CHAPTER 3

b-Chromatic Number of Graphs and Transformation

Graph G^{++-}

In this Chapter, the exact value of b-Chromatic number of different families of graphs such as Cycle, Crown graph, Barbell graph, Lollipop graph, Banana tree, Complete n-ary tree, Hoffmann tree, n-leaf, Bistar are obtained. Also the b-Chromatic number of the Transformation graph G^{++-} for Path, Cycle and Star graph are obtained along with its structural properties.

3.1 Introduction

Let G be a graph without loops and multiple edges with vertex set \( V(G) \) and edge set \( E(G) \). The smallest number \( k \) for which \( G \) admits a colouring with \( k \) colours is the chromatic number \( \chi(G) \) of \( G \). Many graph invariants related to colourings have been defined. Most of them try to minimize the number of colours used to colour the vertices under some constraints. For some invariants, it is meaningful to try to maximize this number. The b-chromatic number is one such example.

A \textit{b-colouring} [6,7,9,44] of a graph \( G \) is a proper colouring of the vertices of \( G \) such that there exists a vertex in each colour class joined to at least a vertex in each other colour class; such a vertex is called a dominating vertex. The \textit{b-chromatic number} of a graph \( G \), denoted by \( \phi(G) \), is the maximal integer \( k \) such that \( G \) may have a b-colouring by \( k \) colours. This parameter has been derived by Irving and Manlove [49] in the year 1999.

3.2 The b-Chromatic Number of Cycle

3.2.1 Theorem

For any Cycle \( C_n \) (\( n \geq 5 \)), \( \phi(C_n)=3 \).

**Proof**

Every Cycle is a 2-regular graph. Therefore \( \phi(C_n) \geq 3 \). Suppose if we assign more than three colours it contradicts the definition of b-colouring because in Cycle \( C_n \), we see that the vertex \( v_i \) is adjacent with the vertices \( v_{i+1} \) and \( v_{i-1} \) for \( i=2,3...n-1 \), \( v_1 \) is adjacent with \( v_2,v_n \) and \( v_n \) is adjacent with \( v_1,v_{n-1} \). Due to the above mentioned adjacency condition; it fails to
produce a b-chromatic colouring. To make the colouring as b-chromatic, we assign three
colours to every cycle $C_n$ for $n \geq 5$. Thus by the colouring procedure the above said colouring
is maximal and b-chromatic.

Example

![Figure 1: $C_n$](image1)

![Figure 2: $\phi(C_8) = 3$](image2)

### 3.3 The b-Chromatic Number of Crown Graph

#### 3.3.1 Theorem

The b-Chromatic number of the Crown graph is $n$ for $n \geq 3$ i.e. $\phi(S_n) = n$, $n \geq 3$

**Proof**

Let $G = S_n$ be the Crown graph. Consider the bipartition of $S_n$ namely $V = \{v_1, v_2, \ldots, v_n\}$
and $U = \{u_1, u_2, \ldots, u_n\}$ respectively. Consider the colour class $C = \{c_1, c_2, \ldots, c_n\}$. Assign the
colour $c_i$ to the vertex $v_i$ for $i = 1, 2, 3, \ldots, n$. Now, if $u_j$ is the vertex in $U$ which is not adjacent
with $v_i$, assign the colour $c_i$ to the vertex $u_i$ where $u_j \in U \setminus \{u_j\}$.

Now in this colouring, $v_i$ will realize its colour but any vertex in $V \setminus \{v_i\}$ will not
realizes its own colour, the only colour which is not adjacent to these vertices is $c_i$. Since all
the vertices in $V-\{v_i\}$ is adjacent to $u_i$. The only possibility is to make the above said colouring as b-chromatic one is to colour $u_i$ as $c_i$ for $i=1,2,3..n$, since $V$ and $U$ are independent sets. Hence by colouring procedure the above said colouring is the maximum and b-chromatic.

