Chapter 4

Riemann–Stieltjes Integrals of Fuzzy Valued Functions

Abstract
The concept of Riemann–Stieltjes integral of fuzzy valued functions is discussed in this chapter. The Riemann’s condition of integrability for fuzzy valued function is introduced. Based on this condition equivalent condition of integrability of fuzzy valued functions in the Riemann–Stieltjes sense are obtained. A result on Riemann–Stieltjes integrability of composite function is obtained.

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4.1 Introduction

Sugeno [87] has introduced the concept of fuzzy integral. Prompted by the work of Sugeno [87] many formulations of fuzzy integrals have been developed. Sims and Wang [82] presented a detailed review on fuzzy measures and fuzzy integrals. The fuzzy integrals of Riemann type is defined through atoms by Hisakichi Suzuki [34] in which he has established that a continuous function is fuzzy integrable in the sense of Riemann. The concepts of fuzzy Riemann integrals and improper fuzzy Riemann integrals based on the closed intervals with numerical integration were introduced by Wu [101, 102]. Since the $\alpha$-level set of the fuzzy Riemann integral is a closed interval whose end points are classical Riemann integrals it is possible to develop the fuzzy analog of various properties of Riemann integrals. Along this line Yuhu Feng [110] has defined the mean square Riemann–Stieltjes integral of two types associated with the class of fuzzy stochastic processes. Problems in Physics which involve mass distributions that are partly discrete and party continuous can be treated using Riemann–Stieltjes integrals. In probability theory Riemann–
Stieltjes integral can be employed as an efficient tool when we bound to simultaneously treat continuous and discrete random variables. Hence the fuzzy analog of Riemann–Stieltjes integral is necessary to deal fuzzy random variables.

In this chapter we present the notion of Riemann–Stieltjes integrals for fuzzy valued functions. The immediate properties of Riemann–Stieltjes integrals of fuzzy valued functions are established. We provide the Riemann’s condition of integrability for fuzzy valued functions and dealt with some extensions of the Riemann–Stieltjes integrability of fuzzy valued functions.

In Section 4.2 some basic notions of fuzzy sets and operations on fuzzy numbers are introduced. In Section 4.3 we present the notions of Riemann–Stieltjes upper and lower sums for fuzzy valued functions, and introduced the notion of Riemann’s condition for fuzzy valued functions. Based on this condition we have obtained equivalent conditions of Riemann–Stieltjes integrability of fuzzy valued functions.
4.2 Preliminaries

We introduce in this section some notions of fuzzy sets and operations of fuzzy numbers.

**Definition 4.2.1.** Let \( X \) be an universal set and \( A \) be a fuzzy subset of \( X \) with membership function \( A(x) \in [0, 1] \). The \( \alpha \)-level set of the fuzzy set \( A \) is defined as

\[
A_\alpha = \{ x ; A(x) \geq \alpha \}
\]

where \( A_0 \) is the closure of the set \( \{ x ; A(X) \neq 0 \} \).

**Definition 4.2.2.** The fuzzy set \( A \) is called a normal fuzzy set if there exists \( x \) such that \( A(x) = 1 \).

\( A \) is called a convex fuzzy set if \( A(\lambda x + (1-\lambda)y) \geq \min\{A(x), A(y)\} \)

for \( \lambda \in [0, 1] \).

**Theorem 4.2.1** ([111]). \( A \) is a convex fuzzy set if and only if

\( \{ x ; A(x) \geq \alpha \} \) is a convex set for each \( \alpha \in [0, 1] \).

**Definition 4.2.3.** Let \( f(x) \) be a real valued function on a topological space. If the set \( \{ x ; f(x) \geq \alpha \} \) is closed for each \( \alpha \), \( f(x) \) is said to be upper semi continuous.
Definition 4.2.4. $\tilde{a}$ is called a fuzzy number if $\tilde{a}$ is a normal convex fuzzy set and the $\alpha$-level set $\tilde{a}_\alpha$ is bounded for each $\alpha \neq 0$.

