Chapter 2

Decompositions of Signed Fuzzy Number Valued Measures and Singular Complements

Abstract
In this Chapter, invoking the notion of signed fuzzy number valued measures on the fuzzy set, we attempt to study the Lebesgue decompositions of a signed fuzzy number valued measures. We introduce in this Chapter a new notion called singular complements to the domain of fuzzy number valued measures and derive some consequential properties.

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2.1 Introduction

The concept of fuzzy valued measures is an efficient tool in mathematical economics, pattern recognition and optimal control. In this line signed fuzzy number valued measures or symbol measures [61] has received much attention in the recent times owing to its importance in several applied areas. To advance this notion Ban [4] has proved Radon-Nikodym theorem on fuzzy valued measures. Stojakovic [84] proved Radon-Nikodym theorems on fuzzy valued measures in a Banach space setting. Xue Xiaoping et. al., [103] have furnished new notions of fuzzy number valued measures in Banach spaces. Jianrong Wu et.al., [39] have established the Radon-Nikodym theorem and the Vitali-Hahn-Saks theorem on fuzzy number measures defined in [103]. In [23] Fen-Xia Zhao et.al., have introduced the notions of signed fuzzy number valued measures and established Hahn and Jordan decompositions of signed measures to the domain of fuzzy number valued measures. In the theoretical setting found in [23], Fen Xia Zhao et.al., in [24] have realized Lebesgue decomposition of signed fuzzy number valued measures and established Radon-Nikodym theorems for fuzzy number valued integrals.
In this chapter, invoking the notion of signed fuzzy number valued measures on the fuzzy set we attempt to study the Lebesgue de-compositions of a totally $\sigma$-finite signed fuzzy number valued measures. We introduce a new notion known as singular complements to the realm of fuzzy number valued measures and derive their properties.

In Section 2.2 we recall some basic terms and results used in the sequel. In Section 2.3, our object is to prove the Hahn and Lebesgue decompositions to the case of fuzzy number valued measure. Section 2.4 is devoted to the notion of singular complements and its properties.

Throughout this chapter let $X$ be a non-empty set, $F(X)$ denote the set of all fuzzy sets on $X$, $P(X)$ the class of the fuzzy sets taking values 0 and 1 only. $A$ be the $\sigma$-algebra on $F(X)$, and $m$ denotes the signed fuzzy number valued measure. We make the convention $O \cdot \infty = \tilde{O}$. 
2.2 Preliminaries

**Definition 2.2.1.** A non-empty class $\mathcal{A}$ of fuzzy subsets of $X$ is a fuzzy algebra if

(i) $\alpha \in [0, \infty), A \in \mathcal{A} \Rightarrow \alpha \odot A = \alpha A \land 1 \in \mathcal{A}$

(ii) $A \in \mathcal{A} \Rightarrow A^C = 1 - A \in \mathcal{A}$

(iii) $A, B \in \mathcal{A} \Rightarrow A \oplus B = (A + B) \land 1 \in \mathcal{A}$

**Definition 2.2.2.** A non-empty class $\mathcal{A}$ of fuzzy subsets of $X$ is a fuzzy $\sigma$-algebra, if

(i) $\alpha \in [0, \infty), A \in \mathcal{A} \Rightarrow \alpha A = \alpha A \land 1 \in \mathcal{A}$

(ii) $A \in \mathcal{A} \Rightarrow A^C = 1 - A \in \mathcal{A}$

(iii) $A, B \in \mathcal{A} \Rightarrow A \oplus B = (A + B) \land 1 \in \mathcal{A}$

(iv) $A_n \in \mathcal{A}, n = 1, 2, \ldots \Rightarrow \bigoplus_{n=1}^{\infty} A_n = \left(\sum_{n=1}^{\infty} A_n\right) \land 1 \in \mathcal{A}$

**Definition 2.2.3.** Let $\mathcal{A}$ be a fuzzy algebra we say that the mapping $\mu : A \rightarrow [0, \infty)$ a fuzzy measure on $A$ if

(i) $A_n \in \mathcal{A}, n = 1, 2, \ldots$ and $\sum_{n=1}^{\infty} A_n \leq 1$

(ii) $\bigoplus_{n=1}^{\infty} A_n \in \mathcal{A} \Rightarrow \mu\left(\bigoplus_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$
If $A$ is a fuzzy $\sigma$-algebra on $X$, then $(X, A)$ will be known as a fuzzy measurable space. If $\mu$ is a fuzzy measure on $A$, then $(X, A, \mu)$ is known as the fuzzy measure space.

