CHAPTER 1

INTRODUCTION

1.1 METRIC FIXED POINT THEORY

In late nineteenth century, set theory was developed mostly by Cantor. During the same period Riemann, Ascoli, Arzela, Hadamard and others inspired by the needs of the calculus of variations introduced the concept of function spaces—sets in which points are functions. Frechet, in 1906, using abstract set theory, introduced an axiomatic approach to function spaces. Abstract sets were considered, using the abstract limit process by him. He also introduced abstract metric spaces. Notions related to compactness, separability, and completeness were introduced and the relation between compactness and total boundedness was discussed by him. The theory of abstract metric spaces was created by Frechet and Hausdorff. The name metric space (metrischer Raum) was introduced by Hausdorff. The neighborhood formulation of topology is due to Hilbert, Riesz, Frechet, and especially Hausdorff. Fixed point theory has become one of the most interesting areas of research in the last fifty years. Fixed point theory has three classifications

(i) Metric Fixed Point Theory

(ii) Topological Fixed Point Theory

(iii) Discrete Fixed Point Theory.

The following three theorems are very important for each of the above theories, respectively.

(i) Banach’s Fixed Point Theorem
(ii) Brouwer’s Fixed Point Theorem

(iii) Tarski’s Fixed Point Theorem.

The scope of the present study is in the area of metric fixed point theory. Istratescu (1979), Dugundji and Granas (1982), Bose and Joshi (1985), Zeidler (1986), Takahashi (2000), Khamisi and Kirk (2001), Kirk and Sims (2001) are very good reference books for fixed point theory and applications.

Definition 1.1.1. A self-mapping $f$ of a metric space $(X, d)$ is said to be Lipschitzian, with a Lipschitz constant $k$, if there exists a non-negative real number $k$ such that

$$d(f(x), f(y)) \leq kd(x, y) \text{ for all } x, y \in X. \quad (1.1)$$

We say that,

(i) $f$ is a contraction if $0 \leq k < 1$ in equation (1.1).

(ii) $f$ is nonexpansive if $k = 1$ in equation (1.1).

(iii) $f$ is contractive if $d(f(x), f(y)) < d(x, y)$ for all $x, y \in X$ with $x \neq y$.

A mapping $f : X \to X$ from a nonempty set $X$ to itself has a fixed point in $X$ if there is an $x \in X$ such that $f(x) = x$. The simplest fixed point theorem is that of a continuous function $f: [a, b] \to [a, b]$ which has at least one fixed point. This is a consequence of the intermediate value theorem from analysis. Since $f(a) \geq a$ and $f(b) \leq b$, we have $f(b) - b \leq 0 \leq f(a) - a$. The difference $f(x) - x$ is continuous. Using the intermediate value theorem, 0 is a value of $f(x) - x$ for some $x \in [a, b]$, and that $x$ is a fixed point of $f$; that is, $f$ may have more than one fixed point in $[a, b]$. There are generalizations of this result, such as the Brouwer (1912) fixed point theorem and the Lefschetz (1923) and (1925) fixed point theorem. These generalizations and their proofs belong to topological fixed point theory. There is a fixed point theorem for contractive mappings provided the space is compact.
Theorem 1.1.2. Let \((M,d)\) be a compact metric space and \(T : M \to M\) be a contractive mapping. Then \(T\) has a unique fixed point \(x_0\). Moreover, for each \(x \in M\), \(\lim_{n \to \infty} T^n(x) = x_0\).

The theory of nonexpansive mappings is basically different from that of contraction and contractive mappings. In the case of a contraction, the sequence of iterations defined by \(x_{n+1} = f^n(x)\) converges strongly to the unique fixed point of the mapping \(f\). In the case of a nonexpansive mapping, this sequence of iterations need not converge, there need not be any fixed point, nor need the fixed point be unique, if it exists at all. It is necessary to additional assumptions regarding the structure of the space to ensure the existence of a fixed point for nonexpansive mappings. The notion of normal structure, introduced by Brodskii and Milman (1948) was employed by Kirk (1965) to prove the fundamental existential fixed point theorem for nonexpansive mappings. Hilbert space and uniformly convex Banach spaces have the normal structure property. The proofs given independently by Browder (1965) and Gohde (1965) of their results do not use this property.

A non empty convex subset \(K\) of a Banach space \(X\) is said to have normal structure if for each bounded convex subset \(H\) of \(K\) which contains more than one point, there is some point \(x \in H\) which is not a diametral point of \(H\).

Theorem 1.1.3. (Kirk (1965)) If \(K\) is a nonempty, weakly-compact, convex subset of a Banach space \(X\) and if it has normal structure property then, any nonexpansive mapping \(T : K \to K\) has a fixed point in \(K\).

Theorem 1.1.4. (Browder and Gohde (1965)) If \(K\) is a nonempty bounded closed convex subset of a uniformly convex Banach space \(X\) and if \(T : K \to K\) is nonexpansive, then \(T\) has a fixed point in \(K\). Moreover, the set of fixed points of \(T\) is a closed convex subset of \(K\).

The result of Browder (1965) and Gohde (1965) is a special case of Kirk (1965).
**Theorem 1.1.5. (Brouwer (1912))** Let $B^n$ be the unit ball in Euclidean $n$-space $\mathbb{R}^n$ and let $f:B^n \to B^n$ be a continuous function. Then $f$ has a fixed point in $B^n$.

We consider the case $n = 1$. We have a map $f:[-1,1] \to [-1,1]$. Let $g:[-1,1] \to \mathbb{R}$ be defined by $g(x) = x - f(x)$, then $g(-1) \leq 0$ and $g(1) \geq 0$. If $g(-1)$ or $g(1)$ is 0, then $f$ certainly has a fixed point because of the intermediate value theorem.

**Theorem 1.1.6. (Bolzano (1817))** If $g:[-1,1] \to \mathbb{R}$ is a mapping such that $g(-1)<0$ and $g(1)>0$, then $g(c)=0$ for some $c$ with $-1<c<1$.

