CHAPTER II
CHAPTER 2
QUADRATIC DIOPHANTINE EQUATIONS WITH THREE UNKNOWNS

This chapter consists of three sections.

In section (A), Patterns of Pythagorean Triangles for each of which the Hypotenuse added with ratio \( \left( \frac{\text{Area}}{\text{Perimeter}} \right) \) is a Perfect square, are presented.

Section (B), deals with the special patterns of Pythagorean triangles which are generated through the solutions of the ternary quadratic diophantine equation

\[
Y^2 = DX^2 + Z^2
\]

In section (C), the ternary Quadratic equation considered for its non-trivial integral solutions is

\[
x^2 + y^2 = z^2 + 8
\]

In each of the sections, a few interesting relations among the solutions \( x, y, z \) are exhibited. In addition, the recurrence relations satisfied by the solutions along with a few examples are illustrated numerically.
SECTION — A

The general solution for the Pythagorean equation,
\[ x^2 + y^2 = z^2 \] (2.1)
is given by
\[ x = 2pq, \ y = p^2 - q^2, \ z = p^2 + q^2 \] (2.2)
where \( p > q > 0 \).

The assumption that the Hypotenuse added with ratio
\[ \left( \frac{\text{Area}}{\text{Perimeter}} \right) \] is a Perfect square, namely \( \alpha^2 \) yields,
\[ (4p + q)^2 + 7q^2 = (4\alpha)^2 \] (2.3)
which is rewritten as,
\[ Y^2 = 7q^2 + Z^2 \] (2.4)
where \[ Z = 4p + q, \ Y = 4\alpha \] (2.5)
The solution of (2.4) are given by
\[ Y = 7r^2 + s^2, \ Z = 7r^2 - s^2, \ q = 2rs \] (2.6)
From (2.5) and (2.6), we get
\[ p = \frac{7r^2 - s^2 - 2rs}{4}, \ q = 2rs, \ \alpha = \frac{7r^2 + s^2}{4} \] (2.7)
As our interest centers on finding integral solutions, the values of \( p \) and \( \alpha \) are integers for the following choices of \( r \) and \( s \)
a) \[ r = 2KS, \ s = 2S(k \neq 1) \] (2.8)
b) \[ r = 2R + 1, \ s = 2S + 1, \ R \neq S \] (2.9)
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For the choice (a), the sides of the Pythagorean triangle are:

\[ x = 16S^4k(7k^2 - 2k - 1) \]
\[ y = S^4(7k^2 + 6k - 1)(7k^2 - 10k - 1) \]
\[ z = S^4\left((7k^2 - 2k - 1)^2 + 64k^2\right) \]

For the choice (b), the sides of the Pythagorean triangle are:

\[ x = 2\left(7R^2 - S^2 + 6R - 2RS - 2S + 1\right)(8RS + 4R + 4S + 2) \]  \hspace{1cm} (2.10)
\[ y = \left(7R^2 - S^2 + 6RS + 10R + 2S + 3\right)(7R^2 - S^2 - 10RS + 2R - 6S - 1) \]  \hspace{1cm} (2.11)
\[ z = \left(7R^2 - S^2 + 6R - 2RS - 2S + 1\right)^2 + (8RS + 4R + 4S + 2)^2 \]  \hspace{1cm} (2.12)

with the condition,

\[ R > S, \hspace{0.5cm} 7R^2 + 6R + 1 > S^2 + 2RS + 2S \]  \hspace{1cm} (2.13)

It is to be noted that Pythagorean triangle is obtained as follows. A few examples of Pythagorean triangles each satisfying the condition that Hypotenuse added with ratio \( \frac{Area}{Perimeter} \) is a square are given below in Table 2.1 (a).
Table 2.1 (a) Examples