Let $A$ and $B$ be two fuzzy sets. For each $x \in X$ we define the sum, difference, conjunction, union, intersection etc., as follows.

$$(A \oplus B)(x) = \min(1, A(x) + B(x))$$

$$(A\&B)(x) = \max(0, A(x) + B(x) - 1)$$

$$(A \cap B)(x) = \min(A(x), B(x))$$

$$(A \cup B)(x) = \max(A(x), B(x))$$

$$(A \Theta B)(x) = \max(0, A(x) - B(x))$$

$$(A \cdot B)(x) = A(x) \cdot B(x)$$

If $A, B$ are fuzzy sets we employ the notation $A \geq B$ to denote $B \subseteq A$. 

2.3 Decomposition of Signed Fuzzy Number Valued Measures

If a fuzzy measure is allowed to take both positive and negative fuzzy number valued measures we obtain a signed fuzzy number valued measure. In this section we prove two results viz., Hahn decomposition theorem and Lebesgue decomposition theorem for the case of signed fuzzy numbers valued measures. We now provide certain definitions which are useful in the sequel.

**Definition 2.3.1** ([24]). Let $R$ be the set of all real numbers and $F(R)$ be all fuzzy subsets defined on $R$. Let $\tilde{a} \in F(R)$, $\tilde{a}$ is called a fuzzy number if

(i) $\tilde{a}$ is normal. i.e., $\exists x \in R$, such that $\tilde{a}(x) = 1$

(ii) whenever $\lambda \in (0, 1]$ then $a_\lambda = \{x; \tilde{a}(x) \geq \lambda\}$ is a closed interval, denoted as $[a^-_\lambda, a^+_\lambda]$.

If for every positive real number $M$, there exists a $\lambda_0 \in (0, 1]$ such that $M < a^+_\lambda$ or $a^-_{\lambda_0} < -M$ then $\tilde{a}$ is called fuzzy infinite written as $\tilde{\infty}$. If $\tilde{a}$ and $\tilde{b}$ are two fuzzy numbers then for $\epsilon > 0$, if $\tilde{a} - \tilde{b} \leq \epsilon$ then $\tilde{a} \leq \tilde{b} + \epsilon$. Let $F^*(R)$ denote the set of all fuzzy
Definition 2.3.2 ([24, 61]). Let $\mathcal{A}$ be a fuzzy algebra. Let $m$ be a mapping from $\mathcal{A}$ to $F^*(R)$. We say $m$ is a signed fuzzy number valued measure if

(i) $m(\emptyset) = \tilde{0}$

(ii) if $\{A_n\}_{n \in N}$ is a disjoint sequence in $A$ then
$$m( \bigoplus_{n \in N} A_n) = \sum_{n \in N} m(A_n)$$

(iii) $\sup\left\{\sum_{i=1}^{m} |m(A_i)|; \ A_i \in \mathcal{A}, \ i = 1, 2 \ldots m, \ \sum_{i=1}^{m} A_i \leq 1\forall m\right\} < \infty$

If for each $A \in \mathcal{A}$ we have $m(A) \notin \infty$ then $m$ is said to be finite.

Definition 2.3.3. A set $E \in \mathcal{A}$ is said to be positive, in the case of a signed fuzzy number valued measure $m : \mathcal{A} \rightarrow F^*(R)$ defined on the measurable space $(X, \mathcal{A})$ if for each $A \subset E$ such that $A$ is measurable implies $m(A) \geq \tilde{0}$.

Definition 2.3.4. A set $E \in \mathcal{A}$ is said to be negative set, in the case of a signed fuzzy number valued measure $m : \mathcal{A} \rightarrow F^*(R)$ defined on a measurable space $(X, \mathcal{A})$ if for each $A \subset E$ such
that $A$ is measurable implies $m(A) \leq 0$.

A set $E \in \mathcal{A}$ is said to be a null set in the case of signed fuzzy number valued measure $m$ if $E$ is both positive and negative. Consequently a measurable set $E$ is a set of measure zero iff every measurable subset of $E$ has signed fuzzy number valued measure zero. The measure of every null set is zero. A set of signed fuzzy number valued measure zero may be viewed as a union of two measurable sets whose measures are not zero but are negative of each other.

**Definition 2.3.5** ([24]). Let $E \in \mathcal{A}$. We say $E$ has the property (S) if whenever $m(E) \leq 0$ then there exists $F \in \mathcal{A}$ such that $E \geq F$ and $m(F) > 0$.

