Chapter -1

INTRODUCTION TO FUZZY &

INTUITIONISTIC FUZZY SET THEORY
1.1 Introduction

Fuzzy logic is a superset of classic (Boolean) logic that has been extended to handle the concept of partial truth-values between "completely true" and "completely false". As its name suggests, it is derived from fuzzy set theory and it is the logic underlying modes of reasoning which are approximate rather than exact. The importance of fuzzy logic derives from the fact that most modes of human reasoning and especially common sense reasoning are approximate in nature.

In this chapter we present in brief, the theory of fuzzy set introduced by Zadeh [11] and the theory of intuitionistic fuzzy set introduced by Atanassov [7-10].

1.2 Fuzzy Set Theory

The concept of fuzziness as described by Zadeh includes imprecision, uncertainty, and degrees of truthfulness of values. Linguistic variables are used for system input and output, and are represented by words such as "size", "age" and "temperature". A fuzzy set is created to describe linguistic variables in more detail. The linguistic variable "temperature", 
for instance, may have categories (members) of "cold", "very cold",
"moderate", "warm", and "very hot". Once these categories or members
are defined, the fuzzy set is obtained, and a membership function is then
developed for each member in the set.

Fuzzy sets were derived by generalizing the concept of set theory. Fuzzy
sets can be thought of as an extension of classical sets. In a classical set
(or crisp set), the objects in the set are called elements or members of the
set. An element $x$ belonging to a set $A$ is defined as $x \in A$, an element that
is not a member in $A$ is noted as $x \notin A$. A characteristic function or
membership function $\mu_A(x)$ is defined as an element in the universe $U$
having a crisp value of 1 or 0. For every $x \in U$,

$$\mu_A(x) = \begin{cases} 1 & \text{for } x \notin A \\ 0 & \text{for } x \in A \end{cases}$$

This can also be expressed as $\mu_A(x) \in \{0,1\}$

In crisp sets the membership function takes a value of 1 or 0. For fuzzy
sets, the membership function takes values in the interval $[0, 1]$. The
range between 0 and 1 is referred to as the membership grade or degree of membership.

A fuzzy set $A$ is defined as:

$$A = \{ (x, \mu_A(x)) | x \in A, \mu_A(x) \in [0, 1] \}$$

Where $\mu_A(x)$ is a membership function belongs to the interval $[0, 1]$. Fuzzy set theory has equivalent operations to those of crisp set theory. It includes functions such as equality, union and intersection etc.

Membership functions can be defined as the degree of the truthfulness of the proposition. Semantically, probability theory and fuzzy logic use different notions: Probability and Degree of Membership. Probabilities are the likelihoods that an event does or does not occur. Fuzzy Logic models the extent to which an event occurred or can occur. Even though fuzzy membership values and proportional values operate over the same functional range $[0, 1]$, the semantic difference between a fuzzy statement and probabilistic statement is significant.
The whole concept can be illustrated with this example. Let's talk about people and "youthfulness". In this case the set $S$ (the universe of discourse) is the set of people.

A fuzzy subset YOUNG is also defined, which answers the question "To what degree is person x young?" To each person in the universe of discourse, we have to assign a degree of membership in the fuzzy subset YOUNG. The easiest way to do this is with a membership function based on the person's age.

$$\text{young}(x) = \begin{cases} 1, & \text{if } \text{age}(x) \leq 20, \\ \frac{30 - \text{age}(x)}{10}, & \text{if } 20 < \text{age}(x) \leq 30, \\ 0, & \text{if } \text{age}(x) > 30 \end{cases}$$
A graph of this looks like:

![Graph of youth degree](image)

Figure: 1.1

Given this definition, here are some example values:

<table>
<thead>
<tr>
<th>Person</th>
<th>Age</th>
<th>degree of youth</th>
</tr>
</thead>
<tbody>
<tr>
<td>Johan</td>
<td>10</td>
<td>1.00</td>
</tr>
<tr>
<td>Edwin</td>
<td>21</td>
<td>0.90</td>
</tr>
<tr>
<td>Laiba</td>
<td>25</td>
<td>0.50</td>
</tr>
<tr>
<td>Arosha</td>
<td>26</td>
<td>0.40</td>
</tr>
<tr>
<td>Ahmad</td>
<td>28</td>
<td>0.20</td>
</tr>
<tr>
<td>Ravi</td>
<td>83</td>
<td>0.00</td>
</tr>
</tbody>
</table>

So using the definition, we'd say that the degree of truth of the statement "Laiba is YOUNG" is 0.50.
1.3 Types of Membership Function

Zadeh has proposed a series of membership function like Triangular Singleton, L function etc. Depending on the types of function different fuzzy sets are obtained.

