CHAPTER – 4

A Class of Selfinvertible Densities:

4.1. Introduction:

In this Chapter we propose a class of densities on $(0, \infty)$ such that the inverse of the density function is same as the density function itself. Some of the properties of proposed class of densities are studied and illustrative examples are given. We obtain maximum likelihood estimator and moment estimator of a parameter $\alpha$, for a family of density given in (4.3.5). Based on simulation studies, performances of the proposed estimators have been studied.

Many well-known classes of densities exhibit properties like symmetry, unimodality, decreasing on the support $(0, \infty)$. These properties can be used to propose models for a given data set. If $f$ is a monotone decreasing continuous density function with support $(0, \infty)$ then $g(u) = f^{-1}(u)$, $0 < u < f(0)$ is also a monotone decreasing continuous density function with the support $(0, f(0))$. A sub class of decreasing densities on $(0, \infty)$ can be obtained by confining to those satisfying the condition $f(.) = g(.)$, that is $f = f^{-1}$, we refer such a class as the class of Selfinvertible densities. Based on a typical decreasing, non-negative function, new classes of parametric families of Selfinvertible densities are generated.
We generate a class of densities \( \{ f(x, \alpha): 0 < \alpha < 1 \} \) on \((0, \infty)\), that depends on a parameter \( \alpha \) and a decreasing function \( h \) defined on \((0, \infty)\), satisfying certain conditions such that the inverse of the density function is same as the density function itself.

### 4.2. Basic Concepts:

Let \( f \) be a continuous monotone decreasing p. d. f. on \((0, \infty)\) then

\[
g(u) = \begin{cases} 
  f^{-1}(u), & 0 < u \leq f(0) \\
  0, & \text{otherwise.}
\end{cases}
\]

is a density function on \((0, f(0))\).

**Definition:** A continuous monotone decreasing p. d. f. on \((0, \infty)\) is said to be Selfinvertible if

\[
f(x) = f^{-1}(x), \quad x > 0.
\]

The class of all Selfinvertible densities be denoted by \( SI \). If \( f \in SI \) then

i) \( f(f(x)) = f(f^{-1}(x)) = x \).

ii) \( f(1) < 1 \).

Hence there exists a unique solution to the equation \( f(x) = x \), say \( \alpha \), \( 0 < \alpha < 1 \).

Thus the form of Selfinvertible density function \( f \) is

\[
f(x) = \begin{cases} 
  g^{-1}(x), & 0 < x < \alpha \\
  g(x), & x \geq \alpha
\end{cases}
\]  

\((4.2.1)\)
where

\[ i) \quad g(\alpha) = \alpha \quad \text{and} \quad ii) \quad \int_\alpha^\infty g(x) \, dx = \frac{1 - \alpha^2}{2}. \quad (4.2.2) \]

Remark: Second conditions in the definition and equation (4.2.2) are required for \( f \) to be a p.d.f..

To define the density \( f \in SI \) we need to identify \( \alpha \) and a non-negative function \( h \) on \((0, \infty)\), such that

\[
\begin{align*}
\text{i)} & \quad h(.) \text{ is decreasing} \\
\text{ii)} & \quad h(0) = \alpha \\
\text{iii)} & \quad \int_0^\infty h(x) \, dx = \frac{1 - \alpha^2}{2}.
\end{align*}
\]

\quad (4.2.3)

The figure(4.2.1) is graph of a typical Selfinvertible density with parameter \( \alpha \).

\[ \text{Fig.4.2.1: Graph of a typical Self invertible density with parameter } \alpha. \]
Figure (4.2.2) shows the $h(.)$ function.

