CHAPTER 2

FIXED POINT THEOREMS FOR METRIC SPACES AND CONE METRIC SPACES

2.1 INTRODUCTION

Kannan (1968) proved a fixed point theorem for mappings satisfying a contractive condition that did not require continuity at each point. Most of the common fixed point theorems for contraction mappings invariably require a compatibility condition besides assuming continuity of at least one of the mappings. Pant (1999) noticed these criteria for fixed points of contraction mappings and introduced a new continuity condition known as reciprocal continuity and obtained a common fixed point theorem by using compatibility in metric spaces. Sessa (1982) defined the concept of weakly commuting. Later, Jungck (1986) generalized weakly commuting to compatible mappings also called asymptotic commutativity by Tivari and Singh (1986).

Pathak (1986) introduced the concept ‘weak* commuting pair’ which generalizes commuting mappings. Jungck (1996) generalized compatible to weakly compatible mappings. Pant (1994) introduced the notion of pointwise $R$-weak commutative and showed that compatible mappings are pointwise $R$-weak commutative but the converse need not be true. Pathak et al (1997) coined $R$-weakly commuting of type ($A_f$) and $R$-weakly commuting of type ($A_g$). Also, Pant (1999) introduced the notion called non-compatible mappings. Al Thagafi and Shahzad (2008) introduced occasionally weakly compatible which generalizes weakly compatible. Rzepecki (1980) introduced a generalized metric by replacing the set of real numbers with a normal cone of a Banach space. Lin (1987) introduced the notion of K-metric spaces by replacing the set of real numbers with a cone in
the metric function. Zabrejko (1997) studied this new revised version of fixed point theory in $K$-metric and $K$-normed linear spaces. Guang and Xian (2007) generalized the notion of a metric space by replacing the set of real numbers with an ordered Banach space, defining in these way a cone metric space. These authors also described the convergence of sequences in these cone metric spaces and introduced the corresponding notion of completeness. The limit of a convergent sequence is unique provided $P$ is a normal cone with normal constant $K$. Recently, non-convex analysis has found some applications in optimization theory, and so there have been some investigations in non-convex analysis, especially ordered normed spaces, normal cones and topical functions; for example, see Guang and Xian (2007), Mohebi (2005), Mohebi et al (2006). Rezapour, Hamilbarani (2008) showed that there are no normal cones with normal constant $M < 1$ and, for each $k > 1$, there are cones with normal constant $M > k$. They also provided examples of non-normal cones and omitted the assumption of normality. The study of fixed points of functions satisfying certain contractive conditions has been at the center of a vigorous research activity. Fixed point theory has a wide range of applications in different areas, such as nonlinear and adaptive control systems, fractal image decoding, computing magnetostatic fields in a nonlinear medium, and convergence of recurrent networks. Shobha Jain et al (2010) introduced the concept of compatibility in a cone metric space. In section 2.4, vectorial versions of pointwise $R$-weakly commuting mappings and reciprocal continuous in a cone metric space have been introduced, and also the results of Sanjay Kumar and Renu Chugh (2002) have been generalized.

2.2 PRELIMINARIES

Definition 2.2.1. (Pant (1999)) Let $A$ and $S$ be self-mappings of a metric space $(X, d)$. $A$ and $S$ are said to be reciprocal continuous if $\lim_{n \to \infty} ASx_n = At$ and $\lim_{n \to \infty} SAx_n = St$, whenever $\{x_n\}$ is a sequence in $X$ such that $\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Sx_n = t$ for some $t \in X$. 
Continuous mappings are reciprocal continuous on \((X,d)\). But the converse may not be true.

**Definition 2.2.2. (Jungck (1976))** Let \(A\) and \(T\) be self-mappings of a metric space \((X,d)\). \(A\) and \(T\) are said to be commuting if \(AT(x) = TA(x)\) for all \(x \in X\).

**Example 2.2.3.** Let \(X = \mathbb{R}\) with the usual metric and define \(f, g : \mathbb{R} \to \mathbb{R}\) by \(f(x) = x^2\) and \(g(x) = x^3\) for all \(x \in X\). Then \(f(g(x)) = f(x^3) = x^6\) and \(g(f(x)) = g(x^2) = x^6\) for all \(x \in X\).