1.3.1 Triangular: Defined by its lower limit a, its upper limit b and the modal value m, so that a<m<b. We call the value b-m margin when it is equal to the value m-a.

\[ A(x) = \begin{cases} 
0 & \text{if } x \leq a \\
(x-a)/(m-a) & \text{if } x \in (a, m) \\
(b-x)/(b-m) & \text{if } x \in (b, m) \\
1 & \text{if } x \geq 0 
\end{cases} \]

![Triangular Fuzzy Sets](image1.png)

![Singleton Fuzzy Sets](image2.png)

Figure 1.2: Triangular Fuzzy Sets

Figure 1.3: Singleton Fuzzy Sets
1.3.2 Singleton: It takes value 0 in all the universe of discourse except in the point \( m \), where it takes the value 1. It is the representation of a crisp value.

\[
SG(x) = \begin{cases} 
0 & \text{if } x \text{ not equal to } m \\
1 & \text{if } x = m
\end{cases}
\]

1.3.3 L function: This function is defined by two parameters \( a \) and \( b \) in the following way.

\[
L(x) = \begin{cases} 
1 & \text{if } x \leq a \\
a-x & \text{if } a < x \leq b \\
b-a & \text{if } x > b
\end{cases}
\]

Figure 1.4: L-fuzzy set

Figure 1.5: Trapezoid fuzzy set
1.3.4 Trapezoid Function

Defined by its lower limit $a$ and its upper limit $d$ and the upper and lower limit of its nucleus, $b$ and $c$ respectively.

$$A(x) = \begin{cases} 
0 & \text{if } (x \leq a), (x \geq d) \\
(x-a)/(b-a) & \text{if } x \in (a, b) \\
1 & \text{if } x \in (b, c) \\
(d-x)/(d-c) & \text{if } x \in (b, d) 
\end{cases}$$

In general the trapezoid function adapts quite well to the definition of any concept, with the advantage that it is easy to define, represent and simple to calculate.

1.4 Membership Function Determination

Membership function must be defined carefully to get the best result. Inappropriate membership function can lead to wrong interpretation of the application. There have been different methods to calculate the membership functions. Which method is chosen will depend on the application in question, the manner in which uncertain information is
presented, and how this is to be measured during the experiment. Some of
the given method by Pedrycz and Gomide is presented in this section.

1.4.1 Horizontal Method

It is based on the answers of a group of N expert

➢ The question takes the following form: "can x be considered
compatible with concept A?"

➢ Only Yes and No answer are acceptable, so

\[ A(x) = \frac{\text{Affirmative Answers}}{N} \]

1.4.2 Vertical Method

The aim is to build the several \( \alpha \)-cut, for which several values are
selected for \( \alpha \).

➢ Now the question that is formulated for theses predetermined
\( \alpha \) value is as follows “Can the element of X that belong to A to a
degree that is not inferior to \( \alpha \) be identified?”

➢ From these \( \alpha \)-cut, the fuzzy set A can be identified, using the so
called Identity Principle or Representation Theorem.
1.4.3 Method based on Problem Specification

This method requires a numerical function that should be approximate. The error is defined as a fuzzy set that measure the quality of the approximation.

1.5 Some Concepts about Fuzzy Sets

In this section some of the important concepts about fuzzy sets are defined.

Equal

Let A and B are two fuzzy sets over X, Then A is equal to B if

\[ A = B \iff \mu_A(x) = \mu_B(x) \quad \forall x \in X \]

Included

Taking two fuzzy sets A and B over X, A is said to be included in B if

A is a subset of B such that that x ∈ X, \( \mu_A(x) \leq \mu_B(x) \).

\( \alpha \)-cut

The \( \alpha \)-cut of a fuzzy set A, denoted by \( A_\alpha \) is a classic subset of elements in x whose membership takes a greater or equal value to any specific \( \alpha \) value that universe of this course that complies with
\[ A_{\alpha} = \{ x : x \in X, \mu_{A}(x) \geq \alpha \in [0,1] \} \]

**Representation Theorem**

Representation Theorem states that any fuzzy set \( A \) can be obtained from the union of its \( \alpha \)-cuts.

\[ A = \bigcup \alpha A_{\alpha} \]

Where \( \alpha \in [0,1] \)

**Convex Fuzzy set**

By using the representation theorem the concept of Convex Fuzzy set can be established as that in which all the \( \alpha \)-cut are convex:

\[ \forall x,y \in X, \forall \lambda \in [0,1]: \mu_{A}(\lambda . x + (1-\lambda).y) \geq \min(\mu_{A}(x), \mu_{A}(y)) \]

This definition means that any point situated between two other points will have a higher membership degree than the minimum of these two points.

