Chapter – VI

A PSO Algorithm and Fuzzy Production Inventory Models
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A PSO ALGORITHM AND FUZZY PRODUCTION INVENTORY MODELS

This chapter is devoted to explain the concept and method of finding the optimal fuzzy production quantity and fuzzy expected value of the total cost by using the expected value and credibility criteria both based on fuzzy simulation. Also a Particle Swarm Optimization (PSO) algorithm based on fuzzy simulation is designed to obtain optimal solution. An example is given to illustrate the feasibility and verify the validity of the proposed algorithm.

6.1. INTRODUCTION

In real situations, the parameters and variables in any inventory model are imprecise. Many authors had treated several fuzzy inventory models in which the parameters were assumed to be either triangular or trapezoidal fuzzy numbers and again the methods applied for defuzzification are not satisfactory. Therefore, an inventory problem was analysed by introducing fuzzy notions by constructing Fuzzy Expected Value Model (FEVM) and a Fuzzy Dependent Change Programming Model (FDCPM) and for solving these models, the fuzzy simulation based on particle swarm optimization algorithm was suggested [24, 38, 62].

The content of this chapter has been communicated to the Far East Journal of Applied Mathematics.
The Analysis of these models brings out satisfactory results which are tested by implementation of the programs in MATLAB.

6.2. ASSUMPTIONS AND NOTATIONS

The discussion of the fuzzy inventory models is done by making certain assumptions and some notations.

6.2.1. ASSUMPTIONS

It is reasonable to characterize the parameters as fuzzy variables and the discussion of fuzzy production inventory model is to be carried under the following assumptions.

1. No shortage is permitted.

2. Production occurs during the intervals in which usage also occurs.

3. The inventory level never is equal to the production lot size.

4. Starting inventory level is zero.

5. While production begins, demand continues.

6. The parameters $c_h$ and $c_s$ are independent fuzzy variables which are defined on the possibility spaces $(S_i, P(S_i), P_s)$ where $i = 1, 2, ..., n$ respectively.
6.2.2. NOTATIONS

In a classical production inventory model the parameters are represented as below.

- \( q_p \) - Production quantity
- \( c_h \) - Inventory holding cost per unit
- \( d_y \) - Yearly demand
- \( c_s \) - Setup cost per setup
- \( d_r \) - Daily demand rate
- \( p_r \) - Daily production rate
- \( p_y \) - Yearly production

6.3. FORMULATION OF FUZZY INVENTORY MODEL

The production inventory model is considered when units of items are produced and sold simultaneously [58, 63]. A typical behaviour of the classical production inventory model without shortage is presented in Figure 6.1, where \( M \) is the maximum inventory quantity, in which the manufacturer produces and sells during the time \( OP \), etc. and he only sells during the time \( PT \), etc.
**Fig. 6.1. A typical production inventory model**

In this model, the setup cost and holding cost are of the form

\[\bar{c}_s \left( \frac{d_y}{q_p} \right) \text{ and } \bar{c}_h \left( \frac{q_p}{2} \right) \left( 1 - \frac{dr}{pr} \right)\]

respectively, where the parameters are crisp real numbers. The optimal production quantity \(q_p^*\) of classical production inventory model is given by

\[q_p^* = \frac{\sqrt{2} c_s d_y}{\sqrt{c_h (1 - d_r /p_r)}}\]

The objective of the fuzzy production inventory model is to find out the optimal production quantity of items at each time such that the combination of the inventory holding cost and the production cost is minimal. In real situations the production cost and the inventory holding cost are usually affected by various uncontrollable factors and often there exist some fluctuations.
In a typical production inventory model without shortage, the production quantity \(q_p\) is greater than or equal to the total demand \(d_y(q_p \geq d_y)\). The total production cost in the plan period \([0, T]\) can be expressed by \(T_c(q_p)\) and is given by

\[
T_c(q_p) = \frac{c_s d_y}{q_p} + \frac{c_h q_p}{2} \left(1 - \frac{d_r}{p_r}\right)
\] … (6.1)

Since the parameters \(c_s\) and \(c_h\) are fuzzy variables defined on the possibility spaces \((S_i, \mathcal{P}(S_i), P_{S_i})\) where \(i = 1, 2\) respectively, the total cost \(T_c(q_p)\) is a fuzzy variable defined on the product possibility space \((S_i, P(S_i), P_{S_i})\), where \(S = S_1 \times S_2\) and \(P_s = P_{S_1} \times P_{S_2}\).