$\therefore \phi(S_n) = n, n \geq 3$

**Example**

![Graph Image]

*Figure 3: $\phi(S_4) = 4$*

### 3.3.2 Structural Properties of Crown Graph

The number of vertices in the Crown graph $S_n, (n>2)$ i.e. $p(S_n) = 2n$, number of edges in the Crown graph $S_n$ i.e. $q(S_n) = n^2 - n$. The maximum and minimum degree of Crown graph $S_n$ are denoted by $\Delta = n-1$ and $\delta = n-1$ respectively. Thus every Crown graph is a $n-1$ regular graph.

### 3.4 The b-Chromatic Number of Barbell Graph

#### 3.4.1 Theorem

The b-Chromatic number of Barbell graph is $n$ for every $n \geq 3$ i.e. $\phi(B(K_n, K_n)) = n$, $n \geq 3$.

**Proof**

Let $G = B(K_n, K_n)$ be the Barbell graph. By definition, a Barbell graph is obtained by connecting two copies of complete graph $K_n$ by a bridge. Let $V = \{v_1, v_2, ..., v_n\}$ be the vertex
set of the first complete graph denoted as $K^l$ and $U = \{u_1, u_2, \ldots, u_n\}$ be the vertex set of the second complete graph denoted as $K^2$. Now assign a proper colouring to $K^l$ and $K^2$ as follows. Since $K^l$ contains exactly $n$ vertices ($n \geq 3$) which are mutually adjacent to each other. So, we can colour all the vertices of $K^l$ by the colour $c_1, c_2, c_3, \ldots, c_n$.

Next suppose if we assign $n$ different colours to $K^2$ other than the colours $c_1, c_2, c_3, \ldots, c_n$, then the vertices of $K^l$ as well as $K^2$ do not realizes its own colours. So we cannot assign any new colours to $K^2$. Hence we should assign only the pre-used colours to $K^2$. Thus by the colouring procedure it is the maximum and b-chromatic colouring.

\[ \therefore \phi[B(K_n, K_n)] = n \text{ for } n \geq 3. \]

**Example**

![Diagram of Barbell graph](image)

*Figure 4: $\phi[B(K_5,K_5)] = 5$*

### 3.4.2 Structural Properties of Barbell graph

The number of vertices in the Barbell graph ($n > 2$), i.e. $p[B(K_n, K_n)] = 2n$, the number of edges in the Barbell graph $B(K_n, K_n)$, ($n > 2$) i.e. $q[B(K_n, K_n)] = n^2 - n + 1$. The maximum and minimum degree of Barbell graph $B(K_n, K_n)$ is denoted as $\Delta = n$ and $\delta = n - 1$ respectively. The number of vertices having maximum degree $\Delta$ in $B(K_n, K_n)$ is denoted by $n(p_\Delta) = 2$ and the number of vertices having minimum degree $\delta$ in $B(K_n, K_n)$ is denoted by $n(p_\delta) = 2n - 2$. 
3.5 b-Chromatic Number of Lollipop Graph

3.5.1 Theorem

The b-Chromatic number of Lollipop graph is \( m \) for every \( m \geq 3 \) i.e. \( \varphi(L_{m,n}) = m \) for every \( m \geq 3 \).

Proof

The \((m,n)\) Lollipop graph is the graph obtained by joining a Complete graph \( K_m \) to a Path graph \( P_n \) with a bridge. Since \( K_m \) is a complete graph on \( m \) vertices, each pair of vertices of \( K_m \) are mutually adjacent to each other. Therefore, we say that it requires \( m \) colours for producing a b-chromatic colouring. Suppose if we assign any new colour to the Path graph \( P_n \), the vertices in \( K_m \) does not realizes the new colour, So there is a possibility of assigning only the existing colours to the Path graph i.e. assign the colour \( c_i \) for \( i = 1,2,3..n \) to the vertices \( v_i \) of Path graph \( P_n \). Thus by the colouring procedure the above said colours produces maximal and b-chromatic colouring.