$\mathcal{A}$ is said to possess the property (S) if every fuzzy set in $\mathcal{A}$ possess the property (S).

**Theorem 2.3.1** ([24]). Let $m$ be finite. $A$ has the property(S) with respect to $m$, then for each $E \in \mathcal{A}$, $m^+(E)$ and $m^-(E)$ can be expressed as

$$m^+(E) = \sup\{m(F), \ E \geq F, \ F \in \mathcal{A}\}$$

$$m^-(E) = \sup\{-m(F), \ E \geq F, \ F \in \mathcal{A}\}.$$
In what follows we provide Hahn decomposition theorem.

**Theorem 2.3.2.** If \( m \) is a signed fuzzy number valued measure on a measurable space \((X, A)\), then there exists a positive set \( P \) and a negative set \( Q \) such that \( P \cap Q = \emptyset \) and \( X = P \cup Q \).

**Proof.** \( A \) is given to be a fuzzy \( \sigma \)-algebra of subsets of \( X \) and \( m \) a signed fuzzy number valued measure. Then for any \( A \in A \), we have

\[
m^+(A) < \tilde{\infty} \quad \text{and} \quad m^-(A) < \tilde{\infty}.
\]

i.e., \( m \) does not take \(-\infty\).

Consider the family \( B \) of all negative subsets of \( X \).

Let \( \tilde{\lambda} = \sup\{-m(E); \ B \geq E \in A\}, B \in A \) \hspace{1cm} (2.3.1)

(2.3.1) implies for any integer \( n \) and \( \epsilon_n > 0 \) there exists \( E_n \in A \) such that

\[
B_n \geq E_n \quad \text{and} \quad \tilde{\lambda} - \epsilon_n \leq m(E_n).
\]

This implies that for any integer \( m \)

\[
\tilde{\lambda} - \sum_{n=1}^{m} \epsilon_n \leq \sum_{n=1}^{m} m(E_n) = \sum_{n=1}^{m} m(E_n).
\]
$$= m \left( \sum_{n=1}^{m} (E_n) \right)$$

$$\Rightarrow m \left( \sum_{n=1}^{m} (E_n) \right) \geq \tilde{\lambda} - \sum_{n=1}^{m} \epsilon_n$$

Letting \( m \to \infty \) and \( \epsilon_n \to 0 \) we have

$$m \left( \sum_{n=1}^{\infty} (E_n) \right) \geq \tilde{\lambda} \quad (2.3.2)$$

If we take \( Q = \bigcup_{n=1}^{\infty} E_1 \), then \( Q \) is a negative subset of \( X \).

Then \( (2.3.2) \Rightarrow m(Q) \geq \tilde{\lambda} \). We consider the subset \( Q - E_n \) of \( Q \).

$$Q = (Q - E_n) \cup E_n$$

$$m(Q) = m(Q - E_n) + m(E_n) \leq m(E_n)$$

$$\Rightarrow m(Q) \leq m(E_n) \quad \forall n \in N$$

$$E_n \in B \quad \forall n \in N$$

In view of \( (2.3.1) \) \( m(Q) \leq \tilde{\lambda} \)

Thus \( m(Q) = \tilde{\lambda} \)

$$\Rightarrow \tilde{\lambda} > -\infty$$

It remains to be proved that \( P = X - Q \) is a positive subset of \( X \). We suppose that \( P \) is not positive, and so \( P \) is negative.

For each measurable set \( E \subset P, m(E) < 0 \).
Since $E$ is a measurable set with finite negative measure.

i.e. $-\infty < m(E) < 0$, then there exists a set $A$ such that $E \geq A$ with $m(A) < 0$. Since $A$ and $Q$ are negative subsets of $X$, $A \cup B$ is negative.

\[ \therefore m(A \cup Q) \geq \tilde{\lambda}. \]

\[ \Rightarrow \tilde{\lambda} \leq m(A \cup Q) \]

\[ = m(A) + m(Q) \]

\[ = m(A) + \tilde{\lambda} \]

\[ \Rightarrow m(A) \geq 0, \]

a contradiction.

Hence our assumption that $P$ is negative is wrong.

$P = X - Q$ is positive and $Q$ is negative which concludes the proof.

\[ \blacksquare \]

**Definition 2.3.6.** Let $m$ and $n$ be two finite signed fuzzy number valued measures and $A$ the $\sigma$- algebra of fuzzy sub sets of $X$ possess the property (s) w.r.t. $m$ and $n$. We say that the measures $m$ and $n$ are singular if there exists a measurable set $A \subset X$ such that $m(A) = \bar{\delta} = m(X - A)$ symbolically we denote it as $m \perp n$.  