This theorem, first proved by Bernard Bolzano in 1817, is a direct consequence of a basic property of the real numbers. This is called the intermediate value theorem. When $n = 1$, the Brouwer’s fixed point theorem and Bolzano’s intermediate value theorem are equivalent. But this equivalence was not established until Miranda in 1940. As a consequence, what we have today called the n-dimensional intermediate value theorem is popularly known as Miranda’s theorem or, more accurately, the Poincare-Miranda theorem. Brouwer published his fixed point theorem for continuous functions on the 3-ball in 1909.

Just as the Brouwer theorem holds in all dimensions, there is a form of the intermediate value theorem that is valid in all dimensions. It concerns a map $f : l^n \to \mathbb{R}^n$, where $l^n = [-1,1] \times [-1,1] \times \ldots \times [-1,1]$. Let $l_k(-) = \{x = (x_1, x_2, \ldots, x_n) \in l^n|x_k = -1\}$ and let $l_k(+) = \{x = (x_1, x_2, \ldots, x_n) \in l^n|x_k = +1\}$ and let $f_k(+) = \{x = (x_1, x_2, \ldots, x_n) \in l^n|x_k = +1\}$ be the same, except that $x_k = 1$. Write $f = (f_1, f_2, \ldots, f_n)$, where $f_k : l^n \to \mathbb{R}$. Denote the origin in $\mathbb{R}^n$ by $0$.

**Theorem 1.1.7. (n-Dimensional Intermediate value theorem Poincare and Miranda 1940)** Let $f:T^n \to \mathbb{R}^n$ be a continuous function such that $f_k(x) \leq 0$ for all $x \in l_k(-)$ and $f_k(x) \geq 0$ for all $x \in l_k(+)$, for $k=1,\ldots,n$, then $f(0)=0$ for some $c \in T^n$.

**Theorem 1.1.8. (Tarski’s fixed point theorem 1955)** Let $(L, \leq)$ be any complete lattice. Suppose that $f:L \to L$ is monotone increasing (for isotone); that is,
for all \( x, y \in L, x \leq y \) implies \( f(x) \leq f(y) \). Then the set of all fixed points of \( f \) is a complete lattice with respect to \( \leq \).

Tarski's fixed point theorem gave birth to discrete fixed point theory.

**Example 1.1.9.** Let \( a, b \in \mathbb{R} \) satisfy \( a \leq b \), where \( \leq \) is the usual order of real numbers. Since the closed interval \([a, b]\) is a complete lattice, every monotone increasing mapping \( f : [a, b] \to [a, b] \) has a greatest fixed point and a least fixed point. Note that \( f \) need not be continuous here.

### 1.2 COMMON FIXED POINT THEOREMS

A solution of the equation \( fx=gx=x \), if it exists, is called a common fixed point of two self mappings \( f \) and \( g \). The first common fixed theorem for two maps would be Markov's (1936).

**Theorem 1.2.1.** (Markov-Kakutani (1936)) Let \( X \) be a topological vector space and \( F \) be a commuting family of continuous linear operators in \( X \), each of which maps a compact convex subset \( K \) of \( X \) into itself. Then there is a point of \( K \) which is fixed under each \( T \in F \).

This theorem was first proved by Markov in 1936 for locally convex spaces, with the aid of the Schauder-Tychonov theorem. In 1938, Kakutani found a direct elementary proof, valid in any linear topological space, but not necessarily a locally convex space. Hence the theorem is called the Markov-Kakutani theorem. Jungck (1976) was the first to prove a theorem for a pair of commuting self-mappings.

**Proposition 1.2.2.** Let \( f \) be a self-mapping of a nonempty set \( X \). Then \( f \) has a fixed point if and only if there is a constant mapping \( h : X \to X \) which commutes with \( f \). That is, \( h(f(x)) = f(h(x)) \) for all \( x \) in \( X \).

This is a sufficient condition for the existence of a fixed point.
Theorem 1.2.3. (Jungck (1976)) Let $f$ be a continuous mapping of a complete metric space $(X, d)$ into itself. Then $f$ has a fixed point in $X$ if and only if there exists an $\alpha \in (0, 1)$ and a mapping $g : X \to X$, which commutes with $f$, and which satisfies
\[ g(X) \subset f(X) \tag{1.2} \]
and
\[ d(g(x), g(y)) \leq \alpha d(f(x), f(y)) \tag{1.3} \]
for all $x, y \in X$. Indeed, $f$ and $g$ have a unique common fixed point if equations (1.2) and (1.3) hold.

Corollary 1.2.4. Let $f$ and $g$ be commuting mappings of a complete metric space $(X, d)$ into itself. Suppose that $f$ is continuous and $g(X) \subset f(X)$. If there exists an $\alpha \in (0, 1)$ and a positive integer $k$ such that $d(g^k(x), g^k(y)) \leq \alpha d(f(x), f(y))$ for all $x$ and $y$ in $X$, then $f$ and $g$ have a unique common fixed point in $X$.

The Banach contraction principle is obtained as a consequence of Corollary 1.2.4 if $k = 1$, and $f$ is the identity mapping defined by $f(x) = x$ for all $x \in X$.

1.3 A SURVEY OF FIXED POINT THEOREMS FOR EXTENSION OF THE BANACH CONTRACTION PRINCIPLE

The well known Banach contraction principle and its generalizations in the setting of metric spaces play a central role in solving many nonlinear analysis problems. For this, the following definitions are needed. Let $X$ and $Y$ be any two nonempty sets. A multi-valued mapping or a set valued mapping $T$ from $X$ to $Y$ is a mapping that associates with each value $x \in X$ a nonempty subset $T(x)$ of $Y$, called the image or the value of $T$ at $x$. It is denoted by $T : X \to 2^Y$, where $2^Y$ denotes the family of all nonempty subsets of $Y$. For any subset $K$ of $X$, the image of $K$ under $T$ is defined by
\[ T(K) = \bigcup_{x \in K} T(x). \]
For any subset $K$ of $Y$, the preimage of $K$ under $T$ is defined by $T^{-1}(K) = \{ x \in X : T(x) \cap K \neq \emptyset \}$.

