CHAPTER V
CHAPTER 5
CUBIC DIOPHANTINE EQUATIONS WITH FOUR UNKNOWNS

This chapter consists of two sections.

Section (A), deals with ternary cubic Diophantine equation with four unknowns. The equation considered for its integral equation is given by

\[ x^3 + y^3 + (x + y)w^2 = 4z^3 \]

In section (B), ternary cubic Diophantine equation with four unknowns is discussed, since finding integral solutions for the ternary cubic homogeneous or non-homogeneous diophantine equations have been an interest to many mathematicians since antiquity. The equation which is taken for a trial is

\[ x^3 + y^3 + (x + y)xy = z^3 + w^3 + (z + w)zw \]

In each of the sections, a few interesting relations among the solutions \( x, y, z \) are exhibited.
SECTION — A

The equation under consideration is

\[ x^3 + y^3 + (x + y)w^2 = 4z^3 \]  

(5.1)

which simplifies to

\[ u^2 = 3w^2 + w^2 \]  

(5.2)

by using the linear transformations,

\[ x = u + v, \quad y = u - v, \quad z = u \]  

(5.3)

we present below three different patterns of non-trivial integral solutions of (5.2) and thus in view of (5.3), we obtain the corresponding integral solutions of (5.1).

PATTERN 1:

It is well known that (5.2) is satisfied by

\[ v = 2rs, \quad w = 3r^2 - s^2, \quad u = 3r^2 + s^2 \]

and hence, using (5.3), the non-trivial solutions of (5.1) are represented by

\[ x = x(r,s) = 3r^2 + s^2 + 2rs \]
\[ y = y(r,s) = 3r^2 + s^2 - 2rs \]
\[ w = w(r,s) = 3r^2 - s^2 \]
\[ z = z(r,s) = 3r^2 + s^2 \]

A few interesting properties observed are given below.

1. Each of the expression is a Nasty number
   
   a) \( z(r,s) + w(r,s) \)
   
   b) \( 3\left(x^2(r,r) - y^2(r,r)\right) \)
   
   c) \( 2\left(z^2(r,r) - w^2(r,r)\right) \)
2. \(x(r,s) - y(r,s)\) is a perfect square.

3. \(z(r,r+1) - D_r - 1 \equiv 0 \pmod{5}\)

4. \(x(r,s) + y(r,s) = 2z(r,s)\) : \(x, y, z\) forms an arithmetic progression.

5. \(w(r+1,s) + w(r-1,s) = 2w(r,s) + 6\)

6. \(3(z(r,s) - y(r,s))^2 = z^2(r,s) - w^2(r,s)\)

7. \(y(r,1) - H_r - 1 = 0\)

8. \(y(r,1) - 1 = O_r\)

9. \(x(r,r+1) - y(r,r+1) = 8T_r\)

**PATTERN 2:**

It is to be noted that the equation (5.2), is also satisfied by

\[v = 2rs, \quad w = r^2 - 3s^2, \quad u = r^2 + 3s^2\]

and the corresponding solutions of (5.1) are found to be

\[x = x(r,s) = r^2 + 3s^2 + 2rs\]
\[y = y(r,s) = r^2 + 3s^2 - 2rs\]
\[w = w(r,s) = r^2 - 3s^2\]
\[z = z(r,s) = r^2 + 3s^2\]

**Properties:**

1. \(z(r+1,s) + z(r-1,s) = 2(z(r,s)+1)\)

2. \(6(x(r,s) - y(r,s))^2 = 8(z^2(r,s) - w^2(r,s))\)

3. \(w(r+1,s) - w(r-1,s) = x(r,1) - y(r,1)\)

4. \(2x(r,1) - HE_r - 6 \equiv 0 \pmod{5}\)

5. \(2y(r,1) - HE_r - 6 \equiv 0 \pmod{5}\)
6. \(4z(r,1)+x(r,1)-2HP_r-15 \equiv 0 \pmod{5}\)

7. \(x(r,1)+w(r,1) = 2T_r\)

8. \(8\left(x\left(r^2, r+1\right) + w\left(r^2, r+1\right) - 4PY_r\right)\) is a quartic integer.