**6.4. FUZZY EXPECTED INVENTORY HOLDING COST**

If it is required to determine the production quantity such that the fuzzy expected value of the total cost is minimal then a fuzzy (FEVPM) expected value production model can be constructed as

\[
\min_{0 < d_l \leq d_q} \mathbb{E} \left[ \frac{c_s d_y}{q_p} + \frac{c_h q_p}{2} \left(1 - \frac{d_r}{p_r}\right) \right]
\] … (6.2)

Since the setup cost and holding cost \(c_s, c_h\) are independent of each other it follows that the fuzzy expected total cost (FETC) is given by
\[
\text{FETC} = E \left[ \frac{c_s d_y}{q_p} + \frac{c_h q_p}{2} \left( 1 - \frac{d_r}{p_r} \right) \right] \\
= \left[ \frac{E(c_s) d_y}{q_p} + \frac{E(c_h) q_p}{2} \left( 1 - \frac{d_r}{p_r} \right) \right] 
\]

Therefore the optimal solution \( q_p^* \) of the model can be determined by the formula

\[
q_p^* = \sqrt{\frac{2E(c_s) d_y}{E(c_h) \left( 1 - \frac{d_r}{p_r} \right)}} 
\]

where \( E(c_s) \) and \( E(c_h) \) are the expected values of \( c_s \) and \( c_h \) respectively.

If the cost parameters \( c_s \) and \( c_h \) are trapezoidal fuzzy variables, then their expected values can be calculated easily. But for other genetic fuzzy variables, it is necessary to estimate their expected values by fuzzy simulation. Therefore the following fuzzy simulation procedure is formulated to find \( E(c_h) \) and similar procedure is to be followed to estimate \( E(c_s) \).

**Remark**

If \( p_r \) is much greater than \( d_r \) (\( p_r > d_r \)) then the formula for optimal production quantity is not significantly different from the EOQ formula.
Expected Holding Cost and Fuzzy Simulation

In finding the expected holding cost $E(c_h)$, the steps below are followed.

**Step: 1**

(i) Generate $s_k$ uniformly from $S_1$ such that $P_s(s_k) > \varepsilon$, $k=1, 2, ..., N$ and $\varepsilon$ is very small number.

(ii) Assign $x_k = h(s_k)$ and rearrange $x_k$ such that $x_1 < x_2 < \ldots < x_N$

(iii) Calculate $c_k = c_h(x_k)$ for $k = 1, 2 ... N$, the membership function of $h$.

**Step: 2**

To estimate $E(c_h)$, $\sum_{k=1}^{N} x_k w_k$ is to determined where

$$w_1 = \frac{1}{2} \left( \mu_1 + \max_{1 \leq j} \mu_j - \min_{1 \leq j} \mu_j \right)$$

$$w_k = \frac{1}{2} \left( \max_{1 \leq j, k \leq k} \mu_j - \max_{1 \leq j, k \leq k} \mu_j + \min_{1 \leq j, k \leq k} \mu_j - \min_{1 \leq j, k \leq k} \mu_j \right) \text{ for } 2 \leq k \leq n - 1$$

$$w_N = \frac{1}{2} \left( \max_{1 \leq j, k \leq N} \mu_j - \min_{1 \leq j, k \leq N} \mu_j + \mu_N \right)$$

**6.5. FUZZY DEPENDENT CHANCE PROGRAMMING MODEL**

In practical situations, any decision assumes that the total cost should not exceed the budget level ‘b’. Therefore it is to maximize the credibility of the event such that the total cost is less than or equal to be ‘b’ and hence the FDCPP model can be formulated as below.
Maximize $c_r \left[ \frac{c_s d_y}{q_p} + \frac{c_h q_p}{2} \left( 1 - \frac{d_r}{p_r} \right) \right] \leq b$ under the condition that $0 < q_p \leq d_y$.

Therefore it is to find

$$\max_{0 \leq q_p \leq d_y} c_r \left[ \frac{c_s d_y}{q_p} + \frac{c_h q_p}{2} \left( 1 - \frac{d_r}{p_r} \right) \right] \leq b.$$ 

**Credibility of Objective Function Based Event**

For solving this model, it is required to calculate the value of

$$c_r \left[ \frac{c_s d_y}{q_p} + \frac{c_h q_p}{2} \left( 1 - \frac{d_r}{p_r} \right) \right] \leq b$$

for each value of $q_p$ by the following steps

**Step: 1**

Assume $e_1 = e_2 = 0$ and $n = 1$.

