CHAPTER 1
Introduction and Basic Definitions

In this Chapter, Introduction, History, Applications of Graph Theory and Basic definitions which are needed for subsequent chapters are given. For graph theoretic terminology, reference is made to Frank Harary [31], Bondy and Murty [12].

1.1 Introduction [21]

Graphs are one of the prime objects of study in Discrete Mathematics. Graph theory is one of the branches of modern mathematics having experienced a most impressive development in recent years. In the beginning, Graph theory was only a collection of recreational or challenging problems like Euler tours or the four colouring of a map, with no clear connection among them, or among techniques used to attach them. The aim was to get a “yes” or “no” answer to simple existence questions. Under the impulse of Game Theory, Management Sciences and Transportation Network Theory, the main concern shifted to the maximum size of entities attached to a graph. It is no coincidence that graph theory has been independently discovered many times, since it may quite properly be regarded as an area of applied mathematics. The basic combinatorial nature of Graph theory and a clue to its wide applicability are indicated in the words of Sylvester, “The theory of ramification is one of the pure colligation, for it takes no account of magnitude or position; geometrical lines are used, but have no more real bearing on the matter than those employed in genealogical tables have in explaining the laws of procreation”. Indeed, the earliest recorded mention of the subject occurs in the works of Euler, and although the original problem he was considering might be regarded as a somewhat frivolous puzzle, it did arise from the physical world. Subsequent rediscoveries of graph theory by Kirchhoff and Cayley also had their roots in the physical world.

Euler (1707–1782) became the father of Graph theory as well as Topology. Graph theory is considered to have begun in 1736 with the publication of Euler’s solution
of the Konigsberg bridge problem. The Graph theory is one of the few fields of mathematics with a definite birth date by Ore. Kirchhoff’s investigations of electric networks led to his development of the basic concepts and theorems concerning trees in graphs, while Cayley considered trees arising from the enumeration of organic chemical isomers. Another puzzle approach to graphs was proposed by Hamilton. After this, the celebrated four colour conjecture came into prominence and has been notorious ever since. In the present century, there have already been a great many rediscoveries of Graph theory.

Graph theory serves as a Mathematical Model to represent any system which has a Binary Relation. It has application to some areas like Computer Engineering, Civil Engineering, Operations Research, Physics, Chemistry, Genetics, Sociology, Mathematics and many other fields.

1.2 History of Graph Theory [21, 82]

As mentioned earlier, Graph theory was born in 1736 with Euler’s paper in which he solved the Konigsberg bridge problem. For the next 100 years nothing more was done in the field.

In 1847, G.R. Kirchhoff (1824-1887) developed the theory of trees for their applications in electrical networks. Ten years later, Cayley (1821-1895) discovered trees while he was trying to enumerate the isomers of saturated hydrocarbons \( C_n H_{2n+2} \).

About the time of Kirchhoff and Cayley, two other milestones in Graph theory were laid. One was the four colour conjecture, which states that four colours are sufficient for colouring any atlas (a map on a plane) such that the countries with common boundaries have different colours.

It is believed that A.F. Mobius (1790-1868) first presented the four colour problem in one of his lectures in 1840. About 10 years later, De Morgan (1806-1871) discussed this problem with his fellow mathematicians in London. De Morgan’s letter was the first authenticated reference to the four colour problem. The problem became
well known after Cayley published it in 1879 in the first volume of the Proceedings of the Royal Geographic Society. To this day, the four colour problem has stimulated an enormous amount of research in the field of Graph theory.

The other milestone is due to Sir W.R. Hamilton (1805-1865). In the year 1859 he invented a puzzle and sold it for 25 guineas to a game manufacturer in Dublin. The puzzle consisted of a wooden, regular dodecahedron. The corners were marked with the names of 20 important cities like London, New York, Delhi, and Paris and so on. The object in the puzzle was to find a route along the edges of the dodecahedron, passing through each of the 20 cities exactly one.

Although the solution of this specific problem is easy to obtain, to date no one has found a necessary and sufficient condition for the existence of such a route (called Hamiltonian circuit) in an arbitrary graph.

