( n \to \infty \) and

\( T \) is also continuous at \( z \), \( TT x_n \to T z \).

Thus, we have \( S z = T z \) by the uniqueness of the limit and,

by Proposition 4.5.4, \( TS z = ST z. \)

Proposition 4.5.6. Let \( S, T : (X, d) \to (X, d) \) be continuous mappings. If \( S \) and \( T \) are Cauchy compatible, then they are Cauchy compatible of type (A)
Proof. Suppose that \( S \) and \( T \) are Cauchy compatible.

Let \( \{x_n\} \) be a Cauchy sequence in \( X \) such that \( S(x_n), T(x_n) \to t \) for some \( t \in X \).

By the triangle inequality of the metric \( d \), we have
\[
d(SSx_n, TSx_n) \leq d(SSx_n, STx_n) + d(STx_n, TSx_n).
\]

Taking the limit as \( n \to \infty \), since \( S \) and \( T \) are Cauchy compatible, and \( S \) is continuous,
we have
\[
\lim_{n \to \infty} d(SSx_n, TSx_n) \leq \lim_{n \to \infty} d(SSx_n, STx_n) + \lim_{n \to \infty} d(STx_n, TSx_n) = 0,
\]
which implies that
\[
d(SSx_n, TSx_n) \to 0 \text{ as } n \to \infty.
\]

Similarly, it can be shown that
\[
d(TTx_n, STx_n) \to 0 \text{ as } n \to \infty.
\]

Therefore, \( S \) and \( T \) are Cauchy compatible of type \( (A) \). \( \square \)

**Proposition 4.5.7.** Let \( S, T : (X, d) \to (X, d) \) be Cauchy compatible mappings of type \( (A) \). If one of \( S \) and \( T \) is continuous, then \( S \) and \( T \) are Cauchy compatible.

Proof. Assume, without loss of generality, that \( T \) is continuous.

To show that \( S \) and \( T \) are Cauchy compatible,
suppose that \( \{x_n\} \) is a Cauchy sequence in \( X \) such that \( S(x_n), T(x_n) \to t \) for some \( t \in X \).

Then \( TS(x_n) \to Ti \) as \( n \to \infty \), since \( T \) is continuous.

But then the triangle inequality implies that
\[
d(TSx_n, TTx_n) \to 0 \text{ as } n \to \infty. \tag{4.12}
\]

But
\[
d(STx_n, TSx_n) \leq d(STx_n, TTx_n) + d(TTx_n, TSx_n) \text{ for all } n.
\]
Since $S$ and $T$ are Cauchy compatible mappings of type $(A)$,
\[ d(ST x_n, TS x_n) \to 0 \text{ as } n \to \infty. \]
Therefore, equation (4.12) and the last inequality imply that
\[ d(ST x_n, TS x_n) \to 0 \text{ as } n \to \infty, \]
so that $S$ and $T$ are Cauchy compatible. \qed

**Proposition 4.5.8.** Let $S, T : (X, d) \to (X, d)$ be continuous mappings. Then $S$ and $T$ are Cauchy compatible if and only if they are Cauchy compatible of type $(A)$.

**Proposition 4.5.9.** Let $S, T : (X, d) \to (X, d)$ be mappings. If $S$ and $T$ are Cauchy compatible of type $(A)$ and $S(t) = T(t)$ for some $t \in X$, then $ST(t) = TT(t) = TS(t) = SS(t)$.

**Proof.** Suppose that $\{x_n\}$ is the Cauchy sequence in $X$ defined by $x_n = t$, $n=1,2,...$, and $S(t) = T(t)$.

Then we have
\[ S(x_n), T(x_n) \to t \text{ as } n \to \infty. \]
Since $S$ and $T$ are Cauchy compatible of type $(A)$,
we have $d(ST(t), TT(t)) = \lim_{n \to \infty} d(ST x_n, TT x_n) = 0$
and so $ST(t) = TT(t)$.

Similarly, we have $TS(t) = SS(t)$.
But $T(t) = S(t)$ implies that $TT(t) = TS(t)$.