**Fig. 4.2.2: Graph of $h(.)$ function**

Thus for $0 < \alpha < 1$ and $h$ function satisfying the conditions (4.2.3), the general form of Selfinvertible p.d.f. $f$ is

$$f(x, \alpha) = \begin{cases} 
\alpha + h^{-1}(x), & 0 < x < \alpha \\
h(x-\alpha), & x \geq \alpha.
\end{cases} \quad (4.2.4)$$

The family $\{f(x, \alpha): 0 < \alpha < 1, h \text{ satisfies (4.2.3)}\}$ of SI densities forms a semi-parametric family of densities. In the following we consider typical $h$ functions and obtain parametric families of selfinvertible densities. For a decreasing function $g$, we define the inverse as $g^{-1}(u) = \sup\{t: g(t) > u\}$. For example if $\alpha = 1/\sqrt{2}$ then the function

$$h(x) = (1/\sqrt{2}) - x, \quad 0 < x < 1/\sqrt{2}$$

satisfies the conditions (4.2.3) and the corresponding Self invertible density is

$$f(x) = \sqrt{2} - x, \quad 0 \leq x \leq \sqrt{2}. \quad (4.2.5)$$
Note that as $\alpha$ increases to 1, every Self invertible density with parameter $\alpha$ tends to the density corresponding to $U(0,1)$ distribution.

### 4.2.1. Properties:

Let $X$ be a random variable with Selfinvertible p.d.f $f$ and $F$ be the distribution function then

i) there exists a unique $\alpha (< 1)$ such that $f(\alpha) = \alpha$ and

\[
\begin{align*}
  & f(x) > \alpha, \quad \text{for } x < \alpha \\
  & f(x) < \alpha, \quad \text{for } x > \alpha 
\end{align*}
\]

ii) the distribution function $F$ is concave

iii) $F(x) = x f(x) + \overline{F}[f(x)], \quad 0 < x < \alpha.$

The proof immediately follows from figure (4.2.4) and by noting that $\wedge(B) = \wedge(C)$, and $F(x) = \wedge(A) + \wedge(B) = \wedge(A) + \wedge(C)$, where $\wedge(S)$ denotes the area of the region $S$. 
Fig. 4.2.4. Graph showing various regions for computing $F(x)$

Equivalently,

$$F(x) = x f^{-1}(x) + F[f(x)], \quad x > \alpha$$

iv) $E[X/ f'(f^{-1}(x))] = E(X)$

Proof: $E(X) = \int_{0}^{\infty} x f(x) \, dx$

Let $y = f(x)$ then $dx = dy/ f'(f^{-1}(y))$ thus

$$E(X) = \int_{0}^{\infty} y f^{-1}(y) \frac{dy}{f'(f^{-1}(y))}.$$

Since $f^{-1}(x) = f(x)$ we have

$$E(X) = \int_{0}^{\infty} x f(x) \frac{dx}{f'(f^{-1}(x))} = \int_{0}^{\infty} \left[ \frac{x}{f'(f^{-1}(x))} \right] f(x) \, dx = E\left[ \frac{x}{f'(f^{-1}(x))} \right].$$
4.3. Illustrations:

For given \( \alpha \), one can obtain \( h(.) \) functions by using a decreasing density function \( g \) on \((0, \infty)\) (apart from the normalizing constant). In the following, to obtain Selfinvertible families of densities, we consider the \( h(.) \) function corresponding to exponential, triangular, Pareto and truncated normal densities.

**Example 4.3.1:**

Consider a function

\[
h(x) = k \exp(-ax), \quad x > 0
\]

for \( h \) to satisfy (4.2.3) we must have \( k = \alpha \) and \( a = 2\alpha / (1-\alpha^2) \), which gives

\[
h(x) = \alpha \exp[-2\alpha x/(1-\alpha^2)]
\]

which satisfies the property (iii) given in sub-section 4.2.1.
The first two raw moments and variance are given by

$$\mu_1 = (\alpha^4 + \alpha^2 + 2)/8\alpha, \quad \mu_2 = (\alpha^6 + 11\alpha^4 - 9\alpha^2 + 9)/36\alpha^2$$ and

Variance = \left[ 131\alpha^4 - 9\alpha^8 - 2\alpha^6 - 180\alpha^2 + 108 \right] / 576\alpha^2.