**Concave Fuzzy set**

The concept of Concave Fuzzy set can be established as that in which all the \( \alpha \)-cut are concave:
∀ x, y ∈ X, ∀ λ ∈ [0, 1]: μₐ(λ·x + (1 - λ)·y) ≤ min(μₐ(x), μₐ(y))

Figure 1.6 : Convex Fuzzy Set  Figure1.7 : Non-convex Fuzzy Set

Cardinality

The cardinality of a fuzzy set A, with finite universe X, is defined as:

Card(A) = Σ μₐ(x) x ∈ X.

1.6 Some basic operation on fuzzy sets

As we know that the fuzzy set theory was generalized from classical set theory so fuzzy set also allows operation like union, intersection and complement.
1.6.1 Union of Fuzzy Set

The union of fuzzy set A and B is denoted by $A \cup B$ and is defined as the smallest fuzzy set that contains both fuzzy set A and fuzzy set B.

The membership function $\mu_{A \cup B}$ of the union $A \cup B$ of the fuzzy sets A and B is defined as follows:

$$\mu_{A \cup B} = \max \{ \mu_A(x), \mu_B(x) \} \text{ for every } x \text{ of } X.$$ 

The symbol $\bigvee$ is often used instead of the symbol max. The union corresponds to the operation “or”.

1.6.2 Intersection of Fuzzy Set

The intersection of fuzzy sets A and B is denoted by $A \cap B$ and defined as the largest fuzzy set contained in both A and B. The intersection corresponds to the operation “and”.

$$\mu_{A \cap B} = \min \{ \mu_A(x), \mu_B(x) \} \text{ for every } x \text{ of } X.$$ 

(where $\land = \max$; $\lor = \min$).

1.6.3 Complementation:

A is complement of B i.e. $A = \overline{B}$ implies
\[ \mu_A(x) = \mu_B(x) = 1 - \mu_B(x) \quad \forall \ x \in X \]

1.7 Fuzzy Relation

A classical relation between two universes \( X \) and \( Y \) is a subset of the cartesian product \( X \times Y \). Like the classical sets, classical relation can be described by using a characteristic function. In the same way, a fuzzy relation \( R \) is a fuzzy set of tuples, where this characteristic function is extended to the interval. In the event of a binary relation, the tuple has two values.

Let \( U \) and \( V \) be two infinite universes and \( \mu_R: U \times V \rightarrow [1,0] \). Then the fuzzy relation \( R \) is defined as:

\[ R = \bigwedge_{u,v} \mu_R(u,v)/(u,v) \]

The function \( \mu_R(x) \) may be used as a similarity or proximity function. Fuzzy relations generalize the generic concept of relation by allowing the notion of partial belonging (association) between points in the universe of discourse.

Example: Takes as an example the fuzzy relation in \( R^2 \) (binary relation), "approximately equal", with the following membership function \( X \) subset of \( R \) with \( X^2 \).
This fuzzy relation may be defined as:

\[ R(x,y) = \begin{cases} 
1 & \text{if } x = y \\
0.8 & \text{if } |x-y| = 1 \\
0.3 & \text{if } |x-y| = 2 
\end{cases} \]

Where \( x, y \in \mathbb{R} \).

The universe of discourse is finite; a matrix notation can be quite useful to represent the relation. So this example would be shown as:

\[
\begin{array}{ccc}
1 & 0.8 & 0.3 \\
0.8 & 1 & 0.8 \\
0.3 & 0.8 & 1 \\
\end{array}
\]

**Operation and Composition of Fuzzy Relation**

Let \( R \) and \( W \) be two fuzzy relations defined in \( X \times Y \):

**Union:** \((R \cup W)(x,y) = R(x,y) \circ W(x,y)\), using a \( s \)-norm.
Intersection: \((R \cup W)(x,y) = R(x,t) \cap W(x,y)\), using a t-norms.

Complement: \((\sim R)(x,y) = 1 - R(x,y)\)

Inclusion: \(R\) is a subset of \(W\) such that \(R(x,y) \leq W(x,y)\).