Therefore $ST(t) = TT(t) = TS(t) = SS(t)$ . \qed

**Proposition 4.5.10.** Let $S, T : (X, d) \to (X, d)$ be mappings. Let $S$ and $T$ be Cauchy compatible of type $(A)$ and $S x_n, T x_n \to t$ for some $t \in X$. Then we have the following:

1. $\lim_{n \to \infty} TS(x_n) = S(t)$ if $S$ is continuous at $t$,
(2) \( ST(t) = TS(t) \) and \( S(t) = T(t) \) if \( S \) and \( T \) are continuous at \( t \).

**Proof.** Use Proposition 4.5.7 and Proposition 2.2(2) of Jungck (1986).

The following theorem is a generalization of Theorem 3.1 of Pathak et al (1995), in the setting of metric spaces.

**Theorem 4.5.11.** Let \( (X, d) \) be a complete metric space and \( A, B, S \) and \( T \) be self-mappings of \( X \). Suppose that \( S \) and \( T \) are continuous mappings satisfying the following conditions:

\[
A(X) \subset T(X) \quad \text{and} \quad B(X) \subset S(X),
\]

the pairs \( (A, S) \) and \( (B, T) \) are Cauchy compatible of type \((P)\),

\[
d(Ax, By) \leq \Phi(\max\{d(Sx, Ty), d(Sx, Ax), d(Ty, By), \frac{1}{2}[d(Sx, By) + d(Ty, Ax)]\})
\]

for every \( x, y \in X \), where \( \Phi : [0, \infty) \to [0, \infty) \) is a nondecreasing and upper semicontinuous function with \( \Phi(t) < t \) for all \( t > 0 \). Then \( A, B, S \) and \( T \) have a unique common fixed point in \( X \).

**Proof.** Since \( A(X) \subset T(X) \) and \( B(X) \subset S(X) \),

we can choose a Cauchy sequence \( \{x_n\} \) in \( X \) such that

\[
Sx_{2n} = Bx_{2n-1} \quad \text{and} \quad Tx_{2n-1} = Ax_{2n-2} \quad \text{for } n = 1, 2, 3, \ldots
\]

Suppose that

\[
y_{2n-1} = Tx_{2n-1} = Ax_{2n-2} \quad \text{and} \quad y_{2n} = Sx_{2n} = Bx_{2n-1}
\]

By using the technique of Chang (1991), we can prove that \( \{y_n\} \) is a Cauchy sequence in \( X \). Since \( X \) is complete, it converges to a point \( z \in X \).

On the other hand, the subsequences \( \{A_x_{2n-2}\}, \{B_x_{2n-1}\}, \{S_x_{2n}\} \) and \( \{T_x_{2n-1}\} \)
of \{y_n\} also converge to the point \(z\).

Since \((A, S)\) and \((B, T)\) are Cauchy compatible of type \((P)\), it follows from the continuity of \(S\) and \(T\), equation (4.16), and Proposition 4.5.5, that

\[
T y_{2n} \rightarrow Tz, B y_{2n} = B B y_{2n-1} \rightarrow Tz, S y_{2n-1} \rightarrow Sz, A y_{2n-1} = A A y_{2n-2} \rightarrow Sz
\]

(4.17)

as \(n \rightarrow \infty\).

From equations (4.15) and (4.16), we have

\[
d(A y_{2n-1}, B y_{2n}) \leq \Phi(\max \{d(S y_{2n-1}, T y_{2n}), d(S y_{2n-1}, A y_{2n-1}), d(T y_{2n}, B y_{2n}),
\frac{1}{2}[d(S y_{2n-1}, B y_{2n-2}) + d(T y_{2n}, A y_{2n-1})]\})
\]

By the upper semicontinuity of \(\Phi(t)\), equations (4.16), and (4.17), if \(Sz \neq Tz\), then we have

\[
d(Sz, Tz) \leq \Phi(\max \{d(Sz, Tz), 0, 0, d(Sz, Tz)\})
= \Phi(d(Sz, Tz) < d(Sz, Tz),
\]

which is a contradiction. Thus it follows that \(Sz = Tz\).

Similarly, from equations (4.15), (4.16), (4.17), and the upper semicontinuity of \(\Phi\), we obtain \(Sz = Bz\) and \(Tz = Az\).