The notion of continuity of a multi-valued mapping is defined as follows:

**Definition 1.3.1.** Let $X$ and $Y$ be any two normed linear spaces and let $2^Y$ denote the family of all nonempty subsets of $Y$. A mapping $g : X \rightarrow 2^Y$ is said to be

(i) upper semicontinuous if and only if $g^{-1}(K) = \{ x \in X : g(x) \cap K \neq \emptyset \}$ is closed for each closed subset $K$ of $Y$;

(ii) lower semicontinuous if and only if $g^{-1}(K) = \{ x \in X : g(x) \cap K \neq \emptyset \}$ is open for each open subset $K$ of $Y$;

(iii) continuous if and only if it is both upper semicontinuous and lower semicontinuous.

For more details on continuity of multi-valued mappings, the reader is referred to Beer (1993).

**Theorem 1.3.2.** *(Banach’s Contraction Mapping Principle 1922)* Let $(M, d)$ be a complete metric space and let $T : M \rightarrow M$ be a contraction. Then, $T$ has a unique fixed point $x_0$. Moreover, for each $x \in M$, $d(x, T^n(x)) \rightarrow 0$ and for each $x \in M$, $d(T^n(x), x_0) \leq k^n/(1-k)d(x, T(x))$, for $n=1, 2, ...$

This is to say that any contraction mapping of a complete metric space has a unique fixed point. There have been numerous extensions of Banach’s contraction principle, which asks only that the fixed point be unique, and that the Picard iterates of the mapping always converge to this fixed point. Only a few theorems of Banach’s contraction principle extensions will be listed here. The first generalization of Banach’s contraction principle was given by Rakotch (1962).

**Theorem 1.3.3.** *(Rakotch (1962))* Let $(M, d)$ be a complete metric space, and suppose that $f : M \rightarrow M$ satisfies $d(f(x), f(y)) \leq \alpha(d(x,y))d(x,y)$ for $x, y \in M$, for some $\alpha \in (0, 1)$.
where $\alpha: R^+ \to [0,1)$ is monotonically decreasing. Then $f$ has a unique fixed point $\bar{x}$ and $\{f^n(\bar{x})\}$ converges to $\bar{x}$ for each $x \in M$.

Boyd and Wong (1969) obtained a more general result. In this theorem, it is assumed that $\psi: R^+ \to R^+$ is upper semicontinuous from the right.

**Theorem 1.3.4. (Boyd and Wong (1969))** Let $(M, \rho)$ be a complete metric space and suppose that $f: M \to M$ satisfies $d(f(x), f(y)) \leq \psi(d(x, y))$ for $x, y \in M$, where $\psi: R^+ \to [0, \infty)$ is upper semicontinuous from the right and satisfies $0 \leq \psi(t) < t$ for $t > 0$. Then $f$ has a unique fixed point $\bar{x}$ and $\{f^n(\bar{x})\}$ converges to $\bar{x}$ for each $x \in M$.

Boyd and Wong (1969) also show that, if the space $M$ is metrically convex, then the upper semicontinuity assumption on $\psi$ can be dropped. The following variant of the Boyd-Wong theorem due to Browder.

**Theorem 1.3.5. (Browder (1968))** Let $(X, d)$ be a complete metric space and $M$ be a bounded subset of $X$. Suppose that $f: M \to M$ satisfies $d(f(x), f(y)) \leq \psi(d(x, y))$ for each $x, y \in M$, where $\psi: [0, \infty) \to [0, \infty)$ is monotone nondecreasing and continuous from the right such that $\psi(t) < t$ for all $t > 0$. Then there is a unique element $\bar{x} \in M$ such that $\{f^n(\bar{x})\}$ converges to $\bar{x}$ for each $x \in M$. Moreover, if $d_0$ is the diameter of $M$, then $d(f^n(\bar{x}), x) \leq \psi^n(d_0)$ and $\psi^n(d_0) \to 0$ as $n \to \infty$.

Another variant was given by Matkowski.

**Theorem 1.3.6. (Matkowski (1975))** Let $(M, d)$ be a complete metric space and suppose that $f: M \to M$ satisfies $d(f(x), f(y)) \leq \psi(d(x, y))$ for $x, y \in M$, where $\psi: (0, \infty) \to (0, \infty)$ is monotone nondecreasing and satisfies $\lim_{t \to \infty} \psi^n(t) = 0$ for $t > 0$. Then $f$ has a unique fixed point $\bar{x}$, and $\lim_{n \to \infty} d(f^n(x), \bar{x}) = 0$ for every $x \in M$.

The following theorem was given by Meir and Keeler.
Theorem 1.3.7. (Meir-Keeler (1969)) Let \((M, d)\) be a complete metric space and suppose that \(f : M \to M\) satisfies the condition: Given an \(\varepsilon > 0\), there exists a \(\delta > 0\) such that \(\varepsilon \leq d(x, y) \leq \varepsilon + \delta \Rightarrow d(f(x), f(y)) < \varepsilon\). Then \(f\) has a unique fixed point \(x\), and \(\lim_{n \to \infty} f^n(x) = x\) for each \(x \in M\).

The Meir-Keeler condition implies that \(f\) is contractive

\[(x \neq y \Rightarrow d(f(x), f(y)) < d(x, y)).\]

Thus \(f\) is continuous, and, if \(f\) has a fixed point, it must be unique. Another variant of this theorem in ordered Banach spaces was given by Chen (2000).

Let \(S\) denote the class of functions \(\alpha : \mathbb{R}^+ \to [0, 1)\) which satisfy the condition \(\alpha(l_n) \to 1\) implies that \(l_n \to 0\).

Theorem 1.3.8. (Geraghty (1973)) Let \((M, d)\) be a complete metric space and suppose that \(f\) is a self mapping of \(M\), and suppose that there exists an \(\alpha \in S\) such that, for each \(x, y \in M\),

\[d(f(x), f(y)) \leq \alpha(d(x,y))d(x,y).\]

Then \(f\) has a unique fixed point \(z \in M\), and \(\{f^n (x)\}\) converges to \(z\) for each \(x \in M\).

Caristi’s extension of the Banach contraction principle.

Theorem 1.3.9. (Caristi (1976)) Let \(M\) be a complete metric space. Suppose that \(\varphi : M \to \mathbb{R}\) is lower semicontinuous and bounded below, and suppose that \(g\) is a self mapping of \(M\) satisfies: \(d(x,g(x)) \leq \varphi(x) - \varphi(g(x)), x \in M\). Then \(g\) has a fixed point in \(M\).

