CHAPTER III
CHAPTER 3

QUADRATIC DIOPHANTINE EQUATIONS WITH FOUR UNKNOWNS

This chapter consists of two sections.

In section A, the Quadratic Diophantine equation with four unknowns considered for its non-trivial integral solutions is

$$x^2 - y^2 = zw$$

In section B, various different patterns of non-zero integral solutions are discussed. The quadratic equations with four unknowns taken for the discussion is

$$xy = 2(2k + 1)(z + w)$$

In each of the sections, a few interesting relations among the solutions $x, y, z$ are exhibited. In addition, the recurrence relations satisfied by the solutions along with a few examples are illustrated numerically.
SECTION — A

The non-homogeneous quadratic diophantine equation with four unknowns under consideration is

\[ x^2 - y^2 = zw \]  \hspace{1cm} (3.1)

substituting the linear transformation

\[ z = p + q, \quad w = p - q, \quad p \neq q \]  \hspace{1cm} (3.2)

in (3.1), it is written in the factorizable form as

\[ (x + iq)(x - iq) = (p + iy)(p - iy) \left( \frac{3 + 4i}{5} \right) \left( \frac{3 - 4i}{5} \right) \]

Define

\[ (x + iq) = (p + iy) \left( \frac{3 + 4i}{5} \right) \]  \hspace{1cm} (3.3)

Equating the real and imaginary parts in (3.3), we get

\[ x = \frac{5p - 4q}{3}, \quad y = \frac{5q - 4p}{3} \]  \hspace{1cm} (3.4)

The values of \( x \) and \( y \) are integers when

\[ p = 3n + 5(k - 1), \quad q = 3n + 4(k - 1) \]  \hspace{1cm} (3.5)

Thus using (3.5) in (3.2) and (3.4), the non-trivial integer values of \( x, y, z \) and \( w \) are given by

\[ x = x(n, k) = n + 3k - 3 \]
\[ y = y(n) = n \]
\[ z = z(n, k) = 6n + 9k - 9 \]
\[ w = w(k) = k - 1 \]
A few interesting properties observed among the solutions, are presented below.

1. \(6y(n)+9w(k)=z(n,k)\).
2. \(y(n)+3w(k)=x(n,k)\).
3. \(6x(n,k)−z(n,k)=9w(k)\).
4. \(216y^3(n)+729w^3(k)+162y(n)w(k)z(n,k)=z^3(n,k)\).
5. \(y^3(n)+27w^3(k)+9y(n)w(k)x(n,k)=x^3(n,k)\).
6. \(216x^3(n,k)+162w(k)x(n,k)z(n,k)=z^3(n,k)+729w^3(k)\).
7. \(x(n+1,1)y(n)=2T^T_n\).
8. \(y^2(n)w(n+2)=2PY_n\).
9. \(x(n-1,1)+y(n)=G_n\).
10. \(y(n)w(2n^2)=SO_n\).
11. \(y^2(n)+w^2(n)=CS_n\).
12. \(y(n^2)w(n+2)=2PY_n\).
13. \(x(k^2-2k,k)w(k)-6TH_{k-1} \equiv 0 \pmod{3}\).
14. \(x(n,k)+z(n,k)-7y(n) \equiv 0 \pmod{12}\).
15. \(TH_\alpha−x(\alpha^3,\alpha^2−\alpha)−3 \equiv 0 \pmod{5}\).
16. \(x(n,D_n)+8n+3 \equiv 0 \pmod{12}\).
17. \(x(2\alpha^3,2T_\alpha)−12TH_\alpha \equiv -3 \pmod{2}\).
18. \(z(n,HE_n)+3n+9 \equiv 0 \pmod{18}\).
19. \(6(x(n,O_n)+5n-3)\) is a Nasty number.

However, we have other patterns of solutions which are exhibited below.
PATTERN 1:
The solutions of quadratic diophantine equation (3.1) are presented below

\[ x = 2rsm \]
\[ y = \left( m^2 - 1 \right) r^2 + s^2 \]
\[ z = 2rsm - \left( m^2 - 1 \right) r^2 - s^2 \]
\[ w = 2rsm + \left( m^2 - 1 \right) r^2 + s^2 \]

where \( m, r, s \) are any non-zero integers.