**Definition 2.2.4. (Sessa (1982))** Let \(A\) and \(T\) be self-mappings of a metric space \((X,d)\). \(A\) and \(T\) are said to be weakly commuting if \(d(ATA(x), TA(x)) \leq d(A(x), T(x))\) for all \(x \in X\).

**Example 2.2.5. (Rhoades and Sessa (1986))** Let \(X = [0,1]\) be endowed with the Euclidean metric and define \(A, T : X \to X\) by \(Ax = x/(x + \alpha a)\) and \(Tx = x/(x + \alpha a)\) for each \(x \in X\), where \(\alpha \geq 1\) and \(a > 1\). Then \(d(ATAx, TAx) = x/(x+\alpha a^2) - x/(ax+\alpha a^2) = (a-1)x^2/(x+\alpha a^2)(ax+\alpha a^2)\) and \(d(Tx, Ax) = x/a - x/(x+\alpha a) = x^2 + a(a-1)x/a(x+\alpha a)\) so that \(A\) and \(T\) satisfy \(d(ATA(x), TAx) \leq d(T(x), A(x))\) but they do not commute, since \(ATAx \neq TAx\) for each \(x \neq 0\).

**Definition 2.2.6. (Pathak (1986))** Let \(A\) and \(S\) be two self-mappings of \(X\). The pair \((A, S)\) is said to be a ‘weak\(^k\) commuting pair’ if \(d(ASx, SAx) \leq d(A^2x, S^2x)\) for all \(x \in X\).

**Example 2.2.7. (Pathak (1986))** Consider \(X = [0,1]\) with the usual metric. Define \(A, S : X \to X\) by \(Ax = x/2\) and \(Sx = x/2 + x\) for every \(x \in X\). For all \(x\) in \(X\), one gets \(d(SAx, ASx) = \left| x/(4+x) - x/(4+2x) \right| = x^2/(4+x)(4+2x) = 3x^2/(12+3x)(4+2x) \leq 3x^2/(12+3x)4 < 3x^2/(4+3x)4 = x/4-x/(4+3x) = d(A^2x, S^2x)\), so \((A, S)\) is a weak\(^k\) commuting pair. But, for any non-zero \(x \in X\), we have \(S Ax = x/(4+x) > x/(4+2x) = ASx\), so that \(SA \neq AS\). Thus \(A\) and \(S\) are not commuting mappings.

**Definition 2.2.8. (Jungck (1986))** Let \(A\) and \(S\) be self-mappings of a metric space \((X,d)\). \(A\) and \(S\) are called compatible mappings if
\[ \lim_{n \to \infty} d(ASx_n, SAx_n) = 0, \] whenever \{x_n\} is a sequence in \(X\) such that
\[ \lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Sx_n = t \] for some \(t \in X\).

**Example 2.2.9. (Jungek (1986))** Let \(X = [0, \infty)\) be endowed with the usual metric. Let \(f, g : X \to X\) be mappings defined by \(fx = x^3\) and \(g(x) = 2x^3\). Then \(fgx \neq gfx\). So \(f\) and \(g\) are not commuting on \(X\) and \(|fgx - gfx| > |fx - gx|\). Therefore \(f\) and \(g\) are not weakly commuting on \(X\). However, \(\lim_{x \to 0} |fx - gx| = 0 \in X\) and it implies \(\lim_{x \to 0} |fgx - gfx| = 0\). Therefore, \(f\) and \(g\) are compatible.

**Example 2.2.10. (Anita Tomar and Singh(2000))** The commutativity and compatibility of a pair of self-mappings at a point is given below. Let \(X = [2, 20]\) with the usual metric. Let \(f, g : X \to X\) be mappings defined by
\[
f(x) = \begin{cases} 
2 & \text{if } x = 2 \\
12 + x & \text{if } 2 < x \leq 5 \\
(x - 3) & \text{if } x > 5
\end{cases}
\]
\[
g(x) = \begin{cases} 
2 & \text{if } x = 2 \text{ or } x > 5 \\
8 & \text{if } 2 < x \leq 5
\end{cases}
\]
Then, \(fg(2) = gf(2)\). A decreasing sequence \(\{x_n\}\) is concerned such that \(\lim_{n} x_n = 5\), then \(\lim_{n} fx_n = \lim_{n} (x_n - 3) = 2\) and \(\lim_{n} gx_n = 2\), \(\lim_{n} gfx_n = 2\), \(\lim_{n} g(x_n - 3) = 8\). So \(\lim_{n} d(fgx_n, gfx_n) = 6\), and \(f\) and \(g\) are not compatible on \(X\). This is very exciting to see that \(f\) and \(g\) are commuting at \(x = 2\) and \(\lim_{n} fx_n = \lim_{n} gx_n = 2\) but \(f\) and \(g\) are not compatible on \(X\). Commutativity at a point doesn’t mean compatibility at the same point. But if one defines the compatibility at a point (say \(x = z\)) by only considering the sequence \(x_n = z\), then commutativity and compatibility at this point are equivalent.