This fertile period was followed by half a century of relative inactivity, then a resurgence of interest in graphs started during the 1920s. One of the pioneers in this period was D. Konig. He organized the work of other mathematicians and his own and wrote the first book on the subject, which was published in 1936.

The past 30 years has been a period of intense activity in graph theory both pure and applied. A great deal of research has been done and is being done in this area. Thousands of papers have been published and more than a dozen books written during the past decade. Among the current leaders in the field are Claude Berge, Oystein Ore, Paul Erdos, William Tutte and Frank Harary.

1.3 What is a Graph? [21]

In Mathematics and Computer Science, Graph theory is the study of graphs, mathematical structures used to model pair wise relations between objects from a certain collection. A graph in this context refers to a collection of vertices or nodes and a collection of edges that connect pairs of vertices. A graph may be undirected, meaning that there is no distinction between the two vertices associated with each edge, or its edges may be directed from one vertex to another.
The word graph has atleast two meanings in Mathematics. In Elementary Mathematics, graph refers to a function graph or graph of a function. In Mathematician's terminology, a graph is a collection of points and lines connecting some subset of them. The points of a graph are most commonly known as graph vertices, but may also be called nodes or simply points. Similarly, the lines connecting the vertices of a graph are most commonly known as graph edges, but may also be called arcs or lines.

1.4 Applications of Graph Theory

Graphs are among the most ubiquitous models of both natural and human-made structures. They can be used to model many types of relations and process dynamics in physical, biological and social systems. Many problems of practical interest can be represented by graphs.

In **Computer Science**, graphs are used to represent networks of communication, data organization, computational devices, the flow of computation, etc. One practical example: The link structure of a website could be represented by a directed graph. The vertices are the web pages available at the website and a directed edge from page \( A \) to page \( B \) exists if and only if \( A \) contains a link to \( B \). A similar approach can be taken to problems in travel, biology, computer chip design, and many other fields. The development of algorithms to handle graphs is therefore a major interest in computer science. The transformation of graphs is often formalized and represented by graph rewrite systems. Complementary to graph transformation systems focusing on rule-based in-memory manipulation of graphs are graph databases geared towards transaction-safe, persistent storing and querying of graph-structured data.

Graph-theoretic methods, in various forms, have proven particularly useful in **Linguistics**, since natural language often lends itself well to discrete structure. Traditionally, syntax and compositional semantics follow tree-based structures, whose expressive power lies in the Principle of Compositionality, modeled in a hierarchical graph. Within lexical semantics, especially as applied to computers, modeling word meaning is easier when a given word is understood in terms of related words; semantic networks are therefore important in computational linguistics. Still other methods in
phonology (e.g. Optimality Theory, which uses lattice graphs) and morphology (e.g. finite-state morphology, using finite-state transducers) are common in the analysis of language as a graph. Indeed, the usefulness of this area of mathematics to linguistics has borne organizations such as Text Graphs, as well as various 'Net' projects, such as WordNet, VerbNet and others.

Graph theory is also used to study molecules in Chemistry and Physics. In condensed matter physics, the three dimensional structure of complicated simulated atomic structures can be studied quantitatively by gathering statistics on graph-theoretic properties related to the topology of the atoms. For example, Franzblau's shortest-path (SP) rings. In chemistry a graph makes a natural model for a molecule, where vertices represent atoms and edges bonds. This approach is especially used in computer processing of molecular structures, ranging from chemical editors to database searching. In statistical physics, graphs can represent local connections between interacting parts of a system, as well as the dynamics of a physical process on such systems.

Graph theory is also widely used in Sociology as a way, for example, to measure actors’ prestige or to explore diffusion mechanisms, notably through the use of social network analysis software. Graph theory is useful in Biology where a vertex can represent regions where certain species exist and the edges represent migration paths or movement between the regions. This information is important when looking at breeding patterns or tracking the spread of disease, parasites or how changes to the movement can affect other species.