$\tilde{a}$ is called a closed fuzzy number if $\tilde{a}$ is a fuzzy number and its membership function is upper semi continuous.

$\tilde{a}$ is called a bounded fuzzy number if $\tilde{a}$ is a fuzzy number and its membership function has a compact support.

We say that $\tilde{a}$ is a crisp number with value $m$ if and only if its membership function is.

$$\tilde{a}(r) = \begin{cases} 1; & r = m \\ 0; & \text{otherwise.} \end{cases}$$

Theorem 4.2.2 ([87]). If $\tilde{a}$ is a closed fuzzy number then the $\alpha$-level set of $\tilde{a}$ is a closed interval which is denoted by

$$\tilde{a}_\alpha = [\tilde{a}_\alpha^L, \tilde{a}_\alpha^U].$$

Let $\odot$ be any binary operation $\oplus, \ominus, \otimes$ between any two fuzzy numbers.

Definition 4.2.5. Let $\tilde{a}$ and $\tilde{b}$ be two fuzzy numbers.

The membership function of $\tilde{a} \odot \tilde{b}$ is defined as

$$\tilde{a} \odot \tilde{b}(z) = \sup_{x:y=z} \min\{\tilde{a}(x), \tilde{b}(y)\} \text{ for } \odot = \oplus, \ominus \text{ or } \otimes \text{ and.}$$

= $+, -, \text{ or } \times$. 
The membership function of the inverse of $\tilde{a}$ is specified as
\[
(\frac{1}{\tilde{a}})(z) = \sup_{z=\frac{1}{x} ; x \neq 0} \min \tilde{a}(x) = \tilde{a}\left(\frac{1}{z}\right).
\]
The quotient of $\tilde{a}$ and $\tilde{b}$ is defined as
\[
\tilde{a} \otimes \tilde{b} = \tilde{a} \otimes \left(\frac{1}{\tilde{b}}\right).
\]

**Definition 4.2.6.** Let $\tilde{a}$ be a fuzzy number.

- $\tilde{a}$ is called non-negative if $\tilde{a}(x) = 0; \forall x < 0$.
- $\tilde{a}$ is called non-positive if $\tilde{a}(x) = 0; \forall x > 0$.
- $\tilde{a}$ is called positive if $\tilde{a}(x) = 0; \forall x \leq 0$
- $\tilde{a}$ is called negative if $\tilde{a}(x) = 0; \forall x \geq 0$.

**Theorem 4.2.3 ([87]).** (i) If $\tilde{a}$ and $\tilde{b}$ are two closed fuzzy numbers then
\[
(\tilde{a} \oplus \tilde{b})_\alpha = [\tilde{a}^L_\alpha + \tilde{b}^L_\alpha, \tilde{a}^U_\alpha + \tilde{b}^U_\alpha]
\]
\[
(\tilde{a} \ominus \tilde{b})_\alpha = [\tilde{a}^L_\alpha - \tilde{b}^U_\alpha, \tilde{a}^U_\alpha - -\tilde{b}^L_\alpha]
\]

(ii) If $\tilde{a}$ and $\tilde{b}$ are two closed fuzzy numbers then
\[
(\tilde{a} \otimes \tilde{b})_\alpha = \{\min\{\tilde{a}^L_\alpha \tilde{b}^L_\alpha, \tilde{a}^L_\alpha \tilde{b}^U_\alpha, \tilde{a}^U_\alpha \tilde{b}^L_\alpha, \tilde{a}^U_\alpha \tilde{b}^U_\alpha\}, \max\{\tilde{a}^L_\alpha \tilde{b}^L_\alpha, \tilde{a}^L_\alpha \tilde{b}^U_\alpha, \tilde{a}^U_\alpha \tilde{b}^L_\alpha, \tilde{a}^U_\alpha \tilde{b}^U_\alpha\}\}
\]
(iii) If \( \tilde{a} \) and \( \tilde{b} \) are two non-negative closed fuzzy number then

\[
(\tilde{a} \otimes \tilde{b})_\alpha = [\tilde{a}_L^L \tilde{b}_L^L, \tilde{a}_U^U \tilde{b}_U^U]
\]

### 4.3 Riemann’s Condition

In this section we present the notion of fuzzy valued functions. Motivated by the earlier works on fuzzy Riemann integration we provide a definition of Riemann–Stieltjes integral of fuzzy valued functions which is followed by the Riemann’s conditions on fuzzy valued functions.