<table>
<thead>
<tr>
<th>( R )</th>
<th>( S )</th>
<th>( x )</th>
<th>( y )</th>
<th>( z )</th>
<th>( z + \frac{A}{P} = \alpha^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1</td>
<td>2040</td>
<td>256</td>
<td>2056</td>
<td>2116</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>6132</td>
<td>3565</td>
<td>7093</td>
<td>7744</td>
</tr>
<tr>
<td>5</td>
<td>2</td>
<td>39160</td>
<td>19584</td>
<td>43784</td>
<td>47524</td>
</tr>
<tr>
<td>6</td>
<td>3</td>
<td>86632</td>
<td>23520</td>
<td>89768</td>
<td>94864</td>
</tr>
<tr>
<td>7</td>
<td>2</td>
<td>105000</td>
<td>100000</td>
<td>145000</td>
<td>160000</td>
</tr>
<tr>
<td>8</td>
<td>3</td>
<td>206584</td>
<td>131712</td>
<td>245000</td>
<td>268324</td>
</tr>
</tbody>
</table>

Properties:

a) \( 3(z \pm y) \) is a Nasty number.

b) \( x \equiv 0 \pmod{4} \)

c) \( \frac{z-y}{2} \) is a perfect square.
In addition to the above two patterns of solutions, we have also the following patterns.

PATTERN 1:

\[
\begin{align*}
  x &= 8AB^3 - 2B^4 - 16A^3B + 4A^2B^2 \\
  y &= 8AB^3 - 20A^2B^2 + 4A^4 \\
  z &= 2B^4 + 4A^4 + 12A^2B^2 - 8AB^3
\end{align*}
\]

(2.14) \hspace{1cm} (2.15) \hspace{1cm} (2.16)

with the condition,

\[
B^2 > A^2 + 2AB, \ 4AB > B^2 > 2A^2
\]

(2.17)

A few examples of Pythagorean triangles each satisfying the condition that Hypotenuse added with ratio \( \frac{\text{Area}}{\text{Perimeter}} \) is a square are given below in Table 2.1 (b).
### Table 2.1 (b) Examples

<table>
<thead>
<tr>
<th>$A$</th>
<th>$B$</th>
<th>$x$</th>
<th>$y$</th>
<th>$z$</th>
<th>$z + \frac{A}{P} = \alpha^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3</td>
<td>42</td>
<td>40</td>
<td>58</td>
<td>64</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
<td>510</td>
<td>64</td>
<td>514</td>
<td>529</td>
</tr>
<tr>
<td>2</td>
<td>6</td>
<td>672</td>
<td>640</td>
<td>928</td>
<td>1024</td>
</tr>
<tr>
<td>3</td>
<td>9</td>
<td>3402</td>
<td>3240</td>
<td>4698</td>
<td>5184</td>
</tr>
<tr>
<td>4</td>
<td>10</td>
<td>8160</td>
<td>1024</td>
<td>8224</td>
<td>8464</td>
</tr>
<tr>
<td>4</td>
<td>13</td>
<td>10686</td>
<td>17248</td>
<td>20290</td>
<td>22201</td>
</tr>
<tr>
<td>5</td>
<td>14</td>
<td>24528</td>
<td>14260</td>
<td>28372</td>
<td>30976</td>
</tr>
<tr>
<td>5</td>
<td>15</td>
<td>26250</td>
<td>25000</td>
<td>36250</td>
<td>40000</td>
</tr>
</tbody>
</table>

Properties:

a) $y \equiv 0 \pmod{4}$, $x \equiv 0 \pmod{2}$, $z \equiv 0 \pmod{2}$

b) $x \equiv 0 \pmod{6}$

c) $3(z \pm y)$ is a Nasty number.
d) Each of the expressions is a perfect square.

1. \( \frac{x^2}{2(z - y)} \)

2. \( x + z \)

PATTERN 2:

\[
\begin{align*}
  x &= 16A^3B - 8A^4 - 8AB^3 + 4A^2B^2 \\
  y &= 16A^3B - 20A^2B^2 + A^4 \\
  z &= 6A^4 + B^4 - 4A^2B^2 + 4A^4B^4 - 16A^3B
\end{align*}
\]

(2.18) (2.19) (2.20)

with the condition,

\[ 4AB > 2A^2 > B^2, \quad 4A^2 > B^2 + 4AB \]  \quad (2.21)