In Mathematics, graphs are useful in geometry and certain parts of topology, e.g. Knot Theory. Algebraic graph theory has close links with group theory. A graph structure can be extended by assigning a weight to each edge of the graph. Graphs with weights or weighted graphs are used to represent structures in which pair wise connections have some numerical values. For example, if a graph represents a road network, the weights could represent the length of each road. A digraph with weighted edges in the context of graph theory is called a network. Network analysis has many practical applications, for example, to model and analyze traffic networks. The field of mathematics plays a vital
role in various fields. One of the important areas in mathematics is graph theory which is used in structural models. This structural arrangements of various objects or technologies lead to new inventions and modifications in the existing environment for enhancement in those fields. Graph theoretical concepts are widely used to study and model various applications, in different areas. They include, study of molecules, construction of bonds in chemistry and the study of atoms. Similarly, graph theory is used in sociology for example to measure actors’ prestige or to explore diffusion mechanisms.

Graph theoretical concepts are widely used in *Operations Research*. For example, the travelling salesman problem, the shortest spanning tree in a weighted graph, obtaining an optimal match of jobs and men and locating the shortest path between two vertices in a graph. It is also used in modeling transport networks, activity networks and theory of games. The network activity is used to solve large number of combinatorial problems. The most popular and successful applications of networks in Operations Research is the planning and scheduling of large complicated projects. The best well known problems are

- PERT (Project Evaluation Review Technique)
- CPM (Critical Path Method)

*Game theory* is applied to the problems in engineering, economics and war science to find optimal way to perform certain tasks in competitive environments. To represent the method of finite game a digraph is used. Here, the vertices represent the positions and the edges represent the moves.

Graphs are used in the field of *Chemistry* to model chemical compounds. In computational biochemistry some sequences of cell samples have to be excluded to resolve the conflicts between two sequences. This is modeled in the form of graph where the vertices represent the sequences in the sample. An edge will be drawn between two vertices if and only if there is a conflict between the corresponding sequences. The aim is to remove possible vertices and to eliminate all conflicts. In brief, Graph theory has its unique impact in various fields and is growing large now a days.
The major role of Graph theory in Computer Applications is the development of Graph Algorithms. Numerous algorithms are used to solve problems that are modeled in the form of graphs. These algorithms are used to solve the graph theoretical concepts which in turn used to solve the corresponding computer science application problems. Some algorithms are as follows:

- Shortest path algorithm in a network
- Finding minimum spanning tree
- Finding graph planarity
- Algorithms to find adjacency matrices.
- Algorithms to find the connectedness
- Algorithms to find the cycles in a graph
- Algorithms for searching an element in a data structure (DFS, BFS).

Various computer languages are used to support the graph theory concepts. The main goal of such languages is to enable the user to formulate operations on graphs in a compact and natural manner. Some graph theoretic languages are

- SPANTREE – To find a spanning tree in the given graph.
- GTPL – Graph Theoretic Language
- GASP – Graph Algorithm Software Package
- HINT – Extension of LISP
- GRASPE – Extension of LISP
- IGTS – Extension of FORTRAN
- GEA – Graphic Extended ALGOL (Extension of ALGOL)
- AMBIT – To manipulate digraphs
- GIRL – Graph Information Retrieval Language
- FGRAAL – FORTRAN Extended Graph Algorithmic Language
1.5 Basic Definitions [12, 21, 31, 34]

Definition 1.5.1

A linear graph (or simply a graph) \( G = (V,E) \) consists of a set of objects \( V = \{v_1, v_2, \ldots \} \) called vertices, and another set \( E = \{e_1, e_2, \ldots \} \), whose elements are called edges, such that each edge \( e_k \) is identified with an unordered pair \( (v_i, v_j) \) of vertices. The vertices \( v_i, v_j \) associated with edge \( e_k \) are called the end vertices of \( e_k \). The most common representation of a graph is by means of a diagram, in which the vertices are represented as points and each edge as a line segment joining its end vertices.

Definition 1.5.2

A vertex is simply drawn as a node or a dot. The vertex set of \( G \) is usually denoted by \( V(G) \) or \( V \).

Definition 1.5.3

The order of a graph is the number of its vertices, i.e. \(|V(G)|\).

Definition 1.5.4

An edge (a set of two elements) is drawn as a line connecting two vertices, called endpoints or end vertices. An edge with end vertices \( x \) and \( y \) is denoted by \( xy \) (without any symbol in between). The edge set of \( G \) is usually denoted by \( E(G) \). The size of a graph is the number of its edges, i.e. \(|E(G)|\).