Banach’s contraction mapping principle was extended nicely to set-valued mappings by Nadler (1969). If \(A\) and \(B\) are nonempty closed bounded subsets of a metric space and if \(x \in A\), then, given an \(\varepsilon > 0\), there must exists a point \(y \in B\) such that \(d(x, y) \leq H(A, B) + \varepsilon\), where \(H(A, B)\) denotes the Hausdorff distance between \(A\) and \(B\).
Theorem 1.3.10. (Nadler (1969)) Let \((M, d)\) be a complete metric space, and let \(\mathcal{R}\) be the collection of all nonempty bounded closed subsets of \(M\) endowed with the Hausdorff metric \(H\). Suppose that \(T : M \to \mathcal{R}\) is a contraction mapping in the sense that for some \(k < 1\): 
\[ H(Tx, Ty) \leq k \ d(x, y), \] 
for all \(x, y \in M\). Then there exists a point \(x \in M\) such that \(x \in T(x)\).

Menger (1951) introduced the concept of probabilistic metric spaces (also often called PM-spaces or Menger spaces). These are spaces in which the distance between points is a probability distribution on \(R_+\) rather than a real number. There is no analogue of Banach’s contraction mapping theorem for complete probabilistic metric spaces in general, but a number of positive results are known. First, precise definitions have to be given. Let \(R_+ = [0, \infty)\). A mapping \(F : R_+ \to [0, 1]\) is called a (distance) distribution function, if it is nondecreasing and left-continuous, with \(F(0) = 0\) and \(\sup_{x \in R_+} F(x) = 1\). The set of all such functions is denoted by \(D^+\) and \(\varepsilon_0\) is used to denote the specific distribution function defined by

\[
\varepsilon_0(x) = \begin{cases} 
0 & \text{if } x = 0 \\
1 & \text{if } x > 0 
\end{cases}
\]

Next we define a \(t\)-norm.

**Definition 1.3.11.** A mapping \(\tau : [0, 1] \times [0, 1] \to [0, 1]\) is called a \(t\)-norm if, for any \(a, b, c, d \in [0, 1]\):

(i) \(\tau(a, 1) = a\);

(ii) \(\tau(a, b) = \tau(b, a)\);

(iii) \(\tau(c, d) \geq \tau(a, b)\) if \(c \geq a\) and \(d \geq b\);

(iv) \(\tau(\tau(a, b), c) = \tau(a, \tau(b, c))\);

**Definition 1.3.12.** Let \(S\) be a nonempty set. A probabilistic metric space is an ordered triple \((S, F, \tau)\) which satisfies the following conditions.
(I) $F : S \times S \to D^+$ is a symmetric function which is denoted by $F(p,q) = F_{p,q}$ for $(p,q) \in S \times S$:

1. $F_{p,q} = \varepsilon_0$ if and only if $p = q$;
2. $F_{p,r}(x) = 1$ and $F_{r,q}(y) = 1 \Rightarrow F_{p,q}(x + y) = 1$ for all $p, q, r \in S$ and all $x, y \in R^+$.

(II) $\tau$ is a $t$-norm on $(S, F)$ which satisfies $F_{p,q}(x + y) \geq \tau(F_{p,r}(x), F_{r,q}(y))$ for every $p, q, r \in S$ and $x, y \in R^+$.

A Hausdorff topology on a probabilistic metric space $(S, F, \tau)$ is given by the neighborhood system $\mathcal{U} = \{ U_q(\varepsilon, \lambda) \}$, $q \in S$, $\varepsilon$, $\lambda > 0$, where $U_q(\varepsilon, \lambda) = \{ p \in S : F_{p,q}(\varepsilon) > 1 - \lambda \}$. The first ‘contraction type’ fixed point theorem in the setting described above is due to Sehgal and Bharucha-Reid (1972).

**Theorem 1.3.13. (Sehgal and Bharucha-Reid (1972))** Let $(S, F, \tau)$ be a complete probabilistic metric space for which the $t$-norm $\tau$ is min. Suppose that $f : S \to S$ is a continuous mapping for which there exists a $k \in (0, 1)$ such that, for all $p, q \in S$ and all $u > 0$, $F_{f_p,f_q}(ku) \geq F_{p,q}(u)$. Then $f$ has a unique fixed point $p \in S$, and $\{ f^n(q) \} \to p$ for each $q \in S$.


**Definition 1.3.14. (Kirk (2003))** Let $(M, d)$ be a metric space. A mapping $T : M \to M$ is said to be an asymptotic contraction if

$$d(T^n(x), T^n(y)) \leq \phi_n(d(x, y)) \quad (1.4)$$

for all $x, y \in M$, where $\phi_n : [0, \infty) \to [0, \infty)$ and $\phi_n \to \phi \in \Phi$ uniformly on the range of $d$.

**Theorem 1.3.15. (Kirk (2003))** Suppose that $(M, d)$ is a complete metric space and suppose a continuous function $T : M \to M$ is an asymptotic contraction for which the mappings $\phi_n$ in equation(1.4) are also continuous. Assume also that
some orbit of $T$ is bounded. Then $T$ has a unique fixed point $z \in M$. Moreover, the Picard sequence $(T^n(x))_{n=1}^{\infty}$ converges to $z$ for each $x \in M$.

For a detailed comparison of various definitions and fixed point theorems for contraction and contractive mappings one can refer to the source article by Rhoades (1977). Many authors have studied contractive type mappings on a complete metric space $X$ which are generalizations of the Banach contraction principle. Kannan (1968) introduced the contractive condition: there exists a $\lambda \in (0, 1/2)$ such that

\[ d(Fx, Fy) \leq \lambda [d(x, Fx) + d(y, Fy)] \]

(1.5)

for all $x, y \in X$, and proved a fixed point theorem using this condition. This condition and Banach’s contraction principle are independent. This was shown by two examples in Kannan (1969). Latif and Beg (1997) introduced the notion of a $K$-multivalued mapping, which is the extension of Kannan mappings to multivalued mappings. Rus (2003) coined the term $K$-multivalued mapping, which is a generalization of a $K$-multivalued mapping. Reich (1971) generalized the Banach and Kannan’s fixed point theorems by using the contractive condition: for each $x, y \in X$,

\[ d(Fx, Fy) \leq \alpha d(x, y) + \beta d(Fx, Fy) + \gamma d(y, Fy) \]

where $\alpha, \beta, \gamma$ are nonnegative reals with $\alpha + \beta + \gamma < 1$. Cric (1971) generalized this result by using the contractive condition: for all $x, y \in X$

\[ d(Fx, Fy) \leq \alpha(x, y)d(x, y) + \beta(x, y)d(Fx, Fy) + \gamma(x, y)d(y, Fy) + \delta(x, y)d(x, Fy) + d(y, Fx) \]