Hence we have

\[
Az = Bz = Sz = Tz
\]

(4.18)

From equations (4.15) and (4.16), we have also

\[
d(A x_{2n}, Bz) \leq \Phi(\max \{d(S x_{2n}, Tz), d(S x_{2n}, A x_{2n}), d(Tz, Bz),
\frac{1}{2}[d(S x_{2n}, Bz) + d(Tz, A x_{2n})]\}),
\]

which implies that, if \(Bz \neq z\), then
\[ d(z, Bz) \leq \Phi(d(z, Bz) < d(z, Bz), \]

which is a contradiction.

Therefore we have \( z = Az = Bz = S z = T z \). The uniqueness of the fixed point \( z \) is obvious from equation (4.14).

The following result is an immediate consequence of Theorem 4.5.11.

**Theorem 4.5.12.** Let \((X, d)\) be a complete metric space and \( S, T \) and \( A_n \) be self-mappings of \( X, n=1,2,3,...\). Suppose further that \( S \) and \( T \) are continuous, and, for every \( n \in \mathbb{N} \), the pairs \( \{ A_{2n-1}, S \} \) and \( \{ A_{2n}, T \} \) are Cauchy compatible of type \((P)\), \( A_{2n-1}(X) \subset T(X) \) and \( A_{2n}(X) \subset S(X) \) and, for any \( n \in \mathbb{N} \), the set of positive integers, the following condition is satisfied:

\[
\begin{align*}
    d(A_n x, A_{n+1} y) & \leq \Phi \left( \max \{ d(S x, T y), d(S x, A_n x), d(T y, A_{n+1} y), \right. \\
    & \left. \frac{1}{2} [d(S x, A_{n+1} y) + d(T y, A_n x)] \} \right) \tag{4.19}
\end{align*}
\]

for all \( x, y \in X \), where \( \Phi : [0, \infty) \to [0, \infty) \) is a nondecreasing and upper semicontinuous function satisfying \( \Phi(t) < t \) for all \( t > 0 \). Then \( S, T \) and \( \{ A_n \}, n \in \mathbb{N} \) have a unique common fixed point in \( X \).

Throughout this section, \( \mathcal{F} \) will denote family of all mappings \( \phi : (\mathbb{R}^+) ^5 \to \mathbb{R}^+ \) such that \( \phi \) is upper semicontinuous, nondecreasing in each coordinate variable and, for any \( t > 0 \), \( \phi(t, t, 0, \alpha t, 0) \leq \beta t, \phi(t, t, 0, 0, \alpha t) \leq \beta t \), where \( \beta = 1 \) for \( \alpha = 2 \) and \( \beta < 1 \) for \( \alpha < 2 \), \( \gamma(t) = \phi(t, t, a_1 t, a_2 t, a_3 t) < t \), where \( \gamma : \mathbb{R}^+ \to \mathbb{R}^+ \) is a mapping and \( a_1 + a_2 + a_3 = 4 \).

**Lemma 4.5.13.** (Singh and Meade (1977)) For every \( t > 0 \), \( \gamma(t) < t \) if and only if \( \lim _{n \to \infty} \gamma^n(t) = 0 \), where \( \gamma^n \) denotes the \( n \) times composition of \( \gamma \).

Let \( A, B, S \) and \( T \) be self-mappings of a metric space \((X, d)\).

\[
A(X) \subset T(X) \quad \text{and} \quad B(X) \subset S(X) \tag{4.20}
\]
\[ d^B(Ax, By) \leq \phi(d^B(Sx, Ty), d(Sx, Ax)d(Ty, By), d(Sx, By)d(Ty, Ax), d(Sx, Ax)d(Ty, Ax), d(Sx, By)d(Ty, By)) \]  

(4.21)

for every \( x, y \in X \), where \( \phi \in \mathcal{F} \). Then, by equation (4.20), since \( A(X) \subset T(X) \) for an arbitrary point \( x_0 \in X \), there exists a point \( x_1 \in X \) such that \( A x_0 = T x_1 \). Since \( B(X) \subset S(X) \), for this point \( x_1 \), we can choose a point \( x_2 \) in \( X \) such that \( B x_1 = S x_2 \) and so on. Continuing in this manner, we can define a sequence \( \{y_n\} \) in \( X \) such that

\[ y_{2n} = Tx_{2n+1} = Ax_{2n} \quad \text{and} \quad y_{2n+1} = Sx_{2n+2} = Bx_{2n+1} \]  

(4.22)

for \( n = 1, 2, 3, \ldots \).