**Step: 2**

Generate uniformly a sequence $(s_{1n}, s_{2n})$ from $S = S_1 \times S_2$ such that $P_s(s_{in}) < c$, where $i = 1, 2$ and $\epsilon$ is a sufficiently small number and find a real pair $(c_s(s_{1n})m c_h(s_{2n}))$.

**Step: 3**

Determine $\frac{c_s(s_{1n}) d_y}{q_p} + \frac{c_h(s_{2n}) q_p}{2} \left( 1 - \frac{d_r}{p_r} \right)$

and $\mu = \min[\mu_s(c_s(s_{1n})), \mu_h(c_h(s_{2n}))]$
Step: 4

\[
\frac{c_s(s_{in})d_y}{q_p} + \frac{c_h(s_{2n})q_p}{2}\left(1 - \frac{d_t}{p_t}\right) \leq b \quad \text{and} \quad e_1 < \mu
\]

then assume \( e_1 = \mu \).

Step: 5

\[
\frac{c_s(s_{in})d_y}{q_p} + \frac{c_h(s_{2n})q_p}{2}\left(1 - \frac{d_t}{p_t}\right) > b \quad \text{and} \quad e_2 < \mu
\]

then assume \( e_2 = \mu \).

Step: 6

Go to step: 2, replace \( n \) by \( n+1 \) and repeat until a desired number of iterations is attained.

Step: 7

Return \( e = \frac{1}{2}(e_1 + 1 - e_2) \) as the credibility value of the objective function.

6.6. FUZZY SIMULATION AND PSO ALGORITHM

The Particle Swarm Optimization (PSO) is based on both population and fuzzy simulation. While comparing the PSO algorithm with the other evolutionary algorithms, such as the genetic algorithm, the convergence rate is faster and has a very few parameters.
The proposed FDCPP model is to be solved by designing an algorithm which is done by embedding the fuzzy simulation into the particle swarm optimization algorithm, where the fuzzy simulation is employed to estimate the credibility of each fuzzy event and the PSO algorithm is used to obtain the optimal solution.

**Particle Swarm Optimization Algorithm**

The Particle Swarm Optimization algorithm is described by the following steps

**Step: 1**

Assign \( k = 1 \).

**Step: 2**

Generate randomly the initial position \((Q_p)_i^k\) in \((0, d_y]\) for particle \( i \), where \( N \) denotes the population size of the swarm and \((V_p)_i^k\) from \([0, v]\), the velocity for particle \( i = 1,2,\ldots,N \).

**Step: 3**

Assume \((P_p)_i^k\) be the position of the particle \( i \) such that

\[
C_r \left[ \frac{c_y d_y}{(P_p)_i^k} + \frac{c_h (P_p)_i^k}{2} \left( 1 - \frac{d_r}{p_r} \right) \right] \leq b
\]

\[
\max_{1 \leq i \leq k} C_r \left[ \frac{c_y d_y}{(Q_p)_i^l} + \frac{c_h (Q_p)_i^l}{2} \left( 1 - \frac{d_r}{p_r} \right) \right] \leq b,
\]

where \( b \) stands for budget level and the value of
\[
C_i \left[ \frac{c_s d_y}{(Q_p)_i^k} + \frac{c_h (Q_p)_i^l}{2} \left( 1 - \frac{d_z}{p_r} \right) \right] \leq b
\]

can be calculated by the fuzzy simulation as in 6.4.1 designed previously for \(i = 1, 2, \ldots, N\). Then the position \( (P_p)_i^k \) is called as the l-best for particle \(i\).

Moreover let the position \( (P_p)_i^k \) be such that
\[
c_i \left[ \frac{c_s d_y}{(P_p)_j^k} + \frac{c_h (P_p)_j^k}{2} \left( 1 - \frac{d_z}{p_r} \right) \right] = \max_{1 \leq i \leq N} \left[ \frac{c_s d_y}{(Q_p)_i^l} + \frac{c_h (Q_p)_i^l}{2} \left( 1 - \frac{d_z}{p_r} \right) \right] \leq b
\]

Then the position \( (P_p)_g^k \) is called as the g-best of all \(N\) particles.