**Example 4.3.2:**

Consider a function

$$h(x) = k(a-x), \quad 0 < x < a$$

for h to satisfy (4.2.3) we must have $k = \alpha^2 / (1-\alpha^2)$ and $a = (1-\alpha^2) / \alpha$, which gives

$$h(x) = \left[ \frac{\alpha^2}{1-\alpha^2} \right] \left[ \frac{(1-\alpha^2)}{\alpha} - x \right] \quad (4.3.4)$$

and the p.d.f. becomes

$$f(x) = \begin{cases} \frac{[\alpha + (\alpha^2 - 1)x]}{\alpha^2}, & 0 < x < \alpha \\ \frac{\alpha (\alpha x - 1)}{(\alpha^2 - 1)}, & \alpha \leq x < 1/\alpha \end{cases} \quad (4.3.5)$$

In the following the graphs of density defined in (4.3.5) are plotted for various values of $\alpha = 0.2, 0.3, 0.4, 0.5$ and $0.6$. 
Fig. 4.3.1: Graph of density (4.3.5) for $\alpha = 0.2, 0.3, 0.4, 0.5$ and $\alpha = 0.6$

The corresponding distribution function is

$$F(x) = \begin{cases} 
(x/\alpha) + [(\alpha^2-1) x^2/2\alpha^2], & 0 < x < \alpha \\
[\alpha^2 x^2 - 2\alpha x + (2\alpha^2-1)] / 2(\alpha^2-1), & \alpha \leq x < 1/\alpha.
\end{cases}$$

(4.3.6)

The $r^{th}$ moment is given by

$$E(X^r) = \frac{\alpha^r}{r+1} + \frac{(\alpha^2-1) \alpha^r}{r+2} + \frac{1}{(r+2)(\alpha^2-1)\alpha^r} - \frac{1}{(r+1)(\alpha^2-1)\alpha^r} - \frac{\alpha^{r+4}}{(r+2)(\alpha^2-1)} + \frac{\alpha^{r+2}}{(r+1)(\alpha^2-1)}$$
\[ E(X) = \frac{(2\alpha^2 + 1)}{6\alpha} = m(\alpha) \text{ (say)} \quad (4.3.7) \]

**Example 4.3.3:**

Consider a function

\[ h(x) = \frac{k}{(\alpha + x)^a} \]

for \( h \) to satisfy (4.2.3) we must have \( k = \alpha^{a+1} = \alpha^{2(1-a^2)} \) and

\[ a = \frac{1 + \alpha^2}{1 - \alpha^2} \]

which gives

\[ h(x) = \frac{\alpha^{2(1-a^2)}}{(\alpha + x)^{(1+\alpha^2)/(1-\alpha^2)}} \quad (4.3.8) \]

and hence the p.d.f. is

\[
\begin{align*}
   f(x) &= \begin{cases} 
     \frac{2}{\alpha^{1+\alpha^2}} \frac{\alpha^2 - 1}{x^{\alpha^2 + 1}}, & x \leq \alpha \\
     \frac{2}{\alpha^{1-\alpha^2}} \frac{\alpha^2 + 1}{x^{\alpha^2 - 1}}, & x \geq \alpha 
   \end{cases}
\end{align*}
\]

and the c. d. f. is given by

\[
\begin{align*}
   F(x) &= \begin{cases} 
     \frac{-2\alpha^2}{\alpha^{\alpha^2 + 1}} \left( \frac{\alpha^2 + 1}{2} \right)^{\frac{2\alpha^2}{x^{\alpha^2 + 1}}}, & x < \alpha \\
     1 - \alpha^{1-\alpha^2} \left( \frac{1-\alpha^2}{2} \right)^{\frac{2\alpha^2}{x^{\alpha^2 + 1}}}, & x \geq \alpha 
   \end{cases}
\end{align*}
\]

The first two moments and variance are

\[ \mu_1 = 4\alpha^5/(9\alpha^4 - 1), \quad \mu_2 = \alpha^6/(4\alpha^4 - 1) \]

and Variance = \[\alpha^5 \left[ 9\alpha^5 - 16\alpha^4 - \alpha + 4 \right] / [36\alpha^8 - 13\alpha^4 + 1].\]
Example 4.3.4:

We consider \( h(x) = k \exp\left\{ -\frac{x^2}{2a^2} \right\} \), \( x > 0 \)

for \( h \) to satisfy (4.2.3) we must have \( k = \alpha \) and \( a = \frac{(1-\alpha^2)}{\alpha \sqrt{2\pi}} \)

which gives \( h(x) = \alpha \exp\{ -\pi \frac{\alpha^2 x^2}{(1-\alpha^2)} \} \) \hspace{1cm} (4.3.11)

thus the self invertible p.d.f. is

\[
f(x) = \begin{cases} 
\frac{\alpha^2 + (\alpha^2 - 1)\log (x/\alpha)/\pi}{\alpha}, & 0 < x < \alpha \\
\frac{-\pi \alpha^2 (x-\alpha)^2}{(1-\alpha^2)}, & x \geq \alpha 
\end{cases} \hspace{1cm} (4.3.12)
\]

4.4. Estimation of the parameter \( \alpha \):

In this section we consider the problem of estimation of \( \alpha \) for given \( h \) and as an illustration, in particular we consider a density \( f \) defined in (4.3.5).

a) Maximum Likelihood Estimator (m. l. e.) of \( \alpha \):

Suppose \( X_1, X_2, \ldots, X_n \) are the random observations from a density \( f \) defined in (4.3.5). Let \( \underline{x} = (x_1, x_2, \ldots, x_n) \) be observed values of \( (X_1, X_2, \ldots, X_n) \). The likelihood function is given by

\[
L(\alpha, \underline{x}) = \prod_{x_i < \alpha} f(x_i) \prod_{x_i \geq \alpha} f(x_i). \hspace{1cm} (4.4.1)
\]

Let \( x_{(0)} = 0 \) and \( x_{(n+1)} = 1 \) and \( x_{(1)}, \ldots, x_{(n)} \) are the ordered observations. To find maximum likelihood estimator for \( \alpha \) we confine \( \alpha \) to the interval \( (x_{(r)}, x_{(r+1)}) \) and find \( \alpha^*_r \) such that \( L(\alpha^*_r, \underline{x}) \geq L(\alpha, \underline{x}) \), for \( \alpha \in (x_{(r)}, x_{(r+1)}) \). Let \( L \)
$(\alpha^*, x) \geq L (\alpha_r^*, x)$ for $r = 0,1,\ldots,n$ then $\alpha^*$ is the m. l. e. of $\alpha$. A MATLAB programme for obtaining m.l.e. of $\alpha$ and its MSE for the density defined in (4.3.5) is given in appendix –B.

**b) Estimator of $\alpha$ based on first moment:**

To obtain moment estimator of $\alpha$ of the density defined in (4.3.5), we study the behavior of its mean function $m(\alpha)$. The graph of the function $m(\alpha)$ is given in figure 4.4.1.

**Remark 4.1:** The function $m(\alpha)$ is a convex function with $m(0+) = \infty$, $m(0.5) = m(1) = 0.5$ and $m(\alpha)$ is minimum at $1/\sqrt{2}$ and the minimum value is $(\sqrt{2}/3)$. Hence the equation $m(\alpha) = b$ has unique solution if $b>0.5$, no solution if $b < (\sqrt{2}/3)$ and there are two solutions if $(\sqrt{2}/3) < b \leq 0.5$.

![Graph of the function $m(\alpha)$](image)

**Fig.4.4.1:** graph of the function $m(\alpha)$
Suppose $X_1, X_2, \ldots, X_n$ are the random observations from a density $f$ defined in (4.3.5). Let $\bar{x}$ be the observed value of the sample mean. Generally moment estimator is obtained by equating population mean $m(\alpha)$ to the sample mean $\bar{x}$ provided it gives a solution (for the parameter). However if no solution exists (or there are more than one solutions) then one can obtain most nearest value of the parameter (or choose one based on some other criterion) and we refer one such estimator as a modified moment estimator.

Since $0 < x_i \leq 1/\alpha$, we have $\max\{x_i\} \leq 1/\alpha$, that is $0 < \alpha \leq 1/\max\{x_i\} = 1/x(n)$.