Let \( A, B, S \) and \( T \) be self-mappings of a metric space \((X, d)\) satisfying equations (4.20) and (4.21). Then we have the following lemmas:

**Lemma 4.5.14.** \( \lim_{n \to \infty} d(y_n, y_{n+1}) = 0 \), where \( \{y_n\} \) is the sequence in \( X \) defined by equation (4.22).

**Proof.** Let \( d_n = d(y_n, y_{n+1}) \), \( n = 0, 1, 2, \ldots \).

We shall prove that the sequence \( \{d_n\} \) is nonincreasing in \( R^+ \); that is, \( d_n \leq d_{n-1} \) for \( n = 1, 2, 3, \ldots \). By equation (4.21),

we have

\[ d^B(Ax_{2n}, Bx_{2n+1}) \leq \phi(d^B(Sx_{2n}, Tx_{2n+1}), d(Sx_{2n}, Ax_{2n})d(Tx_{2n+1}, Bx_{2n+1}), d(Sx_{2n}, Bx_{2n+1})d(Tx_{2n+1}, Ax_{2n}), d(Sx_{2n}, Ax_{2n})d(Tx_{2n+1}, Ax_{2n}), d(Sx_{2n}, Bx_{2n+1})d(Tx_{2n+1}, Bx_{2n+1})). \]
\[ d_{2n}^2 = d^2(y_{2n}, y_{2n+1}) \]
\[ \leq \phi(d^2(y_{2n-1}, y_{2n}), d(y_{2n-1}, y_{2n})d(y_{2n}, y_{2n+1})), \]
\[ d(y_{2n-1}, y_{2n+1})d(y_{2n}, y_{2n})d(y_{2n-1}, y_{2n})d(y_{2n}, y_{2n+1})), \]
\[ \leq \phi(d_{2n-1}^2, d_{2n-1}d_{2n}, 0, 0, (d_{2n-1} + d_{2n})d_{2n}) \quad (4.23) \]

Suppose that \( d_{n-1} < d_n \) for some \( n \).

Then, for some \( \alpha < 2, d_{n-1} + d_n = \alpha d_n \)

Since \( \phi \) is nonincreasing in each variable and \( \beta < 1 \) for some \( \alpha < 2 \), by equation (4.23)

\[ d_{2n}^2 \leq \phi(d_{2n}^2, d_{2n}^2, 0, 0, \alpha d_{2n}^2) \leq \beta d_{2n}^2 < d_{2n}^2, \]

a contradiction. Similarly, for \( n \) odd,

\[ d_{2n+1}^2 \leq \phi(d_{2n+1}^2, d_{2n+1}^2, 0, \alpha d_{2n+1}^2, 0) \leq \beta d_{2n+1}^2 < d_{2n+1}^2. \]

Hence, for every \( n \), \( d_n^2 \leq \beta d_n^2 < d_n^2 \), which is a contradiction.

Therefore, \( \{d_n\} \) is a nonincreasing sequence in \( R^+ \).

Again using equation (4.21), we have

\[ d_1^2 = d^2(y_1, y_2) \]
\[ = d^2(Ax_2, Bx_1) \]
\[ \leq \phi(d_0^2, d_0d_1, 0, (d_0 + d_1)d_1, 0) \]
\[ \leq \phi(d_0^2, d_0^2, 2d_0^2, d_0^2) \]
\[ = \gamma(d_0^2) \]

In general, we have \( d_n^2 \leq \gamma^n(d_0^2) \), which implies, if \( d_0 > 0 \),

by Lemma 4.5.13, that \( \lim_{n \to \infty} d_n^2 \leq \lim_{n \to \infty} \gamma^n(d_0^2) = 0. \)

Therefore \( \lim_{n \to \infty} d_n = 0. \)

If \( d_0 = 0 \), since \( \{d_n\} \) is a nonincreasing, we have \( \lim_{n \to \infty} d_n = 0. \) \( \square \)
Lemma 4.5.15. The sequence \( \{y_n\} \) defined by equation (4.22) is a Cauchy sequence in \( X \).

Proof. We shall use Lemma 4.5.14 to show that \( \{y_n\} \) is a Cauchy sequence.

Suppose that \( \{y_{2n}\} \) is not a Cauchy sequence.