(1.6)

where $\alpha, \beta, \gamma, \delta$ are functions from $X^2$ to $[0, 1)$ such that

\[ \lambda = \sup \{\alpha(x, y) + \beta(x, y) + \gamma(x, y) + 2\delta(x, y) : x, y \in X\} < 1. \]

(1.7)

Mappings which satisfy equations (1.6) and (1.7) are called generalized contractions. As observed in Cric (1972), a self mapping $F$ on a metric space $(X, d)$ is a generalized contraction if and only if $F$ satisfies the following condition:

\[ d(Fx, Fy) \leq \lambda M(x, y), \]

where

\[ M(x, y) = \max \{d(x, y), d(x, Fx), d(y, Fy), \frac{1}{2} [d(x, Fy) + d(y, Fx)]\}, \lambda \in (0, 1) \text{ and } x, y \in X. \]
1.4 AN OUTLINE OF THE RESULTS OF THE THESIS

The scope of the present study is metric fixed point theory. The thesis is divided into five chapters and a brief introduction to each chapter is given below: Chapter 1 deals with the history of metric space and various fixed point theories. It also provides information on common fixed point theorems and their origin. It includes a brief survey of Banach’s contraction principle. In Chapter 2, “Common fixed point theorems using minimal commutativity and reciprocal continuity conditions in metric space” are generalized in two different directions. One generalization “Some Common fixed point theorem using occasionally weakly compatible and reciprocal continuity conditions in metric space” is obtained. Pointwise $R$-weakly commuting and compatible are replaced by occasionally weakly compatible mappings which generalize the results of Sanjay Kumar and Renu Chugh (2002). In section 2.4, the vectorial version of reciprocal continuity and pointwise $R$-weakly commuting are introduced. Another generalization “Common fixed point theorem using compatibility and reciprocal continuity conditions in cone metric space” is also obtained.

Lemma 1.4.1. Let $A, B, S$ and $T$ be self-mappings of a complete metric space $(X, d)$ satisfying equations (1.8) and (1.9)

\[ A(X) \subset T(X), B(X) \subset S(X) \]  \hspace{1cm} (1.8)

\[ d(Ax, By) \leq g(d(Sx, Ty), d(Ax, Sx), d(By, Ty), d(Ax, Ty), d(By, Sx)) \]  \hspace{1cm} (1.9)

for every $x, y \in X$, where $g \in G$. Define a sequence $\{y_n\}$ in $X$ as

\[ y_{2n} = Ax_{2n} = Tx_{2n+1} \text{ and } y_{2n+1} = By_{2n+1} = Sy_{2n+1}, \text{ for } n = 1, 2, 3, ... \]  \hspace{1cm} (1.10)

Then the sequence $\{y_n\}$ defined by equation (1.10) is a Cauchy sequence in $X$.

Theorem 1.4.2. Let $(A, S)$ and $(B, T)$ be pairs of self-mappings of a complete metric space $(X, d)$ satisfying equations (1.8) and (1.9). Suppose that $(A, S)$ and $(B, T)$ are weakly compatible pairs of reciprocal continuous mappings. Then $A, B, S$ and $T$ have a unique common fixed point in $X$. 
Corollary 1.4.3. Let \((A, S)\) and \((B, T)\) be pairs of self-mappings of a complete metric space \((X, d)\) satisfying equations (1.8), (1.10) and (1.11) given below.

\[
d(Ax, By) \leq hM(x, y), 0 \leq h < 1, x, y \in X, \tag{1.11}
\]

where \(M(x, y) := \max\{d(Sx, Ty), d(Ax, Sx), d(By, Ty), [d(Ax, Ty) + d(By, Sx)]/2\}\). Suppose that \((A, S)\) or \((B, T)\) is an occasionally weakly compatible pair of reciprocal continuous mappings. Then \(A, B, S\) and \(T\) have a unique common fixed point in \(X\).

Corollary 1.4.4. Let \((A, S)\) and \((B, T)\) be pairs of self-mappings of a complete metric space \((X, d)\) satisfying equations (1.8), (1.10) and (1.12) given below. Suppose that \((A, S)\) or \((B, T)\) is an occasionally weakly compatible pair of reciprocal continuous mappings satisfying

\[
d(Ax, By) \leq h \max\{d(Ax, Sx), d(By, Ty), 1/2d(Ax, Ty), 1/2d(By, Sx), d(Sx, Ty)\} \tag{1.12}
\]

for all \(x, y \in X\), where \(0 \leq h < 1\). Then \(A, B, S\) and \(T\) have a unique common fixed point in \(X\).

If \((f, S)\) and \((g, T)\) are occasionally weakly compatible, then \(f, g, S, T\) have a unique common fixed point.

Corollary 1.4.5. Let \((f, S)\) and \((h, T)\) be occasionally weakly compatible pairs of self-mappings of a complete metric space \((X, d)\) satisfying \(f(X) \subset T(X)\), \(h(X) \subset S(X)\) and

\[
d(fx, hy) \leq g(d(Sx, Ty), d(fx, Sx), d(hy, Ty), d(fx, Ty), d(hy, Sx)) \tag{1.13}
\]

for all \(x, y \in X\), where \(g \in G\). Suppose that \((f, S)\) or \((h, T)\) is a pair of reciprocal continuous mappings. Then \(f, h, S\) and \(T\) have a unique common fixed point in \(X\).