Properties:
1. \( y \pm x \) is written as the difference of squares.
2. \( (z, x, w) \) form an Arithmetic progression.
3. \( w - z = 2y \).
4. When \( r, s, m \) forms a geometric progression each of the following expressions forms a Nasty number.
   1. \( 3x \)
   2. \( 3(w - y) \)
   3. \( 3(y + z) \)
   4. \( 6(z + w) \)

PATTERN 2:
The solutions of quadratic diophantine equation (3.1) are given below

\[ x = 2rsm \]
\[ y = 2rs \]
\[ z = 2rsm - 2rs \]
\[ w = 2rsm + 2rs \]

where \( m, r, s \) are any non-zero integers.
Properties:

1. \((z, x, w)\) form an Arithmetic progression.
2. \((x - z)(x - w) + y^2 = 0\).
3. \((z + y)(w - y) = x^2\).
4. When \(m\) is a perfect square \(\frac{6(w - y)}{(x - z)}\) is a Nasty number.

PATTERN 3:
The solutions of quadratic diophantine equation (3.1) are listed below

\[
\begin{align*}
  x &= u + v \\
  y &= v - u \\
  z &= 2u \\
  w &= 2v
\end{align*}
\]

where \(u\) and \(v\) are any non-zero integers.

Properties:

1. \(xy\) is written as difference of squares.
2. The difference of square of \(x\) and \(y\) is a perfect square, when \(u = v\).
3. When \(v = u + 1, \ x^2 - y^2 = 8T_u\)

PATTERN 4:
The solutions of quadratic diophantine equation (3.1) are presented below

\[
\begin{align*}
  x &= u + v + 1 \\
  y &= v - u \\
  z &= 2u + 1 \\
  w &= 2v + 1
\end{align*}
\]

where \(u\) and \(v\) are any non-zero integers.
Properties:

1. \( x + y - w = 0 \).
2. \( x - y - z = 0 \).
3. \( z^2 - D_u - 1 \equiv 0 \pmod{7} \).
4. \( z^2 - y^2 - 2C_Tu + 2 \equiv 0 \pmod{9} \).
5. \( y^2 + w - 2 \) is a perfect square

Pattern 5:
The solutions of quadratic diophantine equation (3.1) are presented below

\[
\begin{align*}
x &= \left(m^2 - 1\right)r^2 - s^2 \\
y &= \left(m^2 - 1\right)r^2 + s^2 \\
z &= -2s^2 \\
w &= 2\left(m^2 - 1\right)r^2
\end{align*}
\]

where \( m, r, s \) are any non-zero integers.

Properties:

1. \( x + y = w \).
2. \( x - y = z \).
3. The following expressions forms a Nasty number.
   a) \( 3(w - 2x) \)
   b) \( 3(2y - w) \)
   c) \( 6(y - x - z) \)
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PATTERN 6:
The solutions of quadratic diophantine equation (3.1) are given below

\[
x = \left(1 - m^2\right)r^2 + s^2
\]

\[
y = 2rs
\]

\[
z = \left(1 - m^2\right)r^2 + s^2 - 2rs
\]

\[
w = \left(1 - m^2\right)r^2 + s^2 + 2rs
\]

where \( m, r, s \) are any non-zero integers.

Properties:

1. \((z, x, w)\) form an Arithmetic progression.

2. \( w - z = 2y \).

3. The following expressions forms a Nasty number.

   a) \(6y^2\).

   b) \(3\left((w - z)y\right)\).

   c) \(6\left((x - z)y\right)\).

It is to be noted that, if the values of \( z \) and \( w \) presented in the above patterns are taken as the sides of rectangles, then the area of each rectangle is expressed as the difference of two squares.
SECTION ─ B

The non-homogeneous quadratic diophantine equation with four unknowns under consideration is

\[ xy = 2(2k + 1)(z + w) \]  \hspace{1cm} (3.6)

solving the quadratic diophantine equation by introducing the linear transformations

\[ x = u + v, \quad y = u - v, \quad z = v + s, \quad w = v - s \]  \hspace{1cm} (3.7)

in (3.6), it reduces to

\[ u^2 = (v + 2(2k + 1))^2 - 4(2k + 1)^2 \]  \hspace{1cm} (3.8)

which is in the form of the well known Pythagorean equation.

Employing the standard solutions of the Pythagorean equation, the integral solutions of (3.6) are respectively,

\[ x = x(k) = 8k^2 + 4k \]
\[ y = y(k) = 4k \]
\[ z = z(k, s) = 4k^2 + s \]
\[ w = w(k, s) = 4k^2 - s \]

A few observations are listed below

1. \( x(k) - y(k) = z(k, s) + w(k, s) \).

2. \( x(k) - 6(2^{2n}) - 2 = 2 \, KN_n \), where \( k = 2^n \)

3. \( \left( \frac{x(k)}{y(k)} \right)^2 - 8T_k = 1 \).