Definition 1.5.5

Two vertices are said to be adjacent if they are the end vertices of same edge.

If a vertex \( v \) is an end vertex of an edge \( e \) we say that the vertex \( v \) is incident on the edge \( e \) and also the edge \( e \) is incident on vertex \( v \).

Definition 1.5.6

An edge of a graph that joins a node to itself is called loop or self-loop.
Definition 1.5.7
In a Multigraph, no loops are allowed but more than one edge can join two vertices. These edges are called Multiple or Parallel edges.

Definition 1.5.8
A graph which has neither self-loops nor parallel edges called simple graph.

Definition 1.5.9
A graph is a weighted graph if a number (weight) is assigned to each edge. Such weights might represent, for example, costs, lengths or capacities, etc. depending on the problem at hand. Some authors call such a graph as network.

Definition 1.5.10
The number of edges incident on a vertex \( V_i \) with self-loops counted twice is called degree of a vertex \( V_i \) and is denoted by \( \text{deg } V_i \) or \( d(V_i) \).

Definition 1.5.11
The maximum of the degrees of all the vertices is called the maximum degree of the graph and it is denoted by \( \Delta(G) \) or \( \Delta \).

The minimum of the degrees of all the vertices is called the minimum degree of the graph and it is denoted by \( \delta(G) \) or \( \delta \).

Definition 1.5.12
A vertex having no incident edge is called isolated vertex.

Definition 1.5.13
Any vertex of degree one is called a pendant vertex.

Definition 1.5.14
A graph without any edge is called a null graph.
Definition 1.5.15

A simple graph in which there exists an edge between every pair of vertices is called a complete graph.

Definition 1.5.16

A clique in an undirected graph \( G = (V, E) \) is a subset of the vertex set \( C \subseteq V \), such that for every two vertices in \( C \), there exists an edge connecting the two. This is equivalent to saying that the subgraph induced by \( C \) is complete.

Definition 1.5.17

A graph \( G \) is called labeled if its \( p \) points are distinguished from one another by names such as \( v_1, v_2, \ldots, v_n \).

Definition 1.5.18

A graph \( G \) in which all vertices are of equal degree is called a regular graph.

Definition 1.5.19

A graph \( G \) is connected if for every \( u, v \in G \) there exists a \( uv \)-path in \( G \). Otherwise \( G \) is called disconnected.

Definition 1.5.20

The maximal connected subgraphs of \( G \) are called its components.

Definition 1.5.21

If \( G \) and \( H \) are two graphs with vertex sets \( V(H), V(G) \) and edge sets \( E(H) \) and \( E(G) \) respectively such that \( V(H) \subseteq V(G) \) and \( E(H) \subseteq E(G) \) then we call \( H \) as a subgraph of \( G \) or \( G \) as a supergraph of \( H \).

Definition 1.5.22

A spanning subgraph is a subgraph containing all the vertices of \( G \).
Definition 1.5.23

If $G$ is a graph with vertex set $V$ and $U$ is a subset of $V$ then the subgraph $G(U)$ of $G$ whose vertex set is $U$ and whose edge set comprises exactly the edges of $E$ which join vertices in $U$ is termed as **induced subgraph** of $G$.

Definition 1.5.24

A *walk* is defined as a finite alternating sequence of vertices and edges which begins and ends with vertices such that no edge appears more than once in a sequence, such a sequence is called a walk or Trail in $G$.

The vertex in which a walk begins is called the initial vertex and the vertex in which a walk ends is called the final vertex of the walk. The initial and final vertices are called **terminal vertices**. Non-terminal vertices of a walk are called its **internal vertices**.

Definition 1.5.25

A walk that begins and ends at the same vertex is called a **closed walk**. A walk that is not closed is called an **open walk**.

Definition 1.5.26

The *Length of the Path* is the number of edges in the path.

Definition 1.5.27

A **closed walk** with at least one edge in which no vertex except the terminal vertices appears more than once is called a *cycle* or *circuit*.