Definition 2.3.7 ([24]). Let $m$ and $n$ be two signed fuzzy number valued measures, with $m$ and $n$ are finite. We say $n$ is absolutely continuous with respect to $m$ if $n(A) = \tilde{o}$ for each $A \in \mathcal{A}$ for which $m^*(A) = \tilde{o}$ where $m^*(A) = m^+(A) + m^-(A)$. We denote it as $n \ll m$.

We now state Radon Nikodym theorem for fuzzy number valued integrals established in [24]. For a detailed proof we refer [24].

Theorem 2.3.3 ([24]). If $m_1$ and $m_2$ are finite fuzzy number valued measures, $m_2(A) \subseteq A^*$, where $A^*$ is a non empty class of subsets of $F(R)$, $A$ has the property (s) w.r.t. $m_1$ and $m_2$ then the following conditions are equivalent.

(i) $m_1 \ll m_2$

(ii) there exists non-negative measurable function $f$ such that

$$m_1(A) = \int_A f dm_2; \ A \in \mathcal{A}.$$

we now prove the Lebesgue decomposition theorem.

Theorem 2.3.4. Let $(X, A, \mu)$ be a $\sigma$-finite fuzzy measure space and $m$ a $\sigma$ finite valued measure defined on $A$. then there exists
two uniquely determined measures $m_0$ and $m_1$ such that $m = m_0 + m_1$, $m_0 \perp \mu$ and $m_1 \ll \mu$.

**Proof.** Let $\lambda = \mu + m$

$\mu$ and $\nu$ are $\sigma$-finite, $\lambda$ is $\sigma$-finite.

$\mu \ll \lambda$ and $m \ll \lambda$.

By Radon Nikodym theorem [24], there exists non negative functions $f, g : X \to [-\infty, \infty]$ such that

$$\mu(E) = \int_E f \, d\lambda$$

$$m(E) = \int_E g \, d\lambda; \quad \forall E \in A$$

let $A = \{x \in X; f(x) > 0\}$

$$B = \{x \in X; f(x) = 0\}$$

Then $X = A \cup B, A \cap B = \emptyset$. More over

$$\mu(B) = \int_{P_\nu} f \, d\lambda = 0$$

we define two functions:

$m_0, m_1 : A \to [-\infty, \infty]$ so that

$$m_0(E) = m(E \cap B)$$

$$m_1(E) = m(E \cup A), \quad \forall E \in A$
Then $m_0$ and $m_1$ are measures on $A$, Satisfying the condition

$$m = m_0 + m_1$$

$$m_0(A) = m(A \cap B) = m(\emptyset) = \bar{o}$$

$$\mu(B) = \bar{o} = m_0(A) = m_0(X - B)$$

$$\mu(B) = m_0(X - B)$$

i.e., $m_0$ is mutually singular to $\mu$.

\[ \therefore m_0 \perp \mu. \]

we now prove that $m_1 \ll \mu$.

For this let $E \in \mathcal{A}$ be arbitrary such that $\mu(E) = 0$.

Then $\int_E f d\lambda = \mu(E) = 0$.

or $\int_E f d\lambda = 0$. Also $f(x) \geq 0$ for $x \in E$.

This shows that $f = 0$ a.e on $E$ relative to $\lambda$ since $f > 0$ on $A \cap E$ we have

$$m_1(E) = m(E \cap A) \text{ (by definition of } m_1) \leq \lambda(E \cap A) = \bar{o}.$$

$\Rightarrow m_1(E) = \bar{o}.$

But $m_1(E) \geq 0$.

$\Rightarrow m_1(E) = 0.$

Thus $\mu(E) = 0 \Rightarrow m_1(E) = 0.$
⇒ $m_1 \ll \mu$.

To prove the uniqueness of $m_0$ and $m_1$, we assume that $m = m'_0 + m'_1$ possessing the same property of $m_0$ and $m_1$. Then $m = m_0 + m_1$ and $m = m'_0 + m'_1$ are the two Lebesgue decompositions of $V$.

⇒ $m_0 + m_1 = m'_0 + m'_1$.

$m_0 - m'_0 = m'_1 - m_1$.

Moreover, $m'_1 - m_1$ is absolutely continuous and $m_0 - m'_0$ is singular to $m$. We have $m_0 - m'_0 = 0$.

⇒ $m_0 = m'_0$ and $m_1 = m'_1$ which completes the proof. ■

2.4 Singular Complements

In this section we introduce a new notion known as singular complements and derive its properties.