**Definition 2.2.11. (Pant (1994))** Two self-mappings \(A\) and \(S\) of a metric space \((X, d)\) are called \(R\)-weakly commuting at a point \(x \in X\) if \(d(ASx, SAx) \leq Rd(Ax, Sx)\) for some \(R > 0\).

**Definition 2.2.12. (Pant (1994))** The mappings \(A\) and \(S\) are called pointwise \(R\)-weakly commuting if given \(x \in X\), there exists \(R > 0\) such that \(d(ASx, SAx) \leq Rd(Ax, Sx)\). Obviously, weak commutativity implies \(R\)-weak
commutativity. However, $R$-weak commutativity implies weak commutativity only when $R \leq 1$.

The notion of pointwise $R$-weak commutativity is equivalent to commutativity for coincidence points. That is, two maps are pointwise $R$-weakly commuting if and only if they are weakly compatible.

**Definition 2.2.13. (Pathak et al (1997))** Mappings $f$ and $g$ are $R$-weakly commuting of type $(A_f)$ if there exists an $R > 0$ such that $d(gfx, ggx) \leq Rd(fx, gx)$ for all $x \in X$.

**Definition 2.2.14. (Pathak et al (1997))** Mappings $f$ and $g$ are $R$-weakly commuting of type $(A_g)$ if there exists an $R > 0$ such that $d(gfx, ffx) \leq Rd(fx, gx)$ for all $x \in X$.

**Example 2.2.15. (Singh, Anita Tomar (2003))** Let $X = [1, \infty)$ and $d$ be the usual metric on $X$. Let $f, g : X \to X$ be such that $fx = 2x - 1$ and $gx = x^2$ for all $x \in X$. Thus $d(gfx, ggx) = (x^2 - 1)^2$ and $d(gfx, ggx) > Rd(fx, gx)$ for each $x > 1$ and some $R > 0$ (For example, take $R=3$). Thus an $R$-weakly commuting pair of self-mappings need not be $R$-weakly commutative of type $(A_f)$.

The following example shows that a pair of self-mappings may be $R$-weakly commuting and $R$-weakly commuting of type $(A_f)$.

**Example 2.2.16. (Pant(1999))** Let $X=[2,20]$ endowed with the usual metric. Let $f, g : X \to X$ be mappings defined as

\[
fx = \begin{cases} 
2 & \text{if } x = 2 \\
12 & \text{if } 2 < x \leq 5 \\
(x + 1)/3 & \text{if } x > 5
\end{cases}
\]

and

\[
gx = \begin{cases} 
2 & \text{if } x = 2 \text{ or } x > 5 \\
6 & \text{if } 2 < x \leq 5
\end{cases}
\]
These mappings are $R$-weakly commuting of type $(A_f)$. Also, these mappings are $R$-weakly commuting.

**Definition 2.2.17.** Let $f$ and $g$ be self-mappings of a metric space $(X, d)$ and define $C(f, g) := \{x \in X : f(x) = g(x)\}$. The pair $(f, g)$ is called:

(i) weakly compatible (Jungck (1996)) if $f(g(x)) = g(f(x))$ for all $x \in C(f, g)$.

(ii) occasionally weakly compatible (Al-Thagafi and Shahzad (2008)) if $f(g(x)) = g(f(x))$ for some $x \in C(f, g)$.

**Example 2.2.18.** (Al Thagafi and Shahzad (2009)) Let $X = [0, \infty)$ with the usual metric. Define $f, g : X \to X$ by $f(x) = 2x$ and $g(x) = x^2$ for all $x \in X$. Then $C(f, g) := \{0, 2\}$, $f(g(0)) = g(f(0))$, and $f(g(2)) \neq g(f(2))$. Thus $C(f, g)$ is an occasionally weakly compatible pair but is not weakly compatible.