**Definition 4.3.1.** Let \( F \) be the set of all fuzzy numbers, \( F_{cl} \) be a set of all closed fuzzy numbers and \( F_b \) be a set of all bounded fuzzy numbers. Then

\[
f(x) \text{ is a fuzzy valued function if } f : X \to F
\]

\[
f(x) \text{ is a closed fuzzy valued function if } f : X \to F_{cl}
\]

\[
f(x) \text{ is a bounded fuzzy valued function if } f : X \to F_b
\]

We denote \( f^L_\alpha (x) = (f(x))^L_\alpha \) and \( f^U_\alpha (x) = (f(x))^U_\alpha \).

**Definition 4.3.2.** Let \( f(x) \) be closed and bounded fuzzy valued function on \([a, b]\), and \( g(t) \) be a real function defined on \([a, b]\).
Let $\Delta_n$ be a finite partition of $[a, b]$, specified by
\[ a = t_0 < t_1 < \cdots < t_n = b. \]

Let $(t_i')$ be an arbitrary point of the $i^{th}$ sub interval such that
\[ t_{i-1} \leq t_i' \leq t_i, \quad i = 1, 2 \ldots, n. \]

Let $\Delta g(t_i) = g(t_i) - g(t_{i-1})$,
\[
S_n = \sum_{i=1}^{n} f(t'_i)\Delta g(t_i) \text{and } |\Delta_n| = \max_{1\leq i\leq n} \Delta t_i
\]
where $\Delta t_i = t_i - t_{i-1}$. Then the Riemann–Stieltjes integral of $f(t)$ on the interval $[a, b]$ with respect to $g(t)$ is defined as
\[
\lim_{|\Delta_n| \to 0} (s_n)_{\alpha} = \int_{a}^{b} f_{L}^{\alpha}(t)dg(t), \quad \int_{a}^{b} f_{U}^{\alpha}(t)dg(t)
\]
provided this limit exists and it is independent of the partition as well as the selected points $t_i'$. We say $f(t)$ is Riemann–Stieltjes integrable on $[a, b]$ with respect to $g(t)$. The membership function of $\int_{a}^{b} f(x)dx$ is defined for $r \in A_0$ as
\[
\mu_{\int_{a}^{b} f(x)dx}(r) = \sup_{0 \leq \alpha \leq 1} \alpha 1_{A_{\alpha}(r)}
\]
Let $\Delta$ be a partition of $[a, b]$ and let
\[
M_r(f_{\alpha}) = \sup \{ f_{\alpha}^{L}(x) \lor f_{\alpha}^{U}(x); \ x \in [x_{r-1}, x_r] \}
\]
\[ m_r(f_\alpha) = \inf \{ f^L_\alpha(x) \Lambda f^U_\alpha(x) ; x \in [x_{r-1}, x_r] \} \]

The sums

\[
U(\Delta, f_\alpha, g) = \sum_{r=1}^{n} M_r(f_\alpha) \Delta g(t'_r) \\
L(\Delta, f_\alpha, g) = \sum_{r=1}^{n} m_r(f_\alpha) \Delta g(t'_r)
\]

where \( t_{r-1} \leq t'_r \leq t_r \), are called the upper Riemann–Stieltjes sum and the lower Riemann–Stieltjes sum of the fuzzy valued function \( f \) with respect to \( \alpha \in [0, 1] \) for the partition \( \Delta \). Then \( m_r(f_\alpha) \leq M_r(f_\alpha), r = 1, 2, \ldots, n \). If \( g(t) \) is increasing on \( [a, b] \) then \( \Delta g(t_r) \geq 0 \). We can also write \( m_r(f_\alpha) \Delta g(t_r) \leq M_r(f_\alpha) \Delta g(t_r) \). If \( t'_r \in [t_{r-1}, t_r] \) then

\[ m_r(f_\alpha) \leq f(t_r) \leq M_r(f_\alpha) \]