Then there is an \( \varepsilon > 0 \) such that for each even integer \( 2k \), there exist even integers \( 2\delta(k) \) and \( 2\eta(k) \) with \( 2\delta(k) > 2\eta(k) \geq 2k \) such that

\[
d(y_{2\eta(k)}, y_{2\delta(k)}) > \varepsilon
\]  

(4.24)

For each even integer \( 2k \), let \( 2\eta(k) \) be the least even integer exceeding \( 2\eta(k) \) satisfying equation (4.24); that is,

\[
d(y_{2\eta(k)}, y_{2\eta(k)-2}) \leq \varepsilon \quad \text{and} \quad d(y_{2\eta(k)}, y_{2m(k)}) > \varepsilon.
\]  

(4.25)

Then, for each even integer \( 2k \),

we have

\[
\varepsilon \leq d(y_{2n(k)}, y_{2m(k)}) \\
\leq d(y_{2n(k)}, y_{2m(k)-2}) + d(y_{2m(k)-2}, y_{2m(k)-1}) + d(y_{2m(k)-1}, y_{2m(k)}).
\]

By Lemma 4.5.14 and equation (4.25), it follows that

\[
d(y_{2n(k)}, y_{2m(k)}) \to \varepsilon \quad \text{as} \quad k \to \infty
\]  

(4.26)

By the triangle inequality, we have

\[
|d(y_{2n(k)}, y_{2m(k)-1}) - d(y_{2n(k)}, y_{2m(k)})| \leq d(y_{2m(k)} - 1, y_{2m(k)}) \quad \text{and}
\]

\[
|d(y_{2n(k)+1}, y_{2m(k)-1}) - d(y_{2n(k)}, y_{2m(k)})| \leq d(y_{2m(k)} - 1, y_{2m(k)}) + d(y_{2n(k)}, y_{2n(k)+1}).
\]
From Lemma 4.5.14 and equation (4.26), taking the limit as $k \to \infty$

$$d(y_{2n(k)}, y_{2m(k)-1}) \to \varepsilon \quad \text{and} \quad d(y_{2n(k)+1}, y_{2m(k)-1}) \to \varepsilon$$ (4.27)

Therefore, by equations (4.21) and (4.22), we have

$$d(y_{2n(k)}, y_{2m(k)}) \leq d(y_{2n(k)}, y_{2n(k)+1}) + d(y_{2n(k)+1}, y_{2m(k)})$$

$$= d(y_{2n(k)}, y_{2n(k)+1}) + d(Ax_{2m(k)}, Bx_{2m(k)+1})$$

$$\leq d(y_{2n(k)}, y_{2n(k)+1}) + \phi(d^p(Sx_{2m(k)}, Tx_{2n(k)+1}),$$

$$d(Sx_{2m(k)}, Ax_{2m(k)}); d(Tx_{2n(k)+1}, Bx_{2m(k)+1}),$$

$$d(Sx_{2m(k)}, Ax_{2m(k)}); d(Tx_{2n(k)+1}, Ax_{2m(k)}),$$

$$d(Sx_{2m(k)}, Bx_{2m(k)+1})d(Tx_{2n(k)+1}, Bx_{2m(k)+1})]^{\frac{1}{2}}$$ (4.28)

$$\leq d(y_{2m(k)}, y_{2n(k)+1}) + [\phi(d^p(y_{2m(k)}, y_{2m(k)-1}), y_{2n(k)}),$$

$$d(y_{2m(k)-1}, y_{2m(k)}); d(y_{2m(k)-1}, y_{2n(k)}),$$

$$d(y_{2m(k)-1}, y_{2n(k)+1})d(y_{2n(k)-1}, y_{2m(k)}),$$

$$d(y_{2m(k)-1}, y_{2m(k)}); d(y_{2m(k)}; y_{2m(k)}),$$

$$d(y_{2m(k)-1}, y_{2m(k)+1})d(y_{2n(k)}; y_{2n(k)+1})]^{\frac{1}{2}}.$$

Since $\phi$ is upper semicontinuous, as $k \to \infty$, in equation (4.29),

by Lemma 4.5.14, equations (4.26), (4.27) and (4.28),

we have

$$\varepsilon \leq [\phi(\varepsilon^2, 0, \varepsilon^2, 0.0)]^{\frac{1}{2}} \leq [\gamma(\varepsilon^2)]^{\frac{1}{2}} \varepsilon.$$

which is a contradiction.