We now state some properties of cone metric spaces.

**Definition 2.2.19.** (Huang Long-guang and Zhang Xian (2007)) Let $E$ be a real Banach space. A subset $P$ of $E$ is said to be a cone if it satisfies the following:

(i) $P \neq \emptyset$ and $P$ is closed;

(ii) $ax + by \in P$ for every $x, y \in P$ and $a, b \geq 0$;

(iii) $P \cap (-P) = \{0\}$.

The partial ordering $\leq$ with respect to a cone $P$ is defined by $x \leq y$ if and only if $y - x \in P$, and $x < y$ means that $x \leq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in \text{int}P$, where $\text{int}P$ denotes the interior of $P$. 


Definition 2.2.20. (Huang Long-guang and Zhang Xian(2007)) Suppose that $E$ is a real Banach space, $P$ is a cone in $E$ with $\text{int } P \neq \emptyset$ and $\leq$ is a partial ordering with respect to $P$. The cone $P$ is called a normal cone if there is a number $K > 0$ such that for every $x, y \in E$, $0 \leq x \leq y \Rightarrow \| x \| \leq K \| y \|$. The least positive number satisfying the above is called the normal constant of $P$.

Definition 2.2.21. (Huang Long-guang and Zhang Xian(2007)) Let $X$ be a nonempty set. Suppose that the mapping $d : X \times X \to E$ satisfies:

1. $d(x, y) > 0$ for every $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$
2. $d(x, y) = d(y, x)$ for every $x, y \in X$
3. $d(x, y) \leq d(x, z) + d(y, z)$ for every $x, y, z \in X$.

Then $d$ is called a cone metric on $X$ and $(X, d)$ is called a cone metric space. This is more general than a metric space.

Example 2.2.22. (Huang Long-guang and Zhang Xian(2007)) Let $E = \mathbb{R}^2$, $P = \{(x, y) \in E : x, y \geq 0\}$, $X = \mathbb{R}$ and $d : X \times X \to E$ be defined by $d(x, y) = (\| x - y \|, \alpha \| x - y \|)$, where $\alpha \geq 0$ is a constant. Then $(X, d)$ is a cone metric space.

Definition 2.2.23. (Huang Long-guang and Zhang Xian(2007)) Let $(X, d)$ be a cone metric space. Let $\{x_n\}$ be a sequence in $X$ and $x \in X$.

(i) $\{x_n\}$ converges to $x$ if, for every $c \in E$ with $0 \ll c$, there is an $n_0$ such that $d(x_n, x) \ll c$ for all $n \geq n_0$. We denote this fact by saying that $\lim_{n \to \infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.

(ii) if, for any $c \in E$ with $0 \ll c$, there is an $n_0$ such that $d(x_n, x_m) \ll c$ for all $n, m \geq n_0$, then $\{x_n\}$ is called a Cauchy sequence in $(X, d)$.

Definition 2.2.24. (Huang Long-guang and Zhang Xian(2007)) $(X, d)$ is called a complete cone metric space if every Cauchy sequence in $X$ is convergent in $X$. 
Definition 2.2.25. (Shoba Jain et al (2010)) The self-mappings $A$ and $S$ of a cone metric space $(X, d)$ are called compatible mappings if, for $\{x_n\}$ in $X$, $Ax_n \to u$ and $Sx_n \to u$, for some $u \in X$, then, for every $c \in I^R$, there is a positive integer $N$ such that $d(ASx_n, SAx_n) < < c$, for all $n > N$.

Definition 2.2.26. Let $A$ and $S$ be self-mappings of a cone metric space $(X, d)$. The maps $A$ and $S$ are called reciprocal continuous if for $\{x_n\}$ in $X$, $Ax_n \to u$ and $Sx_n \to u$ for some $u \in X$, then $ASx_n \to Au$ and $SAx_n \to Su$.

Continuous mappings are reciprocal continuous on $(X, d)$, but the converse need not be true.

Definition 2.2.27. Two self-mappings $S$ and $T$ of a cone metric space $(X, d)$ are called pointwise $R$-weakly commuting mappings on $X$ if, for a given $x \in X$, there exists an $R > 0$ such that $d(ASx, SAx) \leq Rd(Ax, Sx)$.