A few examples of Pythagorean triangles each satisfying the condition that Hypotenuse added with ratio \( \left( \frac{\text{Area}}{\text{Perimeter}} \right) \) is a square are given below in Table 2.1 (c).
### Table 2.1 (c) Examples

<table>
<thead>
<tr>
<th>$A$</th>
<th>$B$</th>
<th>$x$</th>
<th>$y$</th>
<th>$z$</th>
<th>$z + \frac{A}{P} = \alpha^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>2</td>
<td>168</td>
<td>160</td>
<td>232</td>
<td>256</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>736</td>
<td>273</td>
<td>785</td>
<td>841</td>
</tr>
<tr>
<td>5</td>
<td>4</td>
<td>2040</td>
<td>256</td>
<td>2056</td>
<td>2116</td>
</tr>
<tr>
<td>7</td>
<td>5</td>
<td>6132</td>
<td>3565</td>
<td>7093</td>
<td>7744</td>
</tr>
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<td>8</td>
<td>6</td>
<td>11776</td>
<td>4368</td>
<td>12560</td>
<td>13456</td>
</tr>
<tr>
<td>9</td>
<td>7</td>
<td>20340</td>
<td>4669</td>
<td>20869</td>
<td>21904</td>
</tr>
<tr>
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<td>8</td>
<td>32640</td>
<td>4096</td>
<td>32896</td>
<td>33856</td>
</tr>
</tbody>
</table>


SECTION — B

The most cited non-trivial integral solutions of the ternary quadratic diophantine equation

\[ Y^2 = DX^2 + Z^2 \]  

where \( D \) is a square free integer are given by

\[ X = 2rs, Z = Dr^2 - s^2, \quad Y = Dr^2 + s^2 \]  

Define

\[ D = \alpha^4 + 2\alpha^2 - 1 \] \hspace{1cm} (2.24)

\[ Y = P + \alpha^2 X \] \hspace{1cm} (2.25)

where \( P, \alpha \) are non-zero integers.

From (2.23), (2.24) and (2.25) we have

\[ P = \left(\alpha^4 + 2\alpha^2 - 1\right)r^2 + s^2 - 2\alpha^2 rs \]

Considering \( P \) and \( X \) to be the generators of a pythagorean triangle \((u,v,w)\) the corresponding sides are given by

\[ u = 2PX = 4\alpha^4 r^3 s + 8\alpha^2 r^3 s - 4r^3 s + 4rs^3 - 8\alpha^2 r^2 s^2 \]

\[ v = p^2 - x^2 \]

\[ = \alpha^8 r^4 + 4\alpha^6 r^4 - 4\alpha^6 r^3 s + 2\alpha^4 r^4 + 6\alpha^4 r^2 s^2 - 8\alpha^4 r^3 s \]

\[ - 4\alpha^2 r^4 + 4\alpha^2 r^2 s^2 + 2\alpha^2 r^3 s - 2\alpha^2 rs^3 - 6r^2 s^2 + r^4 + s^4 \]

\[ w = p^2 + x^2 \]

\[ = \alpha^8 r^4 + 4\alpha^6 r^4 - 4\alpha^6 r^3 s + 2\alpha^4 r^4 + 2\alpha^4 r^2 s^2 - 8\alpha^4 r^3 s \]

\[ - 4\alpha^2 r^4 + 8\alpha^2 r^2 s^2 + 4\alpha^2 r^3 s - 4\alpha^2 rs^3 + 2r^2 s^2 + r^4 + s^4 \]

For different choices of \( \alpha, r, s \) in the above equations, we generate pythagorean triangles where in each of which the relation

\[ \text{Hypotenuse} + 4\alpha^2 \left( \frac{\text{Area}}{\text{Perimeter}} \right) = Z^2 \]

holds good.
In particular for the choices \( \alpha = 2, 3 \) various relations among the sides of the corresponding Pythagorean triangle are exhibited below.

Case(1):

The generators of the Pythagorean triangle for the choice \( \alpha = 2 \) are

\[
P = 23r^2 + s^2 - 8rs, \quad X = 2rs
\]

For obtaining various relations among the sides, one has to go for particular values of \( r \) and \( s \).