Therefore, \{y_{2m}\} is a Cauchy sequence in $X$ and so is \{y_n\}.

\[\square\]
The following theorem is a generalization of Theorem 3.4 of Jungck et al (1993) in the setting of metric spaces.

**Theorem 4.5.16.** Let $A, B, S$ and $T$ be self-mappings of a complete metric space $(X, d)$ satisfying the following:

$$A(X) \subset T(X) \quad \text{and} \quad B(X) \subset S(X)$$

(4.29)

$$d^2(Ax, By) \leq \phi(d^2(Sx, Ty), d(Sx, Ax)d(Ty, By), d(Sx, By)d(Ty, Ax))$$

(4.30)

for every $x, y \in X$, where $\phi \in F$.

one of $A, B, S$ and $T$ is continuous,  

(4.31)

and the pairs $(A, S)$ and $(B, T)$ are Cauchy compatible of type $(A)$  

(4.32)

Then $A, B, S$ and $T$ have a unique common fixed point $z$ in $X$.

**Proof.**

By Lemma 4.5.15, the sequence $\{y_n\}$ defined by equation (4.22) is a Cauchy sequence in $X$.

Since $(X, d)$ is complete, it converges to a point $z$ in $X$.

On the other hand, the subsequences $\{Ax_{2n}\}, \{Bx_{2n+1}\}, \{Sx_{2n}\}$ and $\{Tx_{2n+1}\}$ of $\{y_n\}$ also converges to $z$.

Now suppose that $T$ is continuous.

Since $B$ and $T$ are Cauchy compatible of type $(A)$, 

by Proposition 4.5.10, $BT(x_{2n+1}), TT(x_{2n+1}) \to T(z)$ as $n \to \infty$.

Putting $x = x_{2n}$ and $y = Tx_{2n+1}$ in equation (4.30)
we have
\[
   d(Ax_{2n}, BTx_{2n+1}) \leq [\phi(d^2(Sx_{2n}, TTx_{2n+1}),
   d(Sx_{2n}, Ax_{2n})d(TTx_{2n+1}, BTx_{2n+1}),
   d(Sx_{2n}, BTx_{2n+1})d(TTx_{2n+1}, Ax_{2n}),
   d(Sx_{2n}, BTx_{2n+1})d(TTx_{2n+1}, BTx_{2n+1})]^{1/2}.
\] (433)

Taking the limit as \(n \to \infty\) in equation (433), since \(\phi \in \mathcal{F}\), we have
\[
   d(z, Tz) \leq [\phi(d^2(z, Tz), 0, d^2(z, Tz), 0, 0)]^{1/2} \leq [\gamma(d^2(z, Tz))]^{1/2} < d(z, Tz),
\]
which is a contradiction.

Thus, we have \(Tz = z\).

Again, replacing \(x\) by \(x_{2n}\) and \(y\) by \(z\) in equation (430),
we have
\[
   d(Ax_{2n}, Bz) \leq [\phi(d^2(Sx_{2n}, Tz), d(Sx_{2n}, Ax_{2n})d(Tz, Bz),
   d(Sx_{2n}, Bz)d(Tz, Ax_{2n}), d(Sx_{2n}, Ax_{2n})d(Tz, Ax_{2n}),
   d(Sx_{2n}, Bz)d(Tz, Bz))]^{1/2}.
\] (434)

Taking the limit as \(n \to \infty\) in equation (434)
\[
   d(z, Bz) \leq [\phi(0, 0, 0, 0, d^2(z, Bz))]^{1/2} \leq [\gamma(d^2(z, Bz))]^{1/2} < d(z, Bz),
\]
which means that \(Bz = z\).

Since \(B(X) \subset S(X)\), there exists a point \(u \in X\) such that \(Bz = Su = z\).

By using equation (430) again, we have
\[
   d(Au, Bz) = d(Au, Bz) \leq [\phi(d^2(Su, Tz), d(Su, Au)d(Tz, Bz),
   d(Su, Bz)d(Tz, Au), d(Su, Au)d(Tz, Au),
   d(Su, Bz)d(Tz, Bz))]^{1/2}
\] 
\[
   = [\phi(d^2(0, 0, 0, d^2(Au, z), 0))]^{1/2}
\leq [\gamma(d^2(Au, z))]^{1/2}
\leq d(Au, z),
\]
which is a contradiction and $A_{n} = z$.