9. \(y(1,s) - 2P_s - 2 = 0\)

10. \(y(1,s) - 2C_s \equiv 0 \pmod{2}\)

**PATTERN 3:**

Let \(a, b\) be two non-zero integers so that

\[u = a^2 + 3b^2\quad (5.4)\]

substituting (5.4) in (5.2) and employing the method of factorization, we have,

\[
\left(w + i\sqrt{3}v\right)\left(w - i\sqrt{3}v\right) = \left(a + i\sqrt{3}b\right)^2 \left(a - i\sqrt{3}b\right)^2
\]

\[
= \left(a + i\sqrt{3}b\right)^2 \left(a - i\sqrt{3}b\right)^2 \left(\frac{1+i\sqrt{3}}{2}\right)\left(\frac{1-i\sqrt{3}}{2}\right)
\]

(5.5)

Now, equation (5.5) is equivalent to the following system of double equations

\[
\left(w + i\sqrt{3}v\right) = \left(a + i\sqrt{3}b\right)^2 \left(\frac{1+i\sqrt{3}}{2}\right)
\]

(5.6)

\[
\left(w - i\sqrt{3}v\right) = \left(a - i\sqrt{3}b\right)^2 \left(\frac{1-i\sqrt{3}}{2}\right)
\]

(5.7)

Equating real and imaginary parts in either (5.6) or (5.7) we have

\[w = w(a,b) = \frac{a^2 - 3b^2 - 6ab}{2}\quad (5.8)\]

\[v = v(a,b) = \frac{a^2 - 3b^2 + 2ab}{2}\quad (5.9)\]
From (5.3), (5.4), (5.8) and (5.9) the values of \(x, y, z\) and \(w\) satisfying (5.1) are given by,

\[
\begin{align*}
x &= x(a,b) = \frac{3a^2 + 3b^2 + 2ab}{2} \\
y &= y(a,b) = \frac{a^2 + 9b^2 - 2ab}{2} \\
w &= w(a,b) = \frac{a^2 - 3b^2 - 6ab}{2} \\
z &= z(a,b) = a^2 + 3b^2
\end{align*}
\]

As our interest centers on finding integral solutions of (5.1), it is seen that the above solutions will be in integers when \(a\) and \(b\) are of the same parity. Thus, we have the following two choices of solutions.

Choice 1:
Let \(a = 2k, b = 2m, \quad k, m \in \mathbb{Z} - \{0\}\)

The corresponding solutions are given by,

\[
\begin{align*}
x &= x(k,m) = 6k^2 + 6m^2 + 4km \\
y &= y(k,m) = 2k^2 + 13m^2 - 4km \\
w &= w(k,m) = 2k^2 - 6m^2 - 12km \\
z &= z(k,m) = 4k^2 + 12m^2
\end{align*}
\]

Properties:

1. \(x(k,1) - 2O_k - 6 \equiv 0 \pmod{8}\)
2. \(x(k,1) - 4P_k - 6 \equiv 0 \pmod{6}\)
3. \(x(k,1) - 4C_k + 4 \equiv 0 \pmod{10}\)
4. \(2T_k - w(k,1) - 6 \equiv 0 \pmod{14}\)
5. \(x(k,1) - 2CS_k + 4G_k = 12\)
6. $z(m,m)$ is a perfect square.

7. $x(m,m) + w(m,m) = 0$

8. $4w(k,m) + 2z(k,m)$ is expressed as difference of squares

9. Each of the expression is a Nasty number
   a) $6z(3r^2-s^2,2rs)$
   b) $6z(r^2-3s^2,2rs)$
   c) $(2k+2m)^2 - 2(z(k,m) - y(k,m))$
   d) $\frac{z(m,m)-2w(m,m)}{8}$

Choice 2:
Let $a = 2k - 1, b = 2m - 1$

For this choice, the corresponding solutions are obtained as

$$x = x(k,m) = 6k^2 + 6m^2 - 8k - 8m + 4km + 4$$
$$y = y(k,m) = 2k^2 + 13m^2 - 4km - 16m + 4$$
$$w = w(k,m) = 2k^2 - 6m^2 - 4k + 12m - 12km - 4$$
$$z = z(k,m) = 4k^2 + 12m^2 + 4k + 12m + 4$$

Properties:

1. $x(k,1) - 2H_k \equiv 0 \pmod{2}$
2. $x(k,1) - 4C_k \equiv 0 \pmod{2}$
3. $x(k,1) - 2P_k - 2 \equiv 0 \pmod{2}$
4. $z(1,m) = 4H_{E_m}$
5. $x(1,m) - H_{E_m} \equiv 0 \pmod{2}$
6. $2z(k,m) - x(k,m) - 2(k-m)^2 - 24T_m \equiv 0 \pmod{4}$
7. $2z(k,m) - x(k,m) - y(k,m) - 10T_m \equiv 0 \pmod{23}$
8. Each of the expression is a Nasty number

1. \[ 3\left( x\left( \alpha^2,1 \right) - 2H_{\alpha^2} \right) \]

2. \[ 60(z(k,m) - 8T_{k-1} + 4T_m - 4) \]