Compatible mappings are necessarily pointwise $R$-weakly commuting, since compatible mappings commute at their coincidence points. But the converse may not be true.

Example 2.2.28. (Shoba Jain et al (2010)) Let $E = \mathbb{R}^2$, $P = \{(x, y) : x, y \geq 0\} \subseteq I^R$ be a cone in $E$. Take $X = \mathbb{R}$. Fix a real number $\alpha > 0$ and define $d : X \times X \to E$ by $d(x, y) = |x - y| (1, \alpha)$. Then $(X, d)$ is a complete cone metric space. Define self-mappings $A$ and $S$ on $X$ as follows:

$$Ax = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational} \end{cases}$$

and

$$Sx = \begin{cases} x/2 & \text{if } x \in [0, 2] \\ 2 & \text{otherwise.} \end{cases}$$

If $\{r_n\}$ is a sequence of rationals such that $A(r_n) \to u$ and $S(r_n) \to u$, then $u = 0$ and $SAr_n = S(0) = 0$ and $S(r_n) = r_n/2$ gives $AS(r_n) = 0$. Thus $d(ASr_n, SAx_n) = 0$. Hence the pair of the self-mappings $(A, S)$ is compatible.

Remark 2.2.29. (Delbosco(1976-77)) Consider the set $S$ of all real continuous functions $g : [0, \infty)^3 \to [0, \infty)$ satisfying the following properties:
(a) \( g(1,1,1)=h < 1 \)

(b) if \( u, v \geq 0 \) are such that \( u \leq g(u, v, v) \) or \( u \leq g(v, u, v) \) or \( u \leq g(v, v, u) \), then \( u \leq hv \).

\textbf{(Constantin(1991))} Consider the family \( G \) of all continuous functions \( g \), where \( g:[0,\infty)^5 \rightarrow [0,\infty] \) and satisfies the following properties:

(c) \( g \) is non-decreasing in the 4th and 5th variables,

(d) if \( u, v \in [0,\infty) \) are such that \( u \leq g(v, v, u, u + v, 0) \) or \( u \leq g(v, v, u + v, 0) \) or \( u \leq g(v, u + v, v, 0) \) or \( u \leq g(v, v, 0, u + v) \), then \( u \leq hv \), where \( 0 < h < 1 \) is a given constant.

(e) if \( u \in [0,\infty) \) is such that \( u \leq g(u, 0, 0, u, u) \) or \( u \leq g(0, u, 0, u, u) \) or \( u \leq g(0, 0, u, u, u) \), then \( u = 0 \).

2.3 SOME COMMON FIXED POINT THEOREMS USING WEAKLY COMPATIBLE AND RECIPROCAL CONTINUITY CONDITIONS IN METRIC SPACES

In this section, pointwise R-weakly commuting and compatibility are replaced by weakly compatible mappings, which generalize the results of Sanjay Kumar and Renu Chugh (2002).

\textbf{Lemma 2.3.1.} Let \( A, B, S \) and \( T \) be self-mappings of a complete metric space \((X, d)\) satisfying equations (2.1) and (2.2)

\[
A(X) \subset T(X), \quad B(X) \subset S(X) \tag{2.1}
\]

\[
d(Ax, By) \leq g(d(Sx, Ty), d(Ax, Sx), d(By, Ty), d(Ax, Ty), d(By, Sx)) \tag{2.2}
\]

for every \( x, y \in X \), where \( g \in G \). Define a sequence \( \{y_n\} \) in \( X \) by

\[
y_{2n} = Ax_{2n} = Tx_{2n+1} \quad \text{and} \quad y_{2n+1} = By_{2n+1} = Sx_{2n+2}, \tag{2.3}
\]
for \(n=1,2,3,...\). Then the sequence \(\{y_n\}\) defined by equation (2.3) is a Cauchy sequence in \(X\).