But since $A$ and $S$ are Cauchy compatible of type (A)
and $A_{n} = S_{n} = z$, by Proposition 4.5.9, $d(A S_{n}, S A_{n}) = 0$ and
we have $A z = A S_{n} = S S_{n} = S z$.

Using equation (4.30), we have

\[ d(A z, z) = d(A, B z) \leq \left[ \phi(d^2(S z, T z), d(S z, A z) d(T z, B z), \\
              d(S z, B z) d(T z, A z), d(S z, A z) d(T z, A z), \\
              d(S z, B z) d(T z, B z)) \right]^{\frac{1}{2}} \]

\[ = \left[ \phi(d^2(A z, z), 0, d^2(A z, z), 0, 0) \right] \]

\[ \leq \left[ \gamma(d^2(A z, z)) \right]^{\frac{1}{2}} \]

\[ < d(A z, z) , \]

which is a contradiction, and so $A z = z$.

Therefore $A z = B z = S z = T z = z$; that is, $z$ is a common fixed point of $A, B, S$ and $T$.

Uniqueness of the common fixed point $z$ follows easily from equation (4.30).

The proof when $A$ or $B$ or $T$ is continuous, is similar, and therefore will be omitted. \( \square \)

**Example 4.5.17.** Let $X = [0, \infty)$ with the usual metric. Define $A, B, S$ and $T : X \rightarrow X$ by

\[ A x = \begin{cases} 
 x & \text{if } 0 \leq x < 1 \\
 1 & \text{if } x \geq 1 
\end{cases} \]

\[ B x = \begin{cases} 
 x & \text{if } 0 \leq x < 2 \\
 1 & \text{if } x \geq 2 
\end{cases} \]
\[ Sx = \begin{cases} 
0 & \text{if } x = 0 \\
2 & \text{if } 0 < x \leq 2 \\
x/2 & \text{if } x > 2 
\end{cases} \]

and

\[ Tx = \begin{cases} 
0 & \text{if } x = 0 \\
2 & \text{if } 0 < x \leq 2 \\
(x + 1)/3 & \text{if } x > 2 
\end{cases} \]

Let \( \phi : (R^+)^5 \to R^+ \) defined by \( \phi(a, b, c, d, e) = a + b + c + d + e \). Then \( \phi \) is upper semi continuous and non decreasing for all \( (a, b, c, d, e) \in (R^+)^5 \). Let \( \{x_n\} = \{0, 0, 0, \cdots \} \) be a Cauchy sequence in \( X \). Then \( ASx_n = A(0) = 0, SSx_n = S(0) = 0, SAx_n = S(0) = 0 \) and \( AAx_n = A(0) = 0 \). Therefore the pair \( (A, S) \) is Cauchy compatible of type \( (A) \). Similarly, the pair \( (B, T) \) is Cauchy compatible of type \( (A) \). Let \( \{x_n\} = \{2 + (1/n)\} \) be a sequence in \( X \). Then \( A x_n \to 1, B x_n \to 1, S x_n \to 1, \) and \( T x_n \to 1 \) as \( n \to \infty \). Therefore the pair \( (A, S) \) is not Cauchy compatible of type \( (A) \). Similarly the pair \( (B, T) \) is not Cauchy compatible of type \( (A) \). Equation (4.30) of Theorem 4.5.16 is also true. Clearly, \( A \) is continuous, \( B, S, T \) are discontinuous. The conditions of the Theorem 4.5.16 are satisfied and \( 0 \in X \) is the unique common fixed point of the maps \( A, B, S \) and \( T \). Hence \( A \) and \( S \) are Cauchy compatible. Also, \( B \) and \( T \) are Cauchy compatible. Let \( \{x_n\} = 1/n \) be a sequence in \( X \). Then \( \lim_{n \to \infty} A x_n = 1, \lim_{n \to \infty} S x_n = 1, \lim_{n \to \infty} B x_n = 1, \lim_{n \to \infty} T x_n = 1, \) and \( \lim_{n \to \infty} d(A x_n, S x_n) \neq 0 \). Therefore \( A \) and \( S \) are not compatible. Also, \( \lim_{n \to \infty} d(B T x_n, T B x_n) \neq 0 \). Therefore \( B \) and \( T \) are not compatible.