Lemma 1.4.6. Let \(A,B,S\) and \(T\) be self-mappings from a complete cone metric space \((X, d)\) to itself satisfying the conditions

\[
A(X) \subset T(X), B(X) \subset S(X) \tag{1.14}
\]
\[ d(Ax, By) \leq g(d(Sx, Ty), d(Ax, Sx), d(By, Ty), d(Ax, Ty), d(By, Sx)), \quad (1.15) \]

for every \( x, y \in X \), where \( g \in G \). Define a sequence \( \{y_n\} \) in \( X \) by

\[ y_{2n} = Ax_{2n} = Tx_{2n+1} \quad \text{and} \quad y_{2n+1} = By_{2n+1} = Sx_{2n+1}, \quad \text{for} \quad n = 1, 2, 3, \ldots \quad (1.16) \]

Let \( P \) be a normal cone with normal constant \( K \). Then the sequence \( \{y_n\} \) defined by equation (1.16) is a Cauchy sequence in \( X \).

**Theorem 1.4.7.** Let \((A, S)\) and \((B, T)\) be pointwise \( R \)-weakly commuting pairs of self-mappings of a complete cone metric space \((X, d)\) satisfying equation (1.14) and (1.15). Suppose that \((A, S)\) or \((B, T)\) is a compatible pair of reciprocal continuous mappings. Let \( P \) be a normal cone with normal constant \( K \). Then \( A, B, S \) and \( T \) have a unique common fixed point in \( X \).

In Chapter 3, various compatible type mappings in fixed point theory are discussed. Two new concepts of Cauchy compatible and Cauchy reciprocal continuous are introduced. Cauchy compatible and Cauchy reciprocal continuous generalize compatible and reciprocal continuous, respectively, in the setting of metric spaces. Using this, some common fixed point theorems, which generalize many theorems, are obtained.

**Lemma 1.4.8.** Let \( A, B, S \) and \( T \) be self-mappings of a complete metric space \((X, d)\) into itself satisfying the conditions

\[ A(X) \subset T(X), B(X) \subset S(X) \quad (1.17) \]

\[ d(Ax, By) \leq g(d(Sx, Ty), d(Ax, Sx), d(By, Ty), d(Ax, Ty), d(By, Sx)) \quad (1.18) \]

for every \( x, y \in X \), where \( g \in G \). Define a sequence \( \{y_n\} \) in \( X \) by

\[ y_{2n} = Ax_{2n} = Tx_{2n+1} \quad \text{and} \quad y_{2n+1} = By_{2n+1} = Sx_{2n+1}, \quad \text{for} \quad n = 1, 2, 3, \ldots \quad (1.19) \]

Then the sequence \( \{y_n\} \) defined by equation (1.19) is a Cauchy sequence in \( X \).

**Theorem 1.4.9.** Let \((A, S)\) and \((B, T)\) be pointwise \( R \)-weakly commuting pairs of self-mappings of a complete metric space \((X, d)\) satisfying equations (1.17)
and (1.18). Suppose that \((A, S)\) or \((B, T)\) is a Cauchy compatible pair of Cauchy reciprocal continuous mappings. Then \(A, B, S\) and \(T\) have a unique common fixed point in \(X\).

**Corollary 1.4.10.** Let \((A, S)\) and \((B, T)\) be pointwise \(R\)-weakly commuting pairs of self-mappings of a complete metric space \((X, d)\) satisfying equations (1.17), (1.19) and (1.20)

\[
d(Ax, By) \leq hM(x, y), 0 \leq h < 1, x, y \in X
\]

where \(M(x, y) = \max\{d(Sx, Ty), d(Ax, Sx), d(By, Ty), [d(Ax, Ty) + d(By, Sx)]/2\}\). Suppose that \((A, S)\) or \((B, T)\) is a Cauchy compatible pair of Cauchy reciprocal continuous mappings. Then \(A, B, S\) and \(T\) have a unique common fixed point in \(X\).

**Corollary 1.4.11.** Let \((A, S)\) and \((B, T)\) be pointwise \(R\)-weakly commuting pairs of self-mappings of a complete metric space \((X, d)\) satisfying equations (1.17), (1.19) and (1.21).

\[
d(Ax, By) \leq h\max\{d(Ax, Sx), d(By, Ty), 1/2d(Ax, Ty), 1/2d(By, Sx), d(Sx, Ty)\}
\]

for all \(x, y \in X\), where \(0 \leq h < 1\). Suppose that \((A, S)\) or \((B, T)\) is a Cauchy compatible pair of Cauchy reciprocal continuous mappings. Then \(A, B, S\) and \(T\) have a unique common fixed point in \(X\).

**Corollary 1.4.12.** Let \((f, S)\) and \((g, T)\) be pointwise \(R\)-weakly commuting pairs of self-mappings of a complete metric space satisfying \(f(X) \subseteq T(X), g(X) \subseteq S(X)\) and

\[
d(fx, gy) \leq g(d(Sx, Ty), d(fx, Sx), d(gy, Ty), d(fx, Ty), d(gy, Sx))
\]

for all \(x, y \in X\), where \(g \in G\). Suppose that \((f, S)\) or \((g, T)\) is a pair of Cauchy reciprocal continuous mappings. Then \(f, g, S\) and \(T\) have a unique common fixed point in \(X\).
In chapter 4, some common fixed point theorems are established using Cauchy compatible and Cauchy reciprocal continuous hybrid mappings, which generalize Singh and Mishra (2002). In this section, \( C(T, f) \) stands for the collection of coincidence points of \( T \) and \( f \); that is, \( C(T, f) = \{ v : f v \in T v \} \).

**Theorem 1.4.13.** Let \((X, d)\) be a metric space, \( f \) be a self mapping of \( X \), and \( T : X \to C^I(X) \). If \( T \) and \( f \) are Cauchy reciprocal continuous and nonvacuously Cauchy compatible on \( X \), then \( C(T, f) \) is nonempty. Further, \( T \) and \( f \) have a common fixed point \( ft \) provided \( f f t = ft \) for some \( t \in C(T, f) \).