**Proof.** From equation (2.2), we obtain

\[
d(Ar_{2n+2}, Br_{2n+1}) \leq g(d(Sr_{2n+2}, Tr_{2n+1}), d(Ar_{2n+2}, Sr_{2n+2}), d(Br_{2n+1}, Tr_{2n+1}),
\]

\[
d(Ar_{2n+2}, Tr_{2n+1}), d(Br_{2n+1}, Sr_{2n+2})).
\]

\[
d(y_{2n+2}, y_{2n+1}) \leq g(d(y_{2n+1}, y_{2n}), d(y_{2n+2}, y_{2n+1}), d(y_{2n+1}, y_{2n}),
\]

\[
d(y_{2n+2}, y_{2n}), d(y_{2n+1}, y_{2n+1}))
\]

\[
\leq g(d(y_{2n+1}, y_{2n}), [d(y_{2n+2}, y_{2n}) + d(y_{2n}, y_{2n+1})], d(y_{2n+1}, y_{2n}),
\]

\[
d(y_{2n+2}, y_{2n}), 0)
\]

By condition (d) of Remark 2.2.9, we obtain \(d(y_{2n+2}, y_{2n+1}) \leq h d(y_{2n}, y_{2n+1})\).

But \(d(y_n, y_{n+1}) \leq h d(y_{n-1}, y_n) \leq \ldots \leq h^n d(y_0, y_1)\)

For every integer \(m > 0\), we have

\[
d(y_n, y_{n+m}) \leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + \ldots + d(y_{n+m-1}, y_{n+m})
\]

\[
\leq h^n (d(y_0, y_1) + h^n (d(y_0, y_1) + \ldots + h^{n+m-1})(d(y_0, y_1))
\]

\[
= h^n ((1 + h + h^2 + \ldots + h^{m-1})d(y_0, y_1)
\]

\[
d(y_n, y_{n+m}) \leq h^n / (1 - h) d(y_0, y_1).
\]

Taking limit as \(n \to \infty\) on both sides, we get \(d(y_n, y_{n+m}) \to 0\).

Therefore \(\{y_n\}\) is a Cauchy sequence in \(X\). \(\Box\)

Theorem 2.3.2 is a generalization of Theorem 3.1 of Sanjay Kumar and Renu Chugh (2002)

**Theorem 2.3.2.** Let \((A, S)\) and \((B, T)\) be pairs of self mappings of a complete metric space \((X, d)\) satisfying equations (2.1) and (2.2). Suppose that \((A, S)\) and \((B, T)\) are weakly compatible pairs of reciprocal continuous mappings. Then \(A,B,S\) and \(T\) have a unique common fixed point in \(X\).
Proof. By Lemma 2.3.1, \( \{y_n\} \) is a Cauchy sequence in \( X \).

Since \( X \) is complete, there exists a point \( z \) in \( X \) such that 
\[
\lim_{n \to \infty} y_n = z
\]
\[
\lim_{n \to \infty} Ax_{2n} = \lim_{n \to \infty} Tx_{2n+1} = z
\]
and 
\[
\lim_{n \to \infty} Bx_{2n+1} = \lim_{n \to \infty} Sx_{2n+2} = z.
\]

Suppose \( A \) and \( S \) are weakly compatible and reciprocal continuous.

By the reciprocal continuity of \( A \) and \( S \), we have 
\[
\lim_{n \to \infty} ASx_{2n} = Az \quad \text{and} \quad \lim_{n \to \infty} SAx_{2n} = Sz.
\]

By the weakly compatibility of \( A \) and \( S \), \( Az = Sz \) for some \( t \in X \) implies that 
\( ASz = SAz \).

Similarly, by the reciprocal continuity of \( B \) and \( T \),
we have 
\[
\lim_{n \to \infty} BTx_{2n} = Bv \quad \text{and} \quad \lim_{n \to \infty} TBx_{2n} = Tv.
\]

By the weakly compatibility of \( B \) and \( T \), \( Bv = Tv \) for some \( v \in X \) implies that 
\( BTv = TBv \).

Since \( A(X) \subset T(X) \), it follows that \( Az = Tv \).

Thus \( Az = Tv = Sz = Bv \) and \( AAz = ASz = SAz = SSz \).

We have \( BBv = BTv = TBv = TTv \).

From equation (2.2), we have
\[
d(Az, AAz) = d(AAz, Bv)
\]
\[
\leq g(d(SAz, Tv), d(AAz, SAz), d(Bv, Tv), d(AAz, Tv), d(Bv, SAz))
\]
\[
= g(d(AAz, Bv), 0, 0, d(AAz, Bv), d(AAz, Bv))
\]

By condition (e) of Remark 2.2.29,
\( AAz = Az \), so \( Az = AAz = SAz \).