Hence when \( g(t) \) is increasing we have the following inequalities.

\[
L(\Delta, f_\alpha, g) \leq S^L_\alpha(\Delta, f_\alpha, g) \leq S^U_\alpha(\Delta, f_\alpha, g) \leq U(\Delta, f_\alpha, g)
\]

where \( S^L_\alpha = \sum_{r=1}^{n} f^L_\alpha(t'_r) \Delta g(t_i) \)

\[ S^U_\alpha = \sum_{r=1}^{n} f^U_\alpha(t'_i) \Delta g(t_i) \]
We present below the Riemann’s condition of integrability for a closed and bounded fuzzy valued function.

**Definition 4.3.3.** Let $f$ be a closed and bounded fuzzy valued function. $f$ is said to satisfy the Riemann’s condition with respect to $g$ on $[a, b]$ if for each $\varepsilon > 0$ there exists a partition $\Delta_{\varepsilon}$ such that $\Delta$ finer than $\Delta_{\varepsilon}$ implies

$$0 \leq U(\Delta, f_\alpha, g) - L(\Delta, f_\alpha, g) < \varepsilon.$$  

**Theorem 4.3.1.** Assume that $g$ is increasing on $[a, b]$. Then the following statements are equivalent.

(i) The fuzzy valued function $f_\alpha$ is Riemann–Stieltjes integrable on $[a, b]$ with respect to $g$.

(ii) The fuzzy valued function $f_\alpha$ satisfies Riemann’s condition.

(iii) $\sup\{L(\Delta, f_\alpha, g); \Delta \in P[a, b]\} = \inf\{U(\Delta, f_\alpha, g); \Delta \in P[a, b]\}$

where $P[a, b]$ denote the class of all partitions on $[a, b]$.

**Proof.** Assume that the fuzzy valued function $f$ is Riemann–Stieltjes integrable on $[a, b]$ with respect to $g$. If $g(b) = g(a)$ then (ii) holds trivially. Take $g(a) < g(b)$. 

Given $\varepsilon > 0$, choose the partition $\Delta_\varepsilon$ so that for any refinement $\Delta$ and all choices of $t_k$ and $u_k$ in $[x_{k-1}, x_k]$ we have

\[
\left| \sum_{k=1}^{n} f^L_\alpha(t_k) \Delta g(x_k) - \int_{a}^{b} f^u_\alpha(t_k) dg(x_k) \right| \vee \\
\left| \sum_{k=1}^{n} f^U_\alpha(t_k) \Delta g(x_k) - \int_{a}^{b} f^L_\alpha(t_k) dg(x_k) \right| < \frac{\varepsilon}{3}
\]

and

\[
\left| \sum_{k=1}^{n} f^L_\alpha(u_k) \Delta g(x_k) - \int_{a}^{b} f^u_\alpha(u_k) dg(x_k) \right| \vee \\
\left| \sum_{k=1}^{n} f^U_\alpha(u_k) \Delta g(x_k) - \int_{a}^{b} f^L_\alpha(u_k) dg(x_k) \right| < \frac{\varepsilon}{3}
\]

The above inequalities yield

\[
\left| \sum_{k=1}^{n} [f^L_\alpha(t_k) - f^U_\alpha(u_k)] \Delta g(x_k) \right| \vee \\
\left| \sum_{k=1}^{n} [f^U_\alpha(t_k) - f^L_\alpha(u_k)] \Delta g(x_k) \right| < \frac{2}{3} \varepsilon
\]

since $M_r(f_\alpha) - m_r(f_\alpha) = \sup\{(f^L_\alpha(x) - f^U_\alpha(1)) \vee (f^U_\alpha(x) - f^L_\alpha(1))\};