**Step: 4**

Assign \((V_p)_i^{k+1}\) for particle \(i\) such that
\[
(V_p)_i^{k+1} = \omega (V_p)_i^k + a_1 r_1 [(P_p)_i^k - (Q_p)_i^k] + a_2 r_2 [(p_p)_j^k - (Q_p)_i^k]
\]

where \(\omega\) is the inertia weight \(a_1, a_2\) are acceleration constants and \(r_1, r_2\) are random numbers in the interval \([0, 1]\).

**Step: 5**

Assign \((Q_p)_i^{k+1}\) for particle \(i\) such that
\[
(Q_p)_i^{k+1} = (Q_p)_i^k + (V_p)_i^{k+1}
\]

**Step: 6**

Assign \(k \leftarrow k+1\) and repeat Step: 3 until a required number of cycles are attained.
Step: 7

Return \((P^*_p)^k_j\) as the optimal solution.

6.7. NUMERICAL EXAMPLES

To illustrate the estimation of expected holding cost, calculation of credibility and implementation of PSO algorithm, the following numerical examples are provided respectively.

6.7.1. Example to Estimate Expected Holding Cost

For estimating the expected holding cost, first define

\[
\mu_h(x) = \begin{cases} 
1-(x-3)^2 & \text{if } 2 \leq x < 3 \\
1 & \text{if } 3 \leq x \leq 4 \\
1-(x-4)^2 & \text{if } 4 < x \leq 5 
\end{cases}
\]

Next to generate 100 random numbers \(x_k\) are generated and arranged them in ascending order. The process of random number generation depending upon the fuzzy simulation is done and the respective values for \(k = 1, 2, \ldots, N\) of \(x_k, \mu_k, w_k\) and \(x_k \cdot w_k\) are calculated and tabulated in the following Table 6.1.
# Table - 6.1

<table>
<thead>
<tr>
<th>$k$</th>
<th>$x_k$</th>
<th>$\mu_k$</th>
<th>$w_k$</th>
<th>$x_k \cdot w_k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.48</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.000000</td>
</tr>
<tr>
<td>2</td>
<td>0.56</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.000000</td>
</tr>
<tr>
<td>19</td>
<td>1.80</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.000000</td>
</tr>
<tr>
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<td>2.05</td>
<td>0.0975</td>
<td>0.04875</td>
<td>0.0999375</td>
</tr>
<tr>
<td>21</td>
<td>2.07</td>
<td>0.1351</td>
<td>0.01880</td>
<td>0.0389160</td>
</tr>
<tr>
<td>31</td>
<td>3.05</td>
<td>1.0000</td>
<td>0.00320</td>
<td>0.0097600</td>
</tr>
<tr>
<td>32</td>
<td>3.08</td>
<td>1.0000</td>
<td>0.00000</td>
<td>0.000000</td>
</tr>
<tr>
<td>37</td>
<td>4.06</td>
<td>1.0000</td>
<td>0.00180</td>
<td>0.0065700</td>
</tr>
<tr>
<td>38</td>
<td>4.07</td>
<td>0.9964</td>
<td>0.19575</td>
<td>0.0026390</td>
</tr>
<tr>
<td>48</td>
<td>4.82</td>
<td>0.3276</td>
<td>0.07785</td>
<td>0.3755237</td>
</tr>
<tr>
<td>49</td>
<td>4.91</td>
<td>0.1719</td>
<td>0.08595</td>
<td>0.4220145</td>
</tr>
<tr>
<td>50</td>
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<td>0.0000</td>
<td>0.00000</td>
<td>0.000000</td>
</tr>
<tr>
<td>51</td>
<td>5.12</td>
<td>0.0000</td>
<td>0.00000</td>
<td>0.000000</td>
</tr>
<tr>
<td></td>
<td>.</td>
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<td></td>
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<td>.</td>
<td>.</td>
</tr>
<tr>
<td>100</td>
<td>9.89</td>
<td>0.0000</td>
<td>0.00000</td>
<td>0.000000</td>
</tr>
</tbody>
</table>