**Theorem 1.4.14.** Let \((X, d)\) be a metric space and \( S, T : X \to C^I(X) \) and \( f, g : X \to X \) such that

1. \( S(X) \subset g(X) \) and the pair \((S, f)\) is Cauchy reciprocal continuous and nonvacuously Cauchy compatible.
2. If there exists a \( \phi \in \psi \) such that \( H(Sx, Ty) \leq \phi(M(x, y)) \), for all \( x, y \in X \),

where

\[
M(x, y) = \max\{ d(fx, gy), d(fx, Sy), d(gy, Ty), d(fx, Ty), d(gy, Sx) \},
\]  

then \( C(S, f) \) and \( C(T, g) \) are nonempty. Further,

1a) \( S \) and \( f \) have a common fixed point \( ft \) provided \( f f t = ft \) for some \( t \in C(S, f) \);

1b) \( T \) and \( g \) have a common fixed point \( gu \) provided \( ggu = gu \) and \( T, g \) are IT-commuting at \( u \in C(T, g) \);

1c) \( S, T, f \) and \( g \) have a common fixed point provided (1a) and (1b) are true.

**Corollary 1.4.15.** Let \((X, d)\) be a complete metric space and \( S, T : X \to C^I(X) \) and \( f, g : X \to X \) such that
(3) \(S(X) \subset g(X)\) and \(T(X) \subset f(X)\) and the pair \((S, f)\) is Cauchy compatible and Cauchy reciprocal continuous.

(4) If there exists a \(q \in (0, 1)\) such that \(H(Sx, Ty) \leq q(M(x, y))\) for all \(x, y \in X\) where
\[
M(x, y) = \max\{d(fx, gy), d(fx, Sx), d(gy, Ty), 1/2[d(fx, Ty) + d(gy, Sx)]\} \tag{1.24}
\]
then \(C(S, f)\) and \(C(T, g)\) are nonempty. Further, (Ia), (Ib), (Ic) are also true.

**Corollary 1.4.16.** Let \(f, g, S, T\) be self-mappings of a complete metric space \((X, d)\) such that \(S(X) \subset g(X)\), \(T(X) \subset f(X)\), and the pair \((S, f)\) is Cauchy compatible and Cauchy reciprocal continuous. If there exists a \(q \in (0, 1)\) such that \(\frac{d(Sx, Ty)}{q(M(x, y))}\) where
\[
M(x, y) = \max\{d(fx, gy), d(fx, Sx), d(gy, Ty), 1/2[d(fx, Ty) + d(gy, Sx)]\} \quad \text{for all} \quad x, y \in X.
\]

\[
\text{IIa) } S \text{ and } f \text{ have a common fixed point.}
\]

\[
\text{IIb) } T \text{ and } g \text{ have a coincidence at } x = u \in X.
\]

\[
\text{IIc) } f, g, S \text{ and } T \text{ have a common fixed point if } T \text{ and } g \text{ are weakly compatible.}
\]

**Theorem 1.4.17.** Let \(f\) and \(g\) be continuous self-mappings of a compact metric space \((X, d)\) and let \(S, T : X \to CB(X)\) be continuous such that \(S(X) \subset g(X)\) and \(T(X) \subset f(X)\). The pair \((S, f)\) is Cauchy compatible if \(H(Sx, Ty) < M(x, y)\), where
\[
M(x, y) = \max\{d(fx, gy), d(fx, Sx), d(gy, Ty), 1/2[d(fx, Ty) + d(gy, Sx)]\} > 0.
\]
Then \(C(S, f)\) and \(C(T, g)\) are nonempty. Further, the conditions (Ia)-(Ic) are also true.

**Corollary 1.4.18.** Let \(f, g, S\) and \(T\) be continuous self-mappings of a compact metric space \((X, d)\) such that \(S(X) \subset g(X)\), \(T(X) \subset f(X)\), and the pair \((S, f)\) is Cauchy compatible. If \(d(Sx, Ty) < M(x, y)\), where
\[
M(x, y) = \max\{d(fx, gy), d(fx, Sx), d(gy, Ty), 1/2[d(fx, Ty) + d(gy, Sx)]\} > 0,
\]
then (IIa)-(IIc) are true.
In Section 4.4, the results of Suneel Kumar and Pant (2008) in the setting of probabilistic metric spaces are generalized.

Lemma 1.4.19. Let \( \{u_n\} \) be a sequence in a Menger space \( (X, F, \Delta_M) \). If there exists a constant \( h \in (0, 1) \) such that \( F_{u_n,u_{n+1}}(ht) \geq F_{u_{n-1},u_n}(t) \), for \( n=1,2,3..., \) then \( \{u_n\} \) is a Cauchy sequence in \( X \).

Lemma 1.4.20. Let \( (X, F, \Delta_M) \) be a complete Menger space. Further, let \( (A, S) \) and \( (B, T) \) be pointwise \( R \)-weakly commuting pairs of self mappings of \( X \) satisfying

\[
A(X) \subseteq T(X), B(X) \subseteq S(X),
\]

\[
\varphi(F_{Au,Bv}(ht), F_{Su,Tv}(t), F_{Au,T}(t), F_{Be,Tv}(ht)) \geq 0,
\]

and

\[
\varphi(F_{Au,Bv}(ht), F_{Su,Tv}(t), F_{Au,T}(t), F_{Be,Tv}(ht)) \geq 0
\]

for all \( u, v \in X, t > 0, h \in (0, 1) \) and for some \( \varphi \in \Phi \). Then the continuity of one of the mappings in the Cauchy compatible pairs \( (A, S) \) or \( (B, T) \) on \( (X, F, \Delta_M) \), implies their Cauchy reciprocal continuity.

Theorem 1.4.21. Let \( (X, F, \Delta_M) \) be a complete Menger space. Further, let \( (A, S) \) and \( (B, T) \) be pointwise \( R \)-weakly commuting pairs of self mappings of \( X \) satisfying equations (1.25), (1.26) and (1.27). If one of the mappings in the Cauchy compatible pair \( (A, S) \) or \( (B, T) \) is continuous, then \( A, B, S, T \) have a unique common fixed point in \( X \).

Corollary 1.4.22. Let \( (X, F, \Delta_M) \) be a complete Menger space. Let \( A \) and \( B \) be self-mappings of \( X \) satisfying

\[
\varphi(F_{Au,Bv}(ht), F_{u,v}(t), F_{Au,u}(t), F_{Be,v}(ht)) \geq 0
\]

\[
\varphi(F_{Au,Bv}(ht), F_{u,v}(t), F_{Au,u}(t), F_{Be,v}(ht)) \geq 0
\]

for all \( u, v \in X, t > 0, h \in (0, 1) \) and for some \( \varphi \in \Phi \). If \( A \) and \( B \) are Cauchy reciprocally continuous mappings, then \( A \) and \( B \) have a unique common fixed point in \( X \).
In Section 4.7, the concepts of Cauchy compatible mappings of type (A) and Cauchy compatible mappings of type (P) are introduced and compared with Cauchy compatible mappings, which extend and improve some results of Pathak et al (1995) and Jungek et al (1993).