\]

Then for each $h > 0$ we can choose $t_k$ and $u_k$ such that

\[
(f^L_\alpha(t_k) - f^U_\alpha(u_k)) \vee (f^U_\alpha(t_k) - f^U_\alpha(u_k)) \geq M_r(f_\alpha) - m_r(f_\alpha) - h.
\]

we take $h = \varepsilon/3[g(b) - g(a)]$. 

Then

\[ U(\Delta, f_\alpha, g) - L(\Delta, f_\alpha, g) \]

\[ = \sum_{k=1}^{n} [M_k(f^L_\alpha) - m_k(f^U_\alpha)] \Delta g(x_k) \]

\[ \vee \sum_{k=1}^{n} (M_k(f^U_\alpha) - m_k(f^L_\alpha)) \Delta g(t_k) \]

\[ < \sum_{k=1}^{n} [f^L_\alpha(t_k) - f^U_\alpha(u_k)] \Delta g(x_k) \]

\[ \vee \sum_{k=1}^{n} f^U_\alpha(t_k) - f^L_\alpha(u_k) \Delta g(t_k) + h \sum_{k=1}^{n} \Delta g(t_k) \]

\[ < \varepsilon \]

This shows that \( f_\alpha \) satisfies the Riemann’s condition, and so (i) implies (ii).

Assume that \( f \) satisfies Riemann’s condition. If \( \varepsilon > 0 \) is given, there exists a partition \( \Delta_\varepsilon \) such that \( \Delta \) finer than \( \Delta_\varepsilon \) implies

\[ U(\Delta, f_\alpha, g) < L(\Delta, f_\alpha, g) + \varepsilon \]

Hence for such \( \Delta \) we have

\[ \sup\{L(\Delta, f_\alpha, g); \Delta \in P[a, b]\} \leq U(\Delta, f_\alpha, g) \]

\[ < L(\Delta, f_\alpha, g) + \varepsilon \]

\[ \leq \inf\{U(\Delta, f_\alpha, g); \Delta \in P[a, b]\} + \varepsilon \]
i.e., \( \sup \{ L(\Delta, f_\alpha, g); \Delta \in P[a,b] \} \leq \inf \{ U(\Delta, f_\alpha, g); \Delta \in P[a,b] \} + \varepsilon \),

for each \( \varepsilon > 0 \).

Hence \( \sup \{ L(\Delta, f_\alpha, g); \Delta \in P[a,b] \} \leq \inf \{ U(\Delta, f_\alpha, g); \Delta \in P[a,b] \} \)

But \( \inf \{ U(\Delta, f_\alpha, g); \Delta \in P[a,b] \} \leq \sup \{ L(\Delta, f_\alpha, g); \Delta \in P[a,b] \} \).

for the increasing function \( g \) on \([a,b] \).

\[
\sup \{ L(\Delta, f_\alpha, g); \Delta \in P[a,b] \} = \inf \{ U(\Delta, f_\alpha, g); \Delta \in P[a,b] \}
\]

(ii) implies (iii).

Finally assume that (iii) is valid.

Let \( A_\alpha \) denote their common value. We will prove that \( \left( \int_a^b f \, dg \right)_\alpha \)
exists and equals \( A_\alpha \).