Example

![Figure 5(a) \( \varphi(L_{6,1}) = 6 \) ](image)

![Figure 5(b) \( \varphi(L_{6,2}) = 6 \) ](image)

3.5.2 Structural Properties of Lollipop Graph

The number of vertices in the Lollipop graph i.e. \( p(L_{m,n}) = m + n \), number of edges in the Lollipop graph i.e. \( q(L_{m,n}) = \frac{m^2 - m + 2n}{2} \). The maximum and minimum degree of Lollipop graph \( L_{m,n} \) is denoted as \( \Delta = n \) and \( \delta = 1 \) respectively. The number of vertices having maximum degree \( \Delta \) in \( L_{m,n} \) is denoted by \( n(p_{\Delta}) = l \) and the number of vertices having minimum degree \( \delta \) in \( L_{m,n} \) is denoted by \( n(p_{\delta}) = n \).
3.6 Results under Observation

- The b-chromatic number of Sunlet graph is $n$ for $n \geq 3$ i.e. $\varphi(S_n) = n$, for $n \geq 3$.
- The b-chromatic number of Complete Bipartite graph is bicolourable.
- The b-chromatic number of Circular Ladder graph is 4 for every $n \geq 4$ i.e. $\varphi(CL_n) = 4$, $n \geq 4$.
- The b-chromatic number of the Halin graph is $m$ always $n > m$, i.e. $\varphi(H_{m,n}) = m$, always $n > m$.
- The b-chromatic number of Path graph is 3 for $n \geq 5$ i.e. $\varphi(P_n) = 3$ for $n \geq 5$.
- The b-chromatic number of Strong binary tree is $\leq 3$.
- The b-chromatic number of Rosette graph is $p - \Delta + 1$.
- The b-chromatic number of Dutch Windmill graph is $m$ for $m \geq 3$ i.e. $\varphi(D_3^m) = m$ for $m \geq 3$.
- The b-chromatic number of Wheel Split graph $W_{n,r}$ is $r+3$ for every $r, n \geq 3$.

3.7 b-Chromatic Number of Banana Tree, Star $n$-leaf, Complete $n$-ary Tree, Hoffmann Tree and Bistar [22, 23, 24, 35]

3.7.1 Theorem

$\varphi(B_{n,k}) = n+1$ for $2 \leq n \leq k$

Proof

Consider the Banana tree $B_{n,k}$ which is obtained by connecting one leaf of each copies of $n$-star with a single root vertex that is distinct from all stars. Let the stars are denoted as $s^1, s^2, s^3, ..., s^k$ with the root vertex $S$, as the one to which $n$-copies of star are connected. Let the vertex set $V(G)$ be $\{v, v_1, v_2, v_3, ..., v_n, v_{k, n}, v\}$ where $v$ is the root vertex. Now represent each stars $s^1, s^2, s^3, ..., s^k$ with the vertices $v_{3i-2}, v_{3i-1}, v_{3i}, v_{3i+1}, ..., v_{3i+n}$ named in anticlockwise direction such that there exist $(v_{ki}, u)$ edge.

Now assign a proper colouring to the root vertex and stars. Consider the colour class $C = \{c_1, c_2, ..., c_n\}$. Assign the colour $c_1$ to the root vertex $v$ and assign the colour $c_{i+1}$ to $s_i$ for $i = 1, 2, ..., k$. Where $s_i$ is the root vertex of the stars such as $s^1, s^2, s^3, ..., s^k$. The above colouring produces a b-chromatic colouring.