Definition 1.5.28

The *Double star graph* $K_{1,n,n}$ is a tree obtained from the star graph $K_{1,n}$ by attaching a new pendant edges to the existing $n$ pendant vertices.

Definition 1.5.29

Any $K_2$ with $n$ pendant edges attached at each end point is called *Bistar* denoted as $B_{n,n}$.
Definition 1.5.30

A graph is **bipartite** if its vertex set is partitioned into two nonempty subsets $X$ and $Y$ such that each edge of $G$ has one end in $X$ and the other end in $Y$.

Definition 1.5.31

A **complete bipartite graph** is a simple bipartite graph such that two vertices are adjacent if and only if they are in different partite sets. The complete bipartite graph with partite sets of size $m$ and $n$ is denoted $K_{m,n}$.

Definition 1.5.32

The **complement** $\bar{G}$ of a graph $G$ is a graph with the same vertex set as $G$ but with an edge set such that $xy$ is an edge in $\bar{G}$ if and only if $xy$ is not an edge in $G$.

Definition 1.5.33

A cycle that has odd length is an **odd cycle**; otherwise it is an **even cycle**. A graph is **acyclic** if it contains no cycles; unicyclic if it contains exactly one cycle; and **Pancyclic** if it contains cycles of every possible length.

Definition 1.5.34

A path or cycle is Hamiltonian if it uses all vertices exactly once. A graph that contains a Hamiltonian path is **traceable**; and one that contains a Hamiltonian path for any given pair of (distinct) end vertices is a **Hamiltonian connected graph**. A graph that contains a Hamiltonian cycle is a **Hamiltonian graph**.

Definition 1.5.35

Two paths are **internally disjoint** if they do not have any vertex in common, except the first and last ones.

Definition 1.5.36

A **tree** is a connected acyclic simple graph. A vertex of degree 1 is called a **leaf** or **pendant vertex**.
Definition 1.5.37

A trail or circuit is *Eulerian* if it uses all edges precisely once. A graph that contains an Eulerian trail is *traversable*. A graph that contains an Eulerian circuit is an *Eulerian graph*.

Definition 1.5.38

An edge incident to a leaf is a *leaf edge* or *pendant edge*.

Definition 1.5.39

A non-leaf vertex is an *internal vertex*. Sometimes, one vertex of the tree is distinguished, and called the *root*; in this case, the tree is called *rooted*.

Definition 1.5.40

A *Spanning Tree* is a spanning subgraph that is a tree. Every graph has a spanning forest. But only a connected graph has a spanning tree.

Definition 1.5.41

A special kind of tree called a star $K_{1,k}$. An induced star with 3 edges is a *claw*.

Definition 1.5.42

A *caterpillar* is a tree in which all non-leaf nodes form a single path.

Definition 1.5.43

A *k-ary* tree is a rooted tree in which every internal vertex has $k$ children. A 1-ary tree is just a path. A 2-ary tree is also called a *binary tree*.

Definition 1.5.44

A *maximum clique* is a clique of the largest possible size in a given graph. The *clique number* $\omega(G)$ of a graph $G$ is the number of vertices in a maximum clique in $G$. 
The intersection number of $G$ is the smallest number of cliques that together cover all edges of $G$.

**Definition 1.5.45**

Two (or more) sub graphs $G_1$ and $G_2$ of a graph $G$ are said to be *edge disjoint* if $G_1$ and $G_2$ do not have any edges in common.

**Definition 1.5.46**

Two (or more) sub graphs $G_1$ and $G_2$ of a graph $G$ are said to be *vertex disjoint* if $G_1$ and $G_2$ do not have any vertex in common.

**Definition 1.5.47**

A *complete n-ary* tree is special symmetrical tree in which the degree of root vertex is $n$, the degree of vertices on other levels are $n+1$, except those of vertices of last level which are all one.

**Definition 1.5.48**

The *Hoffmann tree* $P_n \bar{O}K_1$ is the graph obtained from a path $P_n$ (whose length is $n-1$) by attaching pendant edge at each vertex of the path.

**Definition 1.5.49**

The *tree* $\{k_1,n;2\}$ is obtained from $n$-bistars by subdividing the middle edge $uv$ with a new vertex $w$. 