Given \( \varepsilon > 0 \), choose \( \Delta'_\varepsilon \) so that

\[
U(p, f_\alpha, g) < \sup \{ L(\Delta, f_\alpha, g); \Delta \in p[a,b] \} + \varepsilon
\]

for all \( \Delta \) finer than \( \Delta''_\varepsilon \). Also choose \( \Delta''_\varepsilon \) such that

\[
L(p, f_\alpha, g) > \inf \{ U(\Delta, f_\alpha, g); \Delta \in P[a,b] \} - \varepsilon
\]
for all $\Delta$ finer than $\Delta''_\varepsilon$. If

$$\Delta_\varepsilon = \Delta'_\varepsilon \cup \Delta''_\varepsilon.$$ 

We can write

$$\inf\{U(\Delta, f_\alpha, g); \Delta \in P[a, b]\} - \varepsilon < L(\Delta, f_\alpha, g) \leq s(\Delta, f_\alpha, g) \leq U(\Delta, f_\alpha, g) < \sup\{L(\Delta, f_\alpha, g); \Delta \in P[a, b]\} + \varepsilon$$

for each $\Delta$ finer than $\Delta_\varepsilon$.

But since

$$\sup\{L(\Delta, f_\alpha, g); \Delta \in P[a, b]\} = \inf\{U(\Delta, f_\alpha, g); \Delta \in P[a, b]\} = A_\alpha$$

This shows that

$$|S(\Delta, f_\alpha, g) - A_\alpha| < \varepsilon$$

whenever $\Delta$ is finer than $\Delta_\varepsilon$. This proves that $\left(\int_a^b f\,dg\right)_\alpha$ exists and equals $A_\alpha$.

Thus we have established (iii) implies (i) which completes the proof.
Theorem 4.3.2. Let $f$ be a closed and bounded fuzzy valued function, and $f$ be Riemann–Stieltjes integrable on $[a, b]$ with respect to $g$. Let $m_\alpha = \inf (f^L_\alpha(x) \land f^U_\alpha(x)); x \in [a, b]$ and $M_\alpha = \sup (f^L_\alpha(x) \lor f^U_\alpha(x)); x \in [a, b]$. Let $\emptyset : [m_\alpha, M_\alpha] \to \mathbb{R}$ be continuous. Then the composition function $h = \emptyset \cdot f$ is Riemann–Stieltjes integrable on $[a, b]$ with respect to $g$.

Proof. Let $\varepsilon > 0$. since $\emptyset$ is continuous on the closed and bounded interval $[m_\alpha, M_\alpha]$ it is uniformly continuous on $[m_\alpha, M_\alpha]$. Then there exists $\delta_1 > 0$ such that

$$x, y \in [m_\alpha, M_\alpha] \quad \text{and} \quad |x - y| < \delta_1 \Rightarrow |O(x) - O(y)| < \varepsilon \tag{4.3.1}$$

Let $\delta = \min(\delta_1, \varepsilon)$. Since $f$ is Riemann–Stieltjes integrable relative to $g$ on $[a, b]$ corresponding to $\delta^2$, there exists a partition $\Delta$ of $[a, b]$ such that

$$U(\Delta, f_\alpha, g) - L(\Delta, f_\alpha, g) < \delta^2 \tag{4.3.2}$$

Let $(m_r)_\alpha = \inf (f^L_\alpha(x) \land f^U_\alpha(x)); x \in [t_{r-1}, t_r]$ and $(M_r)_\alpha = \sup (f^L_\alpha(x) \lor f^U_\alpha(x)); x \in [t_{r-1}, t_r]$

We divide the numbers 1, 2, . . . , $n$ into two classes $A$ and $B$
defined as

\[ A = \{ i; \left( (M_i)_\alpha^L - (m_i)_\alpha^U \right) \vee \left( (M_i)_\alpha^U - (m_i)_\alpha^L \right) < \delta \} \]

\[ B = \{ i; \left( (M_i)_\alpha^L - (m_i)_\alpha^U \right) \vee \left( (M_i)_\alpha^U - (m_i)_\alpha^L \right) \geq \delta \} \]

Let \((m^*_r)_\alpha = \inf (h^L_\alpha(x) \land h^U_\alpha(x)) \) for \(x \in [t_{r-1}, t_r] \)

and \((M^*_r)_\alpha = \sup (h^L_\alpha(x) \lor h^U_\alpha(x)) \); \(x \in [t_{r-1}, t_r] \).