Next if we assign any new colour to the leaves of stars $s^1, s^2, s^3, ..., s^k$ which does not produces a b-colouring because $s^1, s^2, s^3, ..., s^k$ are pendant vertices. So we should assign only
the existing colours to the leaves of stars. Thus by the colouring procedure it is the maximal and b-chromatic colouring.

\[ \therefore \varphi(B_{n,k}) = n + 1 \text{ for } 2 \leq n \leq k \]

**Example**

![Figure 6: \(\varphi(B_{3,4}) = 4\)](image)

### 3.7.2 Theorem

The b-chromatic number of any complete n-ary \((n \geq 2)\) tree is \(\left\{ \frac{(p+q)-1}{2n} \right\} \)

**Example**

\[ \varphi[3-ary\ tree] = \left\{ \frac{(3+12)-1}{2(3)} \right\} = 4 \]

![Figure 7(a): 3-ary tree](image)
3.7.3 Theorem

The b-Chromatic number of Hoffman tree is four for $n \geq 7$ i.e. $\phi(P_n \Theta K_1) = 4$

Proof

Let $P = 2n$ be the number of vertices and $q = 2n - 1$ the number of edges of given graph $P_n \Theta K_1$ respectively. Here let $V = \{v_1, v_2, v_3, ..., v_n\}$ and $U = \{u_1, u_2, u_3, ..., u_m\}$ be any two distinct vertex set and let $\{e_1, e_2, e_3, ..., e_{2n-1}\}$ be the edges of $P_n \Theta K_1$ respectively. Here we say that the vertices $v_i$ for $i = 2, 3, 4, ..., n - 1$ is incident with at most three edges i.e. with degree 3. Moreover all
the vertices in \( u_j \) for \( j = 1,2,3,...,n \) are pendant vertices. Therefore, the only possibility is to assign four colours to any Hoffmann tree for producing a b-chromatic colouring. Note that any rearrangement of colours to the graph also fails to accommodate the new colour. Thus by the colouring procedure it is the maximum and b-chromatic colouring.

**Example**

![Hoffmann Tree](image)

**Figure 8: Hoffmann Tree**

### 3.7.4 Structural Properties of Hoffmann Tree

The number of vertices in the Hoffmann tree \( P_n \Theta K_i \) i.e. \( p(P_n \Theta K_i) = 2n \), number of edges in the Hoffmann tree \( P_n \Theta K_i \) i.e. \( q(P_n \Theta K_i) = 2n-1 \). The maximum and minimum degree of Hoffmann tree \( P_n \Theta K_i \) is denoted as \( \Delta = 3 \) and \( \delta = 1 \) respectively. The number of vertices having maximum and minimum degree in \( P_n \Theta K_i \) is denoted by \( n(p_{\Delta}) = n-2 \) and \( n(p_{\delta}) = n \) and remaining vertices with degree 2.

### 3.7.5 Theorem

The b-Chromatic number of every \( n \)-leaf is bicolourable.

**Proof**

Every \( n \)-leaf having exactly one vertex of degree \( n \) and other vertices are pendant. Assume the vertex of degree \( n \) as the root vertex \( v \) and all other vertices as \( v_1, v_2, v_3, ..., v_n \). Now assign a proper colouring to these vertices as follows:

Assign the colour \( c_j \) to the vertex \( v \) and \( c_{j+1} \) to the vertices \( v_i \) for \( i = 1,2, ..., n \). Here the vertex with maximum degree is the root vertex which realizes its own colour but the vertices \( v_1, v_2, v_3, ..., v_n \) does not realize its own colour because all the vertices \( v_1, v_2, v_3, ..., v_n \) are pendant vertices. So the only possibility of assigning the colour \( c_2 \) to all \( v_i \) for \( i = 1,2, ..., n \). Now the vertices \( v, v_1, v_2, v_3, ..., v_n \) realize its own colour, which produces a maximum and b-chromatic colouring.
Example

Figure 9: Leaf
The above figure represents 2-leaf, 3-leaf, 4-leaf, 5-leaf respectively.

3.7.6 Theorem

The b-Chromatic number of every Bistar is bicolourable.