4. \( G_k^2 + 4G_k - 8T_k = -3 \).

5. \( 2D_k - x(k) + 4 \equiv 0 \pmod{6} \)

6. \( x(k) - D_k - 8T_k \equiv 0 \pmod{3} \).

7. \( x(k + 1) - 3y(k) - 12 = 16T_k \).
8. \( x(k)y(k + 1) = 96SP_k \)

9. \( x(k)y(k) + 8P_{k-1} = 48TH_k \)

10. \( y(k)w(k,1) + y(3k^2) = 0 \)

11. \( 2(z(k,s) + w(k,s)) = (y(k))^2 \)

12. \( w(2^2,(-1)^n) - 3(2^n) = 3J_n \)

13. Each of the following expressions is a Perfect square
   a) \( x(k)y(k) - 192(SP_k - TH_k) + 32TH_k \)
   b) \( \frac{x(k) + (y(k))^2 - 8T_k}{20} \)
   c) \( (z(k,s) + w(k,s) - x(k))^2 \)

14. Each of the following expressions is a cubic integer.
   a) \( \frac{(z(k,s) + w(k,s))y(k)}{24} \)
   b) \( (y(k))^2 - 192PR_k - 64 \)
   c) \( x(k)y(k) - 32PY_k \)

15. Each of the following expressions is a Nasty number.
   a) \( 3(x(k) - y(k)) \)
   b) \( 3z(k,s) + w(k,s) \)
   c) \( 6(x(k) - 8T_k) \)
   d) \( 6(x(k)y(k) - 12(SO_k + 2T_k)) \)

However, we have other patterns of solutions of (3.6) which are illustrated below.
In (3.6), introducing the linear transformations,
\[ x = u + v, \ y = u - v, \ z = p + q, \ w = p - q \]
we get
\[ u^2 + (p - 2k - 1)^2 = (p + 2k + 1)^2 + v^2 \] (3.9)

PATTERN 1:
The choice
\[ p + 2k + 1 = \alpha (p - 2k - 1), \ \alpha > 1 \]
in (3.9) gives
\[ u^2 = (\alpha^2 - 1)(p - 2k - 1)^2 + v^2 \]
The integral solutions to (3.6) are obtained as follows:
\[ x = x(\alpha, r) = 2\left(\alpha^2 - 1\right)r^2 \]
\[ y = y(s) = -2s^2 \]
\[ z = z(r, s, k, q) = 2(rs + k) + 1 + q \]
\[ w = w(r, s, k, q) = 2(rs + k) + 1 - q \]

PATTERN 2:
Assuming
\[ (p - 2k - 1) = \beta (p + 2k + 1), \ \beta > 1 \]
in (3.9), it becomes
\[ v^2 = (\beta^2 - 1)(p + 2k + 1)^2 + u^2 \]
Thus, the integral solutions of (3.6) are obtained as
\[ x = x(\beta, r) = 2\left(\beta^2 - 1\right)r^2 \]
\[ y = y(s) = -2s^2 \]
\[ z = z(r, s, k, q) = 2(rs - k) - 1 + q \]
\[ w = w(r, s, k, q) = 2(rs - k) - 1 - q \]
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PATTERN 3:

Equation (3.9) is written as

$$\left( (p + 2k + 1) + v^2 \right) = \left( u^2 + (p - 2k - 1)^2 \right) \ast 1$$  \hspace{1cm} (3.10)

Using the unique factorization method in (3.10) and writing

$$1 = \left( \frac{3 + 4i}{5} \right) \left( \frac{3 - 4i}{5} \right)$$

we have,

$$\left( v + i(p + 2k + 1) \right) \left( v - i(p + 2k + 1) \right) =$$

$$\left( u + i(p - 2k - 1) \right) \left( u - i(p - 2k - 1) \right) \left( \frac{3 + 4i}{5} \right) \left( \frac{3 - 4i}{5} \right)$$  \hspace{1cm} (3.11)

Define

$$\left( v + i(p + 2k + 1) \right) = \left( u + i(p - 2k - 1) \right) \left( \frac{3 + 4i}{5} \right)$$  \hspace{1cm} (3.12)

Equating the real and imaginary parts in (3.12) and performing a few calculations, we get

$$u = p + 4k + 2$$
$$v = -p + 4k + 2$$

The corresponding integral solutions of (3.6) are as follows:

$$x = x(k) = 8k + 4$$
$$y = y(P) = 2P$$
$$z = z(P, q) = 2P + q$$
$$w = w(P, q) = 2P - q$$