Therefore \( Az \) is a common fixed point of \( A \) and \( S \).

Similarly, it can be proved that \( Br (= Az) \) is a common fixed point of \( B \) and \( T \).

To prove uniqueness of \( Az \).

Suppose that \( Az \) and \( Aw \), \( Az \neq Aw \) are common fixed points of \( A, B, S \) and \( T \).
By equation (2.2),

\[ d(Az, Aw) = d(AAz, BAw) \]
\[ \leq g(d(SAz, TAw), d(AAz, SAz), d(BAw, TAw), d(AAz, TAw), d(BAw, SAz)) \]
\[ = g(d(Az, Aw), d(Az, Az), d(Aw, Aw), d(Az, Aw), d(Aw, Az)) \]
\[ = g(d(Az, Aw), 0, 0, d(Az, Aw), d(Az, Aw)) \]

By condition (e) of Remark 2.2.29, \( Az = Aw \).

This completes the proof. \( \Box \)

The following Corollary 2.3.3 and Corollary 2.3.4 are generalizations of Corollary 3.1 and Corollary 3.2 of Sanjay Kumar and Renu Chugh(2002)

**Corollary 2.3.3.** Let \((A, S)\) and \((B, T)\) be pairs of self-mappings of a complete metric space \((X, d)\) satisfying equations (2.1), (2.3),(2.4) and

\[ d(Ax, By) \leq h M(x, y), 0 \leq h < 1, x, y \in X, \]  

(2.4)

where

\[ M(x, y) = \max\{d(Sx, Ty), d(Ax, Sx), d(By, T y), [d(Ax, Ty) + d(By, Sx)]/2\}. \]

Suppose that \((A, S)\) or \((B, T)\) is an occasionally weakly compatible pair of reciprocal continuous mappings. Then \( A, B, S \) and \( T \) have a unique common fixed point in \( X \).

**Proof.** Take the function \( g : [0, \infty)^5 \to [0, \infty) \) defined by

\[ g(x_1, x_2, x_3, x_4, x_5) = h \max\{x_1, x_2, x_3, 1/2(x_4 + x_5)\}, \]

where \( x_4 + x_5 \neq 0 \). Then \( g \) is a special case of equation (2.1), and the result follows from Theorem 2.3.2. \( \Box \)
Corollary 2.3.4. Let \((A, S)\) and \((B, T)\) be pairs of self-mappings of a complete metric space \((X,d)\) satisfying equations (2.1), (2.3) and (2.5) given below.

\[
d(Ax, By) \leq h \max \{d(Ax, Sx), d(By, Ty), 1/2d(Ax, Ty), 1/2d(By, Sx), d(Sx, Ty)\}
\]

(2.5)

for all \(x, y \in X\), where \(0 \leq h < 1\). Suppose that \((A, S)\) or \((B, T)\) is an occasionally weakly compatible pair of reciprocal continuous mappings. Then \(A, B, S\) and \(T\) have a unique common fixed point in \(X\).

Proof: Consider the function \(g : [0, \infty)^3 \rightarrow [0, \infty]\) defined by

\[
g(x_1, x_2, x_3, x_4, x_5) = h \max \{x_1, x_2, x_3, 1/2x_4, 1/2x_5\},
\]

where \(x_4, x_5 \neq 0\). Then \(g\) is a special case of equation (2.1), and the result follows from Theorem 2.3.2.

2.4 COMMON FIXED POINT THEOREM USING COMPATIBILITY AND RECIPROCAL CONTINUITY CONDITIONS IN CONE METRIC SPACES

In this section, vectorial version of two new concepts reciprocal continuity and pointwise R-weakly commuting have been introduced. In cone metric space, distance between any two elements is a vector. The main result Theorem 2.4.2 generalizes Sanjay Kumar and Renu Chugh(2002).