Example

Figure 10: $B_{n,n}$

3.7.7 Structural Properties of Bistar

The number of vertices in the Bistar $B_{n,n}$ i.e. $p(B_{n,n})=2n+2$, number of edges in the Bistar $B_{n,n}$ i.e. $q(B_{n,n})=2n+1$. The maximum and minimum degree of Bistar $B_{n,n}$ is denoted as $\Delta = n+1$ and $\delta = 1$ respectively. The number of vertices having maximum degree $\Delta$ in Bistar is denoted by $n(p_{\Delta})=2$ and the number of vertices having minimum degree $\delta$ in $B_{n,n}$ is denoted by $n(p_{\delta})=2n$.

3.7.8 Theorem

The b-Chromatic number of every Tree $<K_{1,2}>$ is Tricolourable.
3.8 b-Chromatic Number of Transformation Graph $G^++$

3.8.1 Introduction [8, 10, 13]

The concept of Total graph was generalized by Behzad.Wu and Meng and they introduced some new graphical transformations. Since there are eight distinct 3-permutations of $\{+,-\}$, they introduced eight graphical transformations of $G$. One of the Transformation graphs is $G^++$. The Transformation graph $G^++$ is defined as follows:

For a graph $G$, let $V(G)$ and $E(G)$ denote the point set, line set of graph $G$ respectively. The Transformation graph $G^++$ of $G$ is the graph with point set $V(G) \cup E(G)$ in which the points $X$ and $Y$ are joined by a line if one of the following conditions hold.

- $x, y \in V(G)$ and $x, y$ are adjacent in $G$.
- $x, y \in E(G)$ and $x, y$ are adjacent in $G$.
- One of $x$ and $y$ is in $V(G)$ and the other is in $E(G)$ and they are not incident in $G$.

3.8.2 b-Chromatic Number of $G^++$ of Path Graph

3.8.2.1 Theorem

The b-Chromatic number of $G^++$ of any Path graph $P_n$ $(n>3)$ has $n$ colours.

Proof

Consider a Path graph of length $n-1$ with vertex set $V= \{v_1, v_2, v_3, \ldots, v_n\}$ and edge set $E=\{e_1, e_2, e_3, \ldots, e_{n-1}\}$. In Path graph $P_n$, we see that each vertex $v_i$ is adjacent with $v_{i-1}$ and $v_{i+1}$
for $i=2,3,...,n-1$, $v_i$ is adjacent with $v_2$ and $v_n$ is adjacent with $v_{n-1}$ and the lines $e_i$ and $e_n$ are non-adjacent with $n-3$ lines and remaining $e_i$ for $i=2,3,...,n-1$ are non-adjacent with $n-4$ lines.

By the definition of Transformation graph $G^{++}$, the vertex set of $G^{++}(P_n)$ corresponds to both vertex set and edge set of Path graph.

i.e. $V[G^{++}(P_n)]=\{v_i: 1 \leq i \leq n\} \cup \{e_i: 1 \leq i \leq n-1\}$

Consider the colour class $C=\{c_1,c_2,c_3,..,c_n\}$. Assign the colour $c_i$ to $v_i$ for $i=1,2,3..n$ and assign the colour $c_{n+i}$ to $e_i$ for $i=1,2,3..n-1$. Due to the above mentioned non-adjacency condition the above colouring does not produce a $b$-chromatic colouring. Thus, to make the above colouring as $b$-chromatic one, assign the colour $c_i$ to $v_i$ for $1 \leq i \leq n$ and assign the colour $c_1$ to $e_1$, $c_{i+1}$ to $e_i$ for $i=2,3..n-1$. Now the vertices $v_i$ for $i \leq 3$ and the vertices $e_i$ for $3 \leq i \leq n-1$ realizes its own colour, which produces a $b$-chromatic colouring.

Thus by the colouring procedure the above said colouring is $b$-chromatic. By the very construction, it is the maximal colour class.