(I): Taking \( r = s \), are obtain the sides of the triangle to be

\[
u = 64r^4, \quad v = 252r^4, \quad w = 260r^4
\]

Properties:

a) Each of the following expressions is a quartic integer.
   
i) \( 4u \)
   
ii) \( 2(w - v) \)

b) \( 4u - v = 4, \quad w - 4u = 4 \)

c) \( 8(w - v) - u = 0 \)

d) \( 512(w^3 - v^3) - u^3 = 192uvw \)

e) \( u \equiv 0(\text{mod}(w - v)) \)

f) \( u + w, \quad u + v + w \) are a perfect square.

g) \( 3(w \pm v) \) is a nasty number.

h) The triple \( \left( \frac{u}{4r^4}, \frac{v}{4r^4}, \frac{w}{4r^4} \right) \) is primitive.

(II): The substitution of \( s = 2r \) gives

\[
u = 88r^4, \quad v = 105r^4, \quad w = 137r^4
\]
Properties:

1. \( \left( \frac{u}{r^4}, \frac{v}{r^4}, \frac{w}{r^4} \right) \) is primitive.

2. Each of the following expressions is a perfect square.
   a) \( \frac{w \pm v}{2} \)
   b) \( 2v - w - 9 \)
   c) \( 2v - w - 8 \)
   d) \( \text{Area} = 4620r^8 \). From the area, the following Second order Ramanuja numbers are obtained:
      3365, 5780, 8468, 17300, 29588, 55700, 80660, 151700, 154388, 1345172, 1387412
   e) \( 21(u + v - w) \) is a Nasty number.
   f) \( u + v - w \) is expressed as the difference of two cubes.

Case(2):

The generators of the Pythagorean triangles for the choice \( \alpha = 3 \) are
\[ P = 98r^2 + s^2 - 18rs, \quad X = 2rs \]

(III) On substituting \( s = r \), we have,
\[ u = 324r^4, \quad v = 6557r^4, \quad w = 6565r^4 \]

Properties:

i) \( 20u + 10v - 10w \) is a perfect square.
ii) \( 2(w - v) \) is a quartic integer.
iii) \( u - 40(w - v) = 4 \)
iv) \( \left( \frac{u}{r^4}, \frac{v}{r^4}, \frac{w}{r^4} \right) \) is primitive.
(IV) Setting $s = 4r$, we get

$$u = 672r^4, \quad v = 1700r^4, \quad w = 1828r^4$$

Properties:

1. $7(u + v + w)$ is a Nasty number.

2. $$\left( \frac{u}{4r^4}, \frac{v}{4r^4}, \frac{w}{4r^4} \right)$$ is primitive.

3. $4(u + v + w) \equiv 0 \pmod{7}$

4. $u - 5(w - v) + 4$ is a perfect square.
SECTION — C

The ternary Quadratic Diophantine equation considered for its non-trivial integral solutions is

\[ x^2 + y^2 = z^2 + 8 \]  \hspace{1cm} (2.26)

Two different patterns of non-zero integral solutions of (2.26) are exhibited.

PATTERN 1:

Taking \( y = x + 3 \) \hspace{1cm} (2.27)

in (2.26), it reduces to

\[ (2x + 3)^2 = 2z^2 + 7 \]  \hspace{1cm} (2.28)

The sequence of values of \( x \) and \( z \) satisfying (2.28) are given by

\[ x_s = \frac{1}{4}[3f + \sqrt{2}g - 6] \]  \hspace{1cm} (2.29)

\[ z_s = \frac{1}{2\sqrt{2}}[\sqrt{2}f + 3g] \]  \hspace{1cm} (2.30)

substituting (2.29) in (2.27), the sequence of values of \( y \) are obtained as

\[ y_s = \frac{1}{4}[3f + \sqrt{2}g + 6] \]  \hspace{1cm} (2.31)

where \[ f = (3 + 2\sqrt{2})^{s+1} + (3 - 2\sqrt{2})^{s+1} \]

\[ g = (3 + 2\sqrt{2})^{s+1} - (3 - 2\sqrt{2})^{s+1} \]