If $\sqrt{2}/3 < \bar{x} \leq 1/2$ then $m_1 < m_2$, the two roots of the equation $m(\alpha) = \bar{x}$ are given by $(1/2) \leq m_1 = 1.5\bar{x} - \sqrt{2.25\bar{x}^2 - 0.5} \leq (1/\sqrt{2})$ and $(1/\sqrt{2}) \leq m_2 = 1.5\bar{x} + \sqrt{2.25\bar{x}^2 - 0.5} \leq 1$. Thus in the light of the Remark 4.1 modified moment estimator of $\alpha$ depending on three cases (i) $1/x(n) < m_1$, (ii) $m_1 \leq 1/x(n) < m_2$ (iii) $m_1 < m_2 \leq 1/x(n)$ using likelihood criterion is given by

$$\hat{\alpha} = \begin{cases} 
1/\sqrt{2} & \text{if } 0 < \bar{x} \leq \sqrt{2}/3 \\
1/x(n) & \text{if } \sqrt{2}/3 < \bar{x} \leq 1/2 \text{ and } 1/x(n) < m_1 < m_2 \\
m_1 & \text{if } \sqrt{2}/3 < \bar{x} \leq 1/2 \text{ and } m_1 \leq 1/x(n) < m_2 \\
m_1 & \text{if } \sqrt{2}/3 < \bar{x} \leq 1/2 \text{ and } m_1 < m_2 \leq 1/x(n) \text{ and } L(m_1, \bar{x}) \geq L(m_2, \bar{x}) \\
m_2 & \text{if } \sqrt{2}/3 < \bar{x} \leq 1/2 \text{ and } m_1 < m_2 \leq 1/x(n) \text{ and } L(m_2, \bar{x}) \geq L(m_1, \bar{x}) \\
m & \text{if } \bar{x} > 1/2 \text{ and } m < 1/x(n) \\
1/x(n) & \text{if } \bar{x} > 1/2 \text{ and } m > 1/x(n)
\end{cases}$$
where \( m \) is the unique solution of the equation \( m(\alpha) = \bar{x}, \bar{x} > 1/2. \)

**4.4.1: Comparison of m.l.e. and modified moment estimator based on Simulation:**

The results of the simulation study for various permissible values of \( \alpha \) based on 1000 samples each of size \( n \) from a density \( f \) defined in (4.3.5) are tabulated below in table No.4.4.1. The first entry in each cell of the rows indicates average of the 1000 estimates each obtained from samples of size \( n \) and the quantities in the parenthesis indicate the MSE based on 1000 estimates. From the table it is clear that the maximum likelihood estimator performs better than that of modified moment estimator.

**Table No. 4.4.1**

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<td>( \hat{\alpha}_m )</td>
<td>( \hat{\alpha}_{ml} )</td>
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<td>0.1082 (0.002)</td>
<td>0.2082 (0.003)</td>
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<tr>
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<td>0.1005 (0.0002)</td>
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<tr>
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<td>0.0991 (0.00002)</td>
<td>0.1027 (0.0002)</td>
<td>0.2036 (0.00007)</td>
</tr>
<tr>
<td>200</td>
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<td>0.1002 (0.0008)</td>
<td>0.1988 (0.00004)</td>
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<tr>
<td>500</td>
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<td>0.0998 (0.0003)</td>
<td>0.2010 (0.00001)</td>
</tr>
<tr>
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<td>0.4</td>
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<tr>
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<td>0.4928 (0.0031)</td>
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<td>( \alpha_m )</td>
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</tr>
<tr>
<td>10</td>
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<td>0.8147 (0.0152)</td>
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<td>50</td>
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<td>0.7690 (0.0271)</td>
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<td>0.7895 (0.00042)</td>
<td>0.9003 (0.0005)</td>
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Conclusions:

In this Chapter we have introduced a class of densities namely Selfinvertible densities and studied their properties. The proposed density function depends on \((\alpha, h(.))\) where \(0 < \alpha < 1\) and \(h\) is a suitable function that depends on \(\alpha\). For a Selfinvertible density function defined in (4.3.5), m. l. e. and modified moment estimates of the parameter \(\alpha\) are obtained. Based on simulation we have studied their performance and it is observed that maximum likelihood estimator performs better than that of modified moment estimator.