**Example**

![Figure 12: $P_n$](image)

![Figure 13: $\varphi [G^{++}(P_n)] = 5$](image)
3.8.2.2 Structural Properties of $G^{++}(P_n)$

The number of vertices in $G^{++}(P_n)$ i.e. $p[G^{++}(P_n)] = 2n-1$, number of edges in the $G^{++}(P_n)$ i.e. $q[G^{++}(P_n)] = n^2-n-1$. The maximum and minimum degree of $G^{++}(P_n)$ are denoted by $\Delta = n$ and $\delta = n-1$ respectively.

3.8.2.3 Observations

- If $G$ is $P_1$ then clearly $G^{++}$ is $K_1$
- If $G$ is $P_2$ then clearly $G^{++}$ is $G \cup K_1$
- If $G$ is $P_3$ then clearly $G^{++}$ is $C_5$

3.8.3 $b$-Chromatic Number of $G^{++}$ of Cycle

3.8.3.1 Theorem

The $b$-Chromatic number of $G^{++}$ of any Cycle $C_n$ is $n$.

Proof

Consider a Cycle of length $n$, whose vertices are $v_1, v_2, v_3, ..., v_n$ and edges $e_1, e_2, e_3, ..., e_n$. We see that every point in Cycle $C_n$ is non-adjacent with $n-2$ lines. Now consider $G^{++}(C_n)$, here there is no non-incident lines. By the definition of Transformation graph $G^{++}$, the vertex set of $G^{++}(C_n)$ corresponds to both vertex set and edge set of Cycle.

i.e. $V[G^{++}(C_n)] = \{v_i: 1 \leq i \leq n\} \cup \{e_i: 1 \leq i \leq n\}$

By observation we see that $G^{++}(C_n)$ forms a $n$-regular graph. Therefore the $b$-chromatic number of $G^{++}(C_n) \geq n$. Now we will prove $\varphi[G^{++}(C_n)] \leq n$, for this consider a proper colouring of $G^{++}(C_n)$ as follows.

Consider the colour class $C = \{c_1, c_2, c_3, ..., c_n\}$. Assign the colour $c_i$ for $i = 1, 2, 3, ..., n$ to the inner cycle of $C_n$. Next, if we assign the colour $c_{n+1}$ to any vertices in outer cycle, it does not realize the colour $c_{n+1}$. So we should assign only the existing colours to the vertices in outer cycle. By the colouring procedure, we cannot assign more than $n$ colours to $G^{++}(C_n)$ i.e. $\varphi[G^{++}(C_n)] \leq n$. Therefore $\varphi[G^{++}(C_n)] = n$. Thus by the colouring procedure the above said colouring is maximum and $b$-chromatic.
Example

3.8.3.2 Structural Properties of $G^{++}(C_n)$

The number of vertices in $G^{++}(C_n)$ i.e. $p[G^{++}(C_n)]=2n$, number of edges in the $G^{++}(C_n)$ i.e. $q[G^{++}(C_n)] = n^2$. The maximum and minimum degree of $G^{++}(C_n)$ are denoted as $\Delta = n$ and $\delta = n$ respectively. Thus $G^{++}(C_n)$ is an $n$-regular graph.

3.8.4. b-Chromatic Number of $G^{++}$ of Star graph

3.8.4.1 Theorem

If $G$ is $K_{1,n}$ then clearly $\varphi[G^{++}(K_{1,n})] = n+1$

Proof

Consider the graph $K_{1,n}$ with pendant vertices $v_1, v_2, v_3..v_n$ and $v$ where $v$ is the root vertex with degree $n$ i.e. $V(K_{1,n})=\{v\} \cup \{v_i: 1 \leq i \leq n\}$ and $E(K_{1,n})=\{e_i: 1 \leq i \leq n\}$ between the vertices $vv_i$ for $i=1,2,3..n$. 
Consider the graph $G^{++}(K_{1,n})$, by the definition of the Transformation graph $G^{++}$ we define the vertex set of $G^{++}(K_{1,n})$ as follows:

$$V[G^{++}(K_{1,n})] = \{v\} \cup \{v_i: 1 \leq i \leq n\} \cup \{e_i: 1 \leq i \leq n\}$$