**Definition 2.4.1.** Let $U$ be the set of all finite signed fuzzy number valued measures. Let $S$ be a subset $U$. The singular complement of $S$ denoted by $S^{\perp}$ is the set of all finite signed fuzzy number valued measures in $U$ which are singular to each
measure of $S$.

$$S^\perp = \{m; m \in U; m \perp n. \ \forall n \in S\}$$

**Theorem 2.4.1.** Let $U$ be the set of all finite signed fuzzy number valued measures. $A$ has the property $(S)$ w.r.t. the elements of $U$. If $S_1$ and $S_2$ are subsets of $U$ then

$$S_1 \subseteq S_2 \Rightarrow S_2^\perp \subseteq S_1^\perp$$

**Proof.** Let $m \in S_2^\perp$.

$$\Rightarrow m \perp n \forall n \in S_2.$$  

$S_1 \subseteq S_2$ implies $m \perp n \forall n \in S_1$.

$$\Rightarrow m \in S_1^\perp.$$  

$S_2^\perp \subseteq S_1^\perp$ which completes the proof.  

**Theorem 2.4.2.** Let $U$ be the set of all finite signed fuzzy number valued measures. $A$ has the property $(s)$. If $S_1$ and $S_2$ are subsets of $U$ then

$$(i) \ (S_1 + S_2)^\perp = S_1^\perp \cap S_2^\perp.$$  

$$(ii) \ (S_1 \cap S_2)^\perp = S_1^\perp \oplus S_2^\perp.$$  

**Proof.** $S_1 \subseteq S_1 + S_2$.  


Then by the above theorem

\[(S_1 + S_2)^\perp \subseteq S_1^\perp\]
\[S_2 \subseteq S_1 + S_2\]
\[(S_1 + S_2)^\perp \subseteq S_2^\perp\]
\[(S_1 + S_2)^\perp \subseteq S_1^\perp \cap S_2^\perp\]

We now let \(m \in S_1^\perp \cap S_2^\perp\)

\[\Rightarrow \quad m \in S_1^\perp \text{ and } m \in S_2^\perp\]
\[\Rightarrow \quad m \perp n \forall n \in S_1 \text{ and } \forall n \in S_2\]
\[\Rightarrow \quad \exists \tilde{E} \in A \text{ such that } m^*(\tilde{E}) = n^*(\tilde{E}^c) = 0\]
\[\Rightarrow \quad m^+(\tilde{E}) + m^-(\tilde{E}) = n^+(\tilde{E}^c) + n^-(\tilde{E}^c)\]

Let \(n \in S_1 + S_2\).

Then \(n = n_1 + n_2\) where \(n_1 \in S_1\) and \(n_2 \in S_2\) such that
\[\exists \tilde{E}_1 \in A \text{ such that } m^+\tilde{E}_1 + m^-\tilde{E}_1 = n^+(\tilde{E}_1^c) + n^-\tilde{E}_1^c\]
\[\exists \tilde{E}_2 \in A \text{ with } \tilde{E}_1 \cap \tilde{E}_2 = \emptyset \text{ and } E_1 \cup E_2 = E. \text{ We have }\]
\[m^+\tilde{E}_2 + m^-\tilde{E}_2 = n^+(\tilde{E}_2^c) + n^-\tilde{E}_2^c\]
\[n^+(\tilde{E}^c) + n^-\tilde{E}^c = (n_1 + n_2)^+(\tilde{E}^c) + (n_1 + n_2)^-(\tilde{E}^c)\]
\[= n_1^+(\tilde{E}^c) + n_2^+(\tilde{E}^c) + n_1^-\tilde{E}^c + n_2^-\tilde{E}^c\]
\[ m^+ (\tilde{E}_1) + m^- (\tilde{E}_1) + m^+ (\tilde{E}_2) + m^- (\tilde{E}_2) \]
\[ = m^+ (\tilde{E}_1 \cup \tilde{E}_2) + m^- (\tilde{E}_1 \cup \tilde{E}_2) \]
\[ = m^+ (\tilde{E}) + m^- (\tilde{E}) \]
\[ = m^* (\tilde{E}) \]
\[ = \tilde{o} \]

\[ \Rightarrow m \in (S_1 + S_2)^\perp \Rightarrow S_1^\perp \cap S_2^\perp \subseteq (S_1 + S_2)^\perp. \]

which completes the proof.

Proof of (ii) is similar. \[ \blacksquare \]