3. \[ 6\left( 2z(k,m) - x(k,m) - 2(k-m)^2 - 24T_m - 4 \right) \]
SECTION — B

The equation under consideration is

\[ x^3 + y^3 + (x + y)xy = z^3 + w^3 + (z + w)zw \]

(5.10)

which simplifies to

\[ u(u^2 + v^2) = r(r^2 + s^2) \]

(5.11)

by using the linear transformations,

\[ x = u + v, \quad y = u - v, \quad z = r + s, \quad w = r - s \]

(5.12)

in which \(u, v, r, s\) are non-zero and distinct integers.

we present below the different patterns of non-trivial integral solutions of (5.11) and thus in view of (5.12), we obtain the corresponding integral solutions of (5.10).

PATTERN 1:

Assume \(u + iv = (A + iB)(r + is)\)

(5.13)

where \(A, B\) are both non-zero integers.

Equating the real and imaginary parts in (5.13), we get

\[ u = Ar - Bs \]

(5.14)

\[ v = Br + As \]

(5.15)

using (5.14) and (5.15) in (5.11), we get after some algebra,

\[ s = \lambda \left( A\left(A^2 + B^2\right) - 1 \right) \]

(5.16)

\[ r = \lambda B \left( A^2 + B^2 \right) \]

(5.17)

where \(\lambda\) is any non-zero integer.
From (5.14), (5.15), (5.16) and (5.17), the values of \(x, y, z\) and \(w\) satisfying (5.10) are given by

\[
x = x(A, B, \lambda) = \lambda B + \lambda \left(A^2 + B^2\right)^2 - A
\]

\[
y = y(A, B, \lambda) = \lambda B - \lambda \left(A^2 + B^2\right)^2 + A
\]

\[
z = z(A, B, \lambda) = \lambda \left(A^2 + B^2\right)(A + B) - \lambda
\]

\[
w = w(A, B, \lambda) = \lambda \left(A^2 + B^2\right)(B - A) + \lambda
\]

A few interesting properties observed are given below.

1. Each of the expressions is a Nasty number.
   a) \(x(A, B, 3\alpha^2) - x(A, B, 3\alpha^2) + 2A\)
   b) \(x(A, 3\lambda, \lambda) + y(A, 3\lambda, \lambda)\)
   c) \(6w\left(6, B, \alpha^2 + 1\right) - Bz\left(6, B, \alpha^2 + 1\right) - y\left(6, B, \alpha^2 + 1\right)\)

2. \(x(A, B, 2) - y(A, B, 2) + 2A\) is a perfect square

3. \(z(A, B, \lambda) + w(A, B, \lambda) = (x(A, B, \lambda) + y(A, B, \lambda))A^2 + B^2\)

4. \(x(1, 1, \lambda) - D_\lambda + G_\lambda \equiv 0 \pmod{2}\)

5. \(z(A, 1, 1) - 2PY_A = 1\)

6. \(w(A, 1, 1) + 2PY_A - HE_A = 2\)

7. \(z(A, 1, 1) - 2PY_A - G_A = 2\)

8. \(z(1, B, 1) = C_B + T_B\)

9. \(2w(1, B, 1) = 3OC_B - HE_B\)

10. \(Aw(A, B, \lambda) - Bz(A, B, \lambda) - y(A, B, \lambda) = A(\lambda - 1)\)

11. \(2w(-1, B, \lambda) - \lambda P y_B = 0 \pmod{2}\)
Chapter 5 - Cubic Diophantine Equations with four unknowns

PATTERN 2:
Here we assume
\[
\frac{u+iv}{r+is} = \frac{(A+iB)(3+4i)}{5}
\] (5.18)

Equating the real and imaginary parts in (5.18), we get
\[
u = \frac{(A(3r-4s) - B(4r+3s))}{5}
\] (5.19)
\[
v = \frac{(A(4r+3s) - B(3r-4s))}{5}
\] (5.20)

There are three choices for obtaining \( u \) and \( v \) in integers.

Choice 1
Let
\[A = 5\tilde{A} \quad B = 5\tilde{B}\]
Equations (5.19) and (5.20) reduces to
\[u = \tilde{A}(3r-4s) - \tilde{B}(4r+3s)
\] (5.21)
\[v = \tilde{A}(4r+3s) + \tilde{B}(3r-4s)
\] (5.22)

Substituting (5.21) and (5.22) in (5.12), the values of \( x, y, z \) and \( w \) are obtained as follows.
\[x = x(\tilde{A}, \tilde{B}, r, s) = \tilde{A}(7r-s) + \tilde{B}(-r-7s)
\]
\[y = y(\tilde{A}, \tilde{B}, r, s) = \tilde{A}(-r-7s) + \tilde{B}(7r-s)
\]
\[z = z(r, s) = r + s
\]
\[w = w(r, s) = r - s
\]

Properties:
1. \( 2(z(r^2, s)+w(r^2, s)) \) is a perfect square.
2. \( z(r, r-1) = G_r \)
3. \( z(r, s)w(r, s) \) is expressed as difference of squares.