**Theorem 4.4.23.** Let \((X, d)\) be a complete metric space and \(A, B, S, T\) be self-mappings of \(X\). Suppose that \(S\) and \(T\) are continuous mappings satisfying the following conditions:

\[ A(X) \subset T(X) \quad \text{and} \quad B(X) \subset S(X), \]

(1.30)

the pairs \((A, S)\) and \((B, T)\) are Cauchy compatible of type \((P)\),

(1.31)

and

\[ d(Ax, By) \leq \Phi( \max \{d(Sx, Ty), d(Sx, Ax), d(Ty, By), \]

\[ \frac{1}{2}[d(Sx, By) + d(Ty, Ax)]) \),

(1.32)

for every \(x, y \in X\), where \(\Phi : [0, \infty) \to [0, \infty)\) is a nondecreasing and upper semicontinuous function and \(\Phi(t) < t\) for all \(t > 0\). Then \(A, B, S, T\) and \(T\) have a unique common fixed point in \(X\).

**Theorem 4.4.24.** Let \((X, d)\) be a complete metric space and \(S, T\) and \(A_n\) be self-mappings of \(X\), \(n=1, 2, 3, \ldots\). Suppose further that \(S\) and \(T\) are continuous and, for every \(n \in N\), the pairs \(\{A_{2n-1}, S\}\) and \(\{A_{2n}, T\}\) are Cauchy compatible of type \((P)\), \(A_{2n-1}(X) \subset T(X)\) and \(A_{2n}(X) \subset S(X)\) and, for any \(n \in N\), the set of positive integers, the following condition is satisfied:

\[ d(A_nx, A_{n+1}y) \leq \Phi( \max \{d(Sx, Ty), d(Sx, A_nx), d(Ty, A_{n+1}y), \]

\[ \frac{1}{2}[d(Sx, A_{n+1}y) + d(Ty, A_nx)]) \),

(1.33)

for all \(x, y \in X\), where \(\Phi : [0, \infty) \to [0, \infty)\) is a nondecreasing and upper semicontinuous function and \(\Phi(t) < t\) for all \(t > 0\). Then \(S, T\) and \(\{A_n\}\), \(n \in N\) have a unique common fixed point in \(X\).
**Theorem 1.4.25.** Let $A, B, S$ and $T$ be self-mappings of a complete metric space $(X, d)$ satisfying the following:

$$A(X) \subset T(X) \quad \text{and} \quad B(X) \subset S(X), \quad (1.34)$$

$$d^2(Ax, By) \leq \phi(d^2(Sx, Ty), d(Sx, Ax)d(Ty, By), d(Sx, By)d(Ty, Ax)$$

$$d(Sx, Ax)d(Ty, Ay), d(Sx, By)d(Ty, By)), \quad (1.35)$$

for every $x, y \in X$, where $\phi \in F$.

one of $A, B, S$ and $T$ is continuous, \quad (1.36)

the pairs $(A, S)$ and $(B, T)$ are Cauchy compatible of type (A). \quad (1.37)

Then $A, B, S$ and $T$ have a unique common fixed point $z$ in $X$.

In Chapter 5, the equivalence between Mann iteration and Ishikawa iteration for a generalized contraction in a cone is established.

**Theorem 1.4.26.** Let $P$ be a cone subset of a Banach space $E$, and $T$ be a self-mappings of $P$ satisfying

$$\|Tx - Ty\| \leq \phi(M(x, y)), \quad (1.38)$$

where $\phi$ and $M(x, y)$ satisfy

(i)

$$\phi : [0, \infty) \to [0, \infty) \text{ is a real-valued, right continuous function at } 0 \quad (1.39)$$

(ii)

$$\phi(t) < t \text{ for each } t > 0; \quad (1.40)$$

(iii)

$$\phi \text{ is nondecreasing on } (0, \infty); \quad (1.41)$$
(iv)  
\[ g(t) = t / (t - \phi(t)) \]  
\[ \text{is nonincreasing on } (0, \infty); \]  
(1.42)

(v)  
\[ M(x, y) = \max\{\|x - y\|, \|x - Tx\|, \|y - Ty\|, \|x - Ty\|, \|y - Tx\|\} \]  
(1.43)

Let \( \{\alpha_n\} \) satisfy the conditions \( \alpha_n > 0 \) for all \( n \geq 0 \) and \( \sum_{n=1}^{\infty} \alpha_n = \infty \). Denote by \( x^k \) the unique fixed point of \( T \). Then for \( u_0 = x_0 \in P \), the following are equivalent:

(i) the Mann iteration converges to \( x^k \)

(ii) the Ishikawa iteration converges to \( x^k \).

The main aim of the section 5.4 is to show that the convergence of Mann iteration is equivalent to the convergence of Ishikawa iteration in \( P \), while assuming \( \phi(t) < t / 2 \) for each \( t > 0 \) and even after dropping the condition \( g(t) = t / (t - \phi(t)) \) being non-increasing as mentioned in Rhoades and Stefan Soltuz (2006).

Theorem 1.4.27. Let \( X \) be a real Banach space, \( P \) a nonempty convex subset of \( X \), \( T \) be a self-mapping of \( P \), satisfying

\[ \|Tx - Ty\| \leq \phi(M(x, y)), \]  
(1.44)

where \( \phi \) and \( M(x, y) \) satisfy the equations (1.39), (1.41), (1.43), (1.45), and

\[ \phi(t) < t / 2 \]  
for each \( t > 0 \).

Let \( \{\alpha_n\} \) satisfy the conditions \( \alpha_n > 0 \) for all \( n \geq 0 \) and \( \sum_{n=1}^{\infty} \alpha_n = \infty \). Denote by \( x^k \) the unique fixed point of \( T \). Then for \( u_0 = x_0 \in P \), the following are equivalent:

(i) the Mann iteration converges to \( x^k \)

(ii) the Ishikawa iteration converges to \( x^k \).