We see that the vertices $\{e_i: 1 \leq i \leq n\}$ forms a clique of order $n$ (say $K_n$) in $G^{++}(K_{1,n})$. Therefore we say that the b-chromatic number of $G^{++}(K_{1,n}) \geq n$. Consider the colour class $C=\{c_1,c_2,c_3...c_{n+1}\}$. Assign a proper colouring to the vertices as follows.

**Case 1**

First assign the proper colouring to the vertices $e_i$.

Assign the colour $c_i$ to the vertex $e_i$ for $i=1,2,3..n$ and assign the colour $c_{n+1}$ to $v_i$ for $i=1,2,3..n$ and assign any colour to root vertex other than the colour $c_{n+1}$. Now the vertices $e_i$ for $i=1,2,3..n$ realize its own colour. Thus, by the colouring procedure the above said colouring produces a maximal and b-chromatic colouring.

**Example**

![Image](image_url)

*Figure 16: $\varphi[G^{++}(K_{1,3})]=4$*

**Case 2**

Next assign proper colouring to the vertex $v$ and $v_i$ for $i=1,2,3..n$.

Assign the colour $c_i$ to the root vertex $v$ and $c_{n+1}$ to $v_i$ for $i=1,2,3..n$ and assign the same set of colour to $e_i$ which is assigned for $v_i$ because $v_i$ is not adjacent with $e_i$ for $i=1,2,3..n$, which produces a b-chromatic colouring. Thus by the colouring procedure the above said colouring is maximum and b-chromatic.
Example

Figure 17: $\varphi[G^{++}(K_{1,3})] = 4$

3.8.4.2 Structural Properties of $G^{++}(K_{1,n})$

The number of vertices in $G^{++}(K_{1,n})$ i.e. $p[G^{++}(K_{1,n})] = 2n+1$, number of edges in the $G^{++}(K_{1,n})$ i.e. $q[G^{++}(K_{1,n})] = \left\lceil \frac{n(3n-1)}{2} \right\rceil$. The maximum and minimum degree of $G^{++}(K_{1,n})$ is denoted as $\Delta = n+1$ and $\delta = n-1$ respectively. The number of vertices having maximum degree $\Delta$ in $G^{++}(K_{1,n})$ is denoted by $n(p_{\Delta}) = n$ and the number of vertices having minimum degree $\delta$ in $G^{++}(K_{1,n})$ is denoted by $n(p_{\delta}) = n+1$.

3.8.4.3 Theorem

For any Star graph $K_{1,n}$, the number of edges in $G^{++}(K_{1,n})$ i.e. $q[G^{++}(K_{1,n})] = \left\lceil \frac{n(3n-1)}{2} \right\rceil$

Proof

$q[G^{++}(K_{1,n})] = \text{Number of edges in } K_{1,n} + \text{Number of edges in } K_n + \text{Number of edges in crown graph } S_n$

$= \left( \binom{n}{1} + \binom{n}{2} \right) + n(n-1)$

$= n + \left\lceil \frac{n(n-1)}{2} \right\rceil + n(n-1)$

$= \left\lceil \frac{2n + n(n-1) + 2n(n-1)}{2} \right\rceil$

$= \left\lceil \frac{2n + 3n^2 - 3n}{2} \right\rceil$

$= \left\lceil \frac{3n^2 - n}{2} \right\rceil$

$= \left\lceil \frac{n(3n-1)}{2} \right\rceil$

Therefore $q[G^{++}(K_{1,n})] = \left\lceil \frac{n(3n-1)}{2} \right\rceil$