65
Chapter 5 - Cubic Diophantine Equations with four unknowns

4. \(3z\left((r+1)^2,1\right) - 2H_r - 5 \equiv 0 \pmod{9}\)

5. \(3z\left((r+1)^2,1\right) - 2CT_r - 3 \equiv 0 \pmod{9}\)

6. \(x(1,1,r,s) + y(1,1,r,s) + 8z(r,s) = 6w(r,s)\)

7. \(z(r^2,(r-1)^2) = CS_r\)

8. Each of the expressions is a Nasty number.
   a) \(3\left(z(r^2,s) + w(r^2,s)\right)\)
   b) \(x(1,1,r^2,1) - 8\)

9. \(2P_r + 4HP_r - x\left(2,1,r^2,r\right) \equiv 0 \pmod{2}\)

10. \(z\left(r^3,(r+1)^2\right) = 6TH_r + G_r - T_r\)

11. \(SO_r + x\left(1,1,r^3,r\right) + 15r = 0\)

12. \(3O_r + x\left(1,1,r^3,r\right) + 13r = 0\)

Choice 2:
Assume
\(s = 2k - 1, \ r = 5n + k - 3\) \ (5.23)

Substituting (5.23) in (5.19) and (5.20), we get,
\(u = A(3n-k-1) - B(4n+2k-3)\)
\(v = A(4n+2k-3) + B(3n-k-1)\)

The values of \(x, y, z\) and \(w\) satisfying (5.10) are given by
\(x = x(A,B,n,k) = A(7n+k-4) + B(-n-3k+2)\)
\(y = y(A,B,n,k) = A(-n-3k+2) + B(-7n-k+4)\)
\(z = z(n,k) = 5n + 3k - 4\)
\(w = w(n,k) = 5n - k - 2\)
Properties:

1. \( y(1,1,n^2,n) + z(n^2,n) + 2CT_n + 2G_n = 2 \)

2. \( w(k^2,k) + 1 = H_k + O_k + G_k \)

3. \( z(n,k) - w(n,k) = 2G_k \)

4. \( z(k^2,k) + w(k^2,k) - 4HP_k + 6 \equiv 0 \pmod{8} \)

5. \( z(k^2,k) + w(k^2,k) - 4CT_k - 2H_k + 10 \equiv 0 \pmod{10} \)

6. \( z(k^2,k) + w(k^2,k) - 2HE_k - D_k + 8 \equiv 0 \pmod{11} \)

7. \( w(n^2,3n) + 2 = 2HP_n \)

8. \( w(n^2,3n) + 2 = D_n + S_n \)

9. \( w(n^2,n^2) + 2 \) is a perfect square.

10. \( 6\left(w(n^2,n^2) + 2\right) \) is a Nasty number.

Choice 3:

Assume

\[ s = 2k, \quad r = 5n + k - 5 \]  \hspace{1cm} (5.24)

Employing (5.24) in (5.19) and (5.20), we get

\[ u = A(3n - k - 3) - B(4n + 2k - 4) \]

\[ v = A(4n + 2k - 4) + B(3n - k - 3) \]

The values of \( x, y, z \) and \( w \) satisfying (5.10) are given by

\[ x = x(A, B, n, k) = A(7n + k - 7) + B(-n - 3k + 1) \]

\[ y = y(A, B, n, k) = A(-n - 3k + 1) + B(-7n - k + 7) \]

\[ z = z(n, k) = 5n + 3k - 5 \]

\[ w = w(n, k) = 5n - k - 5 \]
Properties:

1. \( w(n^2,n) + 4 = 2HP_n + G_n \)

2. \( z(n,k^2) - w(n,k^2) \) is a perfect square.

3. Each of the following expressions is a Nasty number
   a) \( 6\left(z(n,k^2) - w(n,k^2)\right) \)
   b) \( 6\left(w(n^2,n^2) + 5\right) \)

4. \( x(1,1,n^2,n) + 8 = 4CT_n + 2G_n \)

5. \( y(1,1,n,k) + 4(G_n + G_k) \equiv 0 \pmod{4} \)

6. \( z(n^2,n) + 2 = HP_n + 3G_n \)

7. \( 2x(1,1,n,k) - y(1,1,n,k) + 20 = 0 \pmod{20} \)