Let \(A, B, S\) and \(T\) be self-mappings of a cone metric space \((X, d)\). Let \(P\) be a normal cone with normal constant \(K\) satisfying the following conditions: Let \((A, S)\) and \((B, T)\) be pointwise R-weakly commuting pairs of self mappings of a complete cone metric space \((X, d)\) such that

\[
A(X) \subset T(X), B(X) \subset S(X)
\]

(2.6)

\[
d(Ax, By) \leq g(d(Sx, Ty), d(Ax, Sx), d(By, Ty), d(Ax, Ty), d(By, Sx))
\]

(2.7)
for every $x, y \in X$, where $g \in G$. Then, for an arbitrary point $x_0$ in $X$, by equation (2.8), a point $x_1$ has been chosen such that $T x_1 = A x_0$ and for this point $x_1$, there exists a point $x_2$ in $X$ such that $S x_2 = B x_1$ and so on. Continuing in this manner, sequence $\{y_n\}$ can defined in $X$ such that

$$y_{2n} = A x_{2n} = T x_{2n+1}, \quad y_{2n+1} = B x_{2n+1} = S x_{2n+2},$$

for $n = 1, 2, 3, \ldots$.

**Theorem 2.4.1.** Let $(A, S)$ and $(B, T)$ be pointwise $R$-weakly commuting pairs of self mappings of a complete cone metric space $(X, d)$ satisfying equations (2.6) and (2.7). Suppose that $(A, S)$ or $(B, T)$ is a compatible pair of reciprocal continuous mappings. Let $P$ be a normal cone with normal constant $K$. Then $A, B, S$ and $T$ have a unique common fixed point in $X$.

**Proof.** By Lemma 2.3.1, $\{y_n\}$ is a Cauchy sequence in $X$.

Since $X$ is a complete cone metric space,

there exist a point $z$ in $X$ such that $\lim_{n \to \infty} y_n = z$, $\lim_{n \to \infty} A x_{2n} = \lim_{n \to \infty} T x_{2n+1} = z$ and $\lim_{n \to \infty} B x_{2n+1} = \lim_{n \to \infty} S x_{2n+2} = z$.

Suppose $A$ and $S$ are compatible and reciprocal continuous.

By the reciprocal continuity of $A$ and $S$, we have $A S x_{2n} \to A z$ and $S A x_{2n} \to S z$.

By the compatibility of $A$ and $S$, we have $A z = S z$.

Since $A(X) \subseteq T(X)$, there exists a point $v$ in $X$ such that $A z = T v$.

$$d(A z, B v) \leq g(d(S z, T v), d(A z, S z), d(B v, T v),$$

$$d(A z, T v), d(B v, S z))$$

$$= g(d(T v, T v), d(S z, S z), d(A z, B v), d(A z, A z), d(A z, B v))$$

$$= g(0, 0, d(A z, B v), 0, d(A z, B v)).$$

By condition (e) of Remark 2.2.29, we have $A z = B v$. 
Thus $Az = Sz = T v = B v$.

Since $A$ and $S$ are pointwise $R$-weak commutativity, there exists $R > 0$ such that

$$d(A S z, S A z) \leq R d(A z, S z) = 0$$

$$\Rightarrow A S z = S A z$$

and $A A z = A S z = S A z = S S z$.

Also, $B$ and $T$ are pointwise $R$-weak commutative,

we have $B B v = B T v = T B v = T T v$.

From equation (2.7),

$$d(A z, A A z) = d(A A z, B v)$$

$$\leq g(d(S A z, T v), d(A A z, S A z), d(B v, T v), d(A A z, T v), d(B v, S A z))$$

$$= g(d(A A z, B v), 0, 0, d(A A z, Br), d(A A z, Br))$$

By condition (e) of Remark 2.2.29, we have

$A A z = A z$,

so $A z = A A z = S A z$.

Therefore $A z$ is a common fixed point of $A$ and $S$.

Similarly, it can be proved that $B v (= A z)$ is a common fixed point of $B$ and $T$.

To prove uniqueness of $A z$.

Suppose that $A z$ and $A w$, $A z \neq A w$ are common fixed points of $A, B, S$ and $T$.

By equation(2.7),

$$d(A z, A w) = d(A A z, B A w)$$

$$\leq g(d(S A z, T A w), d(A A z, S A z), d(B A w, T A w), d(A A z, T A w), d(B A w, S A z))$$

$$= g(d(A z, A w), d(A z, A z), d(A w, A w), d(A z, A w), d(A w, A z))$$

$$= g(d(A z, A w), 0, 0, d(A z, A w), d(A z, A w))$$

By condition (e) of Remark 2.2.29, we obtain $A z = A w$. 

$\square$