Thus (2.29), (2.30) and (2.31) represent the non-zero integral solutions of (2.26). The recurrence relations satisfied by the values of \( x_s, y_s \) and \( z_s \) are expressed as follows.
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\[ x_{s+2} - 6x_{s+1} + x_s = 6, \quad x_0 = 5, \quad x_1 = 36 \] \hspace{1cm} (2.32)
\[ y_{s+2} - 6y_{s+1} + y_s = -6, \quad y_0 = 8, \quad y_1 = 39 \] \hspace{1cm} (2.33)
\[ z_{s+2} - 6z_{s+1} + z_s = 0, \quad z_0 = 9, \quad z_1 = 53 \] \hspace{1cm} (2.34)

Some numerical examples are given below in Table 2.1(d).

Table 2.1 (d)

<table>
<thead>
<tr>
<th>S</th>
<th>( x_s )</th>
<th>( y_s )</th>
<th>( z_s )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>5</td>
<td>8</td>
<td>9</td>
</tr>
<tr>
<td>1</td>
<td>36</td>
<td>39</td>
<td>53</td>
</tr>
<tr>
<td>2</td>
<td>217</td>
<td>220</td>
<td>309</td>
</tr>
<tr>
<td>3</td>
<td>1272</td>
<td>1275</td>
<td>1801</td>
</tr>
<tr>
<td>4</td>
<td>7421</td>
<td>7424</td>
<td>10497</td>
</tr>
<tr>
<td>5</td>
<td>43260</td>
<td>43263</td>
<td>61181</td>
</tr>
<tr>
<td>6</td>
<td>252145</td>
<td>252148</td>
<td>356589</td>
</tr>
<tr>
<td>7</td>
<td>1469616</td>
<td>1469619</td>
<td>2078353</td>
</tr>
</tbody>
</table>
A few observations among the solutions are represented as follows.

1. \( y_{2s+1}, x_{2s+1}, z_{2s} \equiv 0 \pmod{3}, \ s = 0,1,2,3,\ldots \)

2. \( y_{2s} \equiv 0 \pmod{4}, \ s = 0,1,2,3,\ldots \)

3. \( z_s + (2\alpha + 1) \equiv 0 \pmod{2}, \ \alpha , s = 0,1,2,3,\ldots \)

4. For each value of \( y_s (s = 1,2,3,\ldots) \), one can generate second order Ramanujan numbers.

Example:

\[
7424 = 2 \cdot 3712 = 4 \cdot 1856 = 8 \cdot 928 = 16 \cdot 464 = 58 \cdot 128
\]

\[
= 232 \cdot 32 = 116 \cdot 64
\]

\[
= 1857^2 - 1855^2 = 930^2 - 926^2 = 468^2 - 460^2 = 240^2 - 224^2
\]

\[
= 93^2 - 35^2 = 132^2 - 100^2 = 90^2 - 26^2
\]

\[
1857^2 - 1855^2 = 930^2 - 926^2 \Rightarrow 1857^2 + 926^2 = 930^2 + 1855^2 = 4305925
\]

\[
90^2 - 26^2 = 93^2 - 35^2 \Rightarrow 90^2 + 35^2 = 93^2 + 26^2 = 9325
\]

\[
90^2 - 26^2 = 132^2 - 100^2 \Rightarrow 90^2 + 100^2 = 132^2 + 26^2 = 18100
\]

\[
93^2 - 35^2 = 132^2 - 100^2 \Rightarrow 93^2 + 100^2 = 132^2 + 35^2 = 18649
\]

\[
90^2 - 26^2 = 240^2 - 224^2 \Rightarrow 90^2 + 224^2 = 240^2 + 26^2 = 58276
\]

\[
93^2 - 35^2 = 240^2 - 224^2 \Rightarrow 93^2 + 224^2 = 240^2 + 35^2 = 58825
\]

Here we observe that 9325, 18100, 18649, 58276, 58825, 4305925 are second order Ramanujan numbers.

5. It is observed that \( \beta = 6y_s - 2z_s - 9, \alpha = 2y_s - 3z_s - 3 \) satisfy the Diophantine equation \( \beta^2 = 2\alpha^2 + 7^2 \).