$E(c_h) = 3.4966182$
Thus, by fuzzy simulation process, the expected holding cost can be determined using formula $E(c_h) = \frac{\sum_{k=1}^{N} x_k \cdot W_k}{N}$. And by similar fuzzy simulation process $E(c_s)$ can be estimated by defining $\mu_s$ as

$$
\mu_s(y) = \begin{cases} 
\frac{y-15}{5} & \text{if } 15 \leq y < 20 \\
1 & \text{if } 20 \leq y \leq 25 \\
\frac{30-y}{5} & \text{if } 25 \leq y \leq 30 \\
0 & \text{elsewhere}
\end{cases}
$$

and using the respective formula $E(c_s) = \frac{\sum_{k=1}^{N} y_k \cdot u_k}{N}$

### 6.7.2. Example to Calculate Credibility

For finding the credibility of the FDCPP model, first 1000 random numbers are generated with respect to the holding cost and setup cost, then the respective membership values $\mu_s$ and $\mu_h$ are calculated. Following the steps in 6.5.1, the required credibility measure ‘e’ of the objective function is calculated by identifying the possibility and necessity measures namely $e_1$ and $e_2$ respectively. Having processed, the values of $x_k, y_k, \mu_h, \mu_s, \mu_{\min}, (FETC)_k, e_1$ and $e_2$ are given in the following Table 6.2.
### Table - 6.2

<table>
<thead>
<tr>
<th>k</th>
<th>x_k</th>
<th>y_k</th>
<th>µ_n</th>
<th>µ_s</th>
<th>µ_min</th>
<th>(FETC)_k</th>
<th>e_1</th>
<th>e_2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
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<td>16.6693</td>
<td>0</td>
<td>0.3339</td>
<td>0</td>
<td>51.5353</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
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<td>19.2554</td>
<td>1.0000</td>
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<td>0.8511</td>
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<td>0</td>
</tr>
<tr>
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<td>0.4020</td>
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<td>0.6249</td>
<td>49.3775</td>
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<td>0.6249</td>
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<tr>
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<td>0.8544</td>
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<td>0.8544</td>
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<td>0.9209</td>
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<td>0.9209</td>
<td>0.8544</td>
</tr>
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<td>0.9430</td>
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<tr>
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<td>0</td>
<td>56.7101</td>
<td>0.9912</td>
<td>0.9430</td>
</tr>
</tbody>
</table>

\[ e = 0.5241 \]

#### 6.7.3. Example to Implement PSO Algorithm

Consider a fuzzy production inventory system without shortage in which the plan period is one year, the yearly demand quantity during the plan period \( d_y = 70 \), the fuzzy setup cost = (15, 20, 25, 30) and the fuzzy holding cost = (2, 3, 4, 5). By running the programs in MATLAB as given in Annexure, the following results are outputted

(i) \( E(c_h) = 3.5097 \)

(ii) \( E(c_s) = 18.5629 \)
If it is required to maximize the credibility that the total cost does not exceed 42, a fuzzy dependent chance programming model can be constructed and the credibility of the event is found as

$$\max C_r \left[ \frac{c_s q_p^*}{q_p} + \frac{c_h q_p^*}{2} \leq 42 \right] = 0.5241$$

The PSO algorithm based on the fuzzy simulation is designed and is used to solve the model. By setting the largest velocity $v=2$ the inertia weight $\omega = 0.95$, the acceleration constants $a_1 = a_2 = 2$ and running simulation the optimal production quantity, the maximal credibility and the fuzzy expected total cost are obtained as

(i) $$q_p^* = \sqrt{\frac{2E(c_s) d_y}{E(c_h) \left( 1 - \frac{d_c}{p_r} \right)}} = 62.8422$$

(ii) $$C_r \left[ \left( \frac{c_s d_y}{q_p^*} + \frac{c_h q_p^*}{2} \left( 1 - \frac{d_c}{p_r} \right) \right) \leq 42 \right] = 0.4963$$

(iii) $$\text{FETC} \left( q_p^* \right) = 43.034$$
In order to verify the feasibility of our proposed algorithm, one can design a genetic algorithm based on fuzzy simulation or a bee algorithm based on fuzzy simulation to solve the same problem and it can be found that the performance of the PSO algorithm based on fuzzy simulation is not only acceptable but also it is preferable because of its wonderful performance.

The attempt made in this chapter, by characterizing the setup cost and inventory holding cost as fuzzy variables of the fuzzy production inventory systems, provides satisfactory results. The fuzzy simulations have been handled successfully to solve the model. Examples related to fuzzy production model are presented by which the algorithm has been tested for its validity and the results conclude that the PSO algorithm is performed well and suited for the fuzzy production model.