For \(r \in A\), and \(t_{r-1} \leq x \leq y \leq t_r\), We have

\[
|f^L_\alpha(x) - f^U_\alpha(y)| \lor |f^U_\alpha(x) - f^L_\alpha(y)|
\leq \left( (M^*_r)_\alpha^L - (m^*_r)_\alpha^U \right) \lor \left( (M^*_r)_\alpha^U - (m^*_r)_\alpha^L \right)
< \delta
\leq \delta_1
\]

This shows that

\[
|O(f^L_\alpha(x)) - O(f^U_\alpha(y))| \lor |O(f^U_\alpha(x)) - O(f^L_\alpha(y))| < \epsilon
\]

This shows that

\[
|h^L_\alpha(x) - h^U_\alpha(y)| \lor |h^U_\alpha(x) - h^L_\alpha(y)| < \epsilon
\]

\[
| (M^*_r)_\alpha^U - (m^*_r)_\alpha^L | \lor | (M^*_r)_\alpha^L - (m^*_r)_\alpha^U | < \epsilon
\]

\[
\sum_{r \in A} \left( (M^*_r)_\alpha^U - (m^*_r)_\alpha^L \right) \lor \left( (M^*_r)_\alpha^L - (m^*_r)_\alpha^U \right) \Delta g(t_r)
\]
\[ \leq \epsilon \sum_{r \in A} \Delta g(t_r) \]
\[ \leq \epsilon \sum_{r=1}^{n} \Delta g(t_r) \]
\[ \Rightarrow [g(b) - g(a)] \quad (4.3.4) \]

For \( r \in B \)
\[ \delta \sum_{r \in B} \Delta g(t_r) \leq \sum_{r \in B} \left( (M_r^*)_U^\alpha - (m_r^*)_L^\alpha \right) \lor \left( (M_r^*)_L^\alpha - (m_r^*)_U^\alpha \right) \Delta g(t_r) \leq \delta \sum_{r=1}^{n} \left( (M_r^*)_U^\alpha - (m_r^*)_L^\alpha \right) \lor \left( (M_r^*)_L^\alpha - (m_r^*)_U^\alpha \right) \Delta g(t_r) \]
\[ = U(\Delta, f_\alpha, g) - L(\Delta, f_\alpha, g) \]
\[ < \delta^2. \]

This shows that
\[ \delta \sum_{r \in B} \Delta g(t_r) < \delta^2 \]
\[ \Rightarrow \sum_{r \in B} \Delta g(t_r) < \delta \leq \epsilon \quad (4.3.5) \]

Take, \( K = \sup\{|\psi(x)|; m_\alpha \leq x \leq M_\alpha\} \). Then
\[ U(\Delta, h_\alpha, g) - L(\Delta, h_\alpha, g) \]
\[ = \sum_{r \in A} \left( (M_r^*)_U^\alpha - (m_r^*)_L^\alpha \right) \lor \left( (M_r^*)_L^\alpha - (m_r^*)_U^\alpha \right) \Delta g(t_r) \]
\begin{align*}
&+ \sum_{r \in B} \left((M_r)_\alpha^U - (m_r)_\alpha^L \right) \lor \left((M_r)_\alpha^L - (m_r)_\alpha^U \right) \Delta g(t_r) \\
&\leq \varepsilon [g(b) - g(a)] + \sum_{r \in B} 2K \Delta g(t_r) \\
&< \varepsilon [g(b) - g(a)] + 2K \varepsilon \\
&= \varepsilon [g(b) - g(a) + 2K]
\end{align*}

Since \( \varepsilon \) is arbitrary and \( [g(b) - g(a) + 2K] \) is a constant, \( h_\alpha = \Phi \cdot f_\alpha \) is Riemann–Stieltjes integrable on \([a, b]\) and with respect to \( g(t) \). \( \blacksquare \)