6. 42(12y_s - 4z_s - 4) is a Nasty number.

7. 7(12y_{2s+1} - 4z_{2s+1} - 4) = (12y_s - 4z_s - 18)^2

8. 7(12y_{2s+1} - 4z_{2s+1} - 32) = 2(4y_s - 6z_s - 6)^2
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9. \((2y_s - 3)^2 - 2z_s^2 = 7\).

PATTERN 2:
Using the linear transformations,
\[ z = u + v, \quad y = u - v \]  \hspace{1cm} (2.35)

(2.26) is reduced to
\[ x^2 = 4uv + 8 \]  \hspace{1cm} (2.36)

Here \(u, v\) are distinct non-zero parameters. It is possible to choose \(u\) and \(v\) such that \(uv + 2\) is a square and the value of \(x\) is obtained. Thus knowing \(u, v\) and using (2.35), the corresponding values of \(y\) and \(z\) are found. The above process is illustrated through the following examples in table 2.1 (e).

**Table 2.1 (e)**

<table>
<thead>
<tr>
<th>(u)</th>
<th>(v)</th>
<th>(x)</th>
<th>(y)</th>
<th>(z)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>7</td>
<td>8</td>
<td>-5</td>
<td>9</td>
</tr>
<tr>
<td>(n^2 + 2n - 1)</td>
<td>1</td>
<td>(2n + 2)</td>
<td>(n^2 + 2n - 2)</td>
<td>(n^2 + 2n)</td>
</tr>
<tr>
<td>2</td>
<td>(2n^2 - 1)</td>
<td>(4n)</td>
<td>(3 - 2n^2)</td>
<td>(2n^2 + 1)</td>
</tr>
<tr>
<td>(2n^2 + 4n + 1)</td>
<td>2</td>
<td>(4n + 4)</td>
<td>(2n^2 + 4n - 1)</td>
<td>(2n^2 + 4n + 3)</td>
</tr>
</tbody>
</table>
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GENERALIZATION OF SOLUTIONS

Let \((x_0, y_0, z_0)\) be the initial non-zero integral solutions to (2.26) and let \((x_1, y_1, z_1)\) be its second solution.

write \(x_1 = h - x_0, \ y_1 = h - y_0, \ z_1 = h - z_0\) (2.37)

where \(h\) is any non-zero integer.

substituting (2.37) in (2.26) we get,

\[ h = 2x_0 + 2y_0 + 2z_0 \]

Therefore, from (2.37), we have

\[
\begin{align*}
x_1 &= x_0 + 2y_0 + z_0 \\
y_1 &= 2x_0 + y_0 + 2z_0 \\
z_1 &= 2x_0 + 2y_0 + 3z_0
\end{align*}
\]

which is represented in the matrix form

\[
(x_1, y_1, z_1)^T = A (x_0, y_0, z_0)^T
\]

where \(A\) is \(3 \times 3\) matrix

\[
\begin{pmatrix}
1 & 2 & 2 \\
2 & 1 & 2 \\
2 & 2 & 3
\end{pmatrix}
\]

and \(T\) refers to the transpose.

The repetition of the above process leads to the general solutions of (2.26) given below

\[
\begin{pmatrix}
x_n \\
y_n \\
z_n
\end{pmatrix} =
\begin{pmatrix}
\frac{y_n + (-1)^n}{2} & \frac{y_n + (-1)^{n+1}}{2} & x_n \\
\frac{y_n + (-1)^{n+1}}{2} & \frac{y_n + (-1)^n}{2} & x_n \\
x_n & x_n & y_n
\end{pmatrix} \begin{pmatrix}
x_0 \\
y_0 \\
z_0
\end{pmatrix} n = 1, 2, 3, \ldots
\]

(2.38)
Here \( x_n, y_n \) represents the general form of integral solutions of the Pellian equation

\[ y^2 = 2x^2 + 1 \]

Thus, knowing \((x_0, y_0, z_0)\) and substituting \( n = 1, 2, 3, \ldots \) in (2.38), one can generate an infinitely many integral solutions to (2.26).