Equality: \(R = W\) such that \(R(x,y) = W(x,y)\)

1.8 Fuzzy Number

The concept of fuzzy number was first introduced by Zadeh with the purpose of analyzing and manipulating approximate numeric values for example "near to 0", "almost 5". Later on Dubois and Prade redefined this concept.

Let \(A\) be a Fuzzy set in \(X\) and \(\mu_A(x)\) be its membership function with \(x\) belongs to \(X\). \(A\) is a fuzzy number if its membership function satisfies that:

1. \(\forall x,y \in X, \forall \mu_A(t) \geq \min(\mu_A(x), \mu_A(y))\), i.e. \(\mu_A(x)\) is convex.

2. \(\mu_A(x)\) is upper semi continuity

3. Support of \(A\) is bounded.

These requirements can be relaxed. Some author includes the necessity for the fuzzy set being normalized in the definition.
1.9 Composition of Fuzzy Relations

Fuzzy relations in different product spaces can be combined with each other by the operation "composition". Different versions of "composition" have been suggested, which differ in their result and also with respect to their mathematical properties. The max-min composition has become the best known and the most frequently used one.

1.9.1 Max-Min composition: Let $R_1(x,y), (x,y) \in X \times Y$ and $R_2(y,z), (y,z) \in Y \times Z$ be the two fuzzy relations. The max-min combination $R_1 \text{ max-min } R_2$ is the fuzzy set.

$$R_1 \circ R_2 = \{[(x,y), \max\{\min\{\mu_{R_1(x,y)}, \mu_{R_2(y,z)}\}\}] \mid x \in X, y \in Y, z \in Z\}$$

1.10 Intuitionistic Fuzzy Set Theory (IFS)

Intuitionistic fuzzy set (IFS), developed by Atanassov is a powerful tool to deal with vagueness. A prominent characteristic of IFS is that it assigns to each element a membership degree and a non-membership degree, and thus, the IFS constitutes an extension of Zadeh’s fuzzy set, which only assigns to each element a membership degree. In the last two decades, many authors have paid attention to the IFS theory has been successfully
applied in different areas such as; logic programming [21-22], decision making problems, medical diagnosis etc. Recently various applications of IFS to artificial intelligence have appeared - intuitionistic fuzzy expert systems, intuitionistic fuzzy neural networks, intuitionistic fuzzy decision making, intuitionistic fuzzy machine learning, intuitionistic fuzzy semantic representations etc.

**Intuitionistic fuzzy sets:** Let a set $E$ be fixed. An IFS $A$ in $E$ is an object of the following form:

$$A = \{ (x, \mu_A(x), v_A(x)) \mid x \in E \},$$

When $v_A(x) = 1 - \mu_A(x)$ for all $x \in E$ is ordinary fuzzy set.

In addition, for each IFS $A$ in $E$, if

$$\pi_A(x) = 1 - \mu_x - v_x$$

Then $\pi_A(x)$ is called the degree of indeterminacy of $x$ to $A$, or called the degree of hesitancy of $x$ to $A$.

Especially, if $\pi_A(x) = 0$, for all $x \in E$ then the IFS, $A$ is reduced to a fuzzy set.
1.10 Some basic operation on Intuitionistic Fuzzy sets

1. $A \subseteq B \iff (\forall x \in E) (\mu_A(x) \leq (\mu_B(x) \land \nu_A(x)) \geq \nu_B(x))$

2. $A=B \iff A \subseteq B \land B \subseteq A$

3. $\bar{A} = \{ (x, \mu_A(x) \land \nu_A(x)) \mid x \in E \}$

4. $A \lor B = \{ <x, \min(\mu_A(x), \mu_B(x)), \max(\nu_A(x), \nu_B(x))> \mid x \in E \}$

5. $A \land B = \{ <x, \max(\mu_A(x), \mu_B(x)), \min(\nu_A(x), \nu_B(x))> \mid x \in E \}$

6. $A + B = \{ <x, \mu_A(x) + \mu_B(x) - \mu_A(x) \cdot \mu_B(x)), \nu_A(x) \cdot \nu_B(x) > \mid x \in E \}$

7. $A \cdot B = \{ <x, \mu_A(x) \cdot \mu_B(x), \nu_A(x) + \nu_B(x) - \nu_A(x) \cdot \nu_B(x) > \mid x \in E \}$

(where $\land = \max$; $\lor = \min$).

1.11 Basic Definitions and Operations of Intuitionistic Fuzzy Relational Calculus

In this section we quote the concept of IFR, mainly from Intuitionistic Fuzzy Relational Calculus: an Overview by Chris Cornelis, Glad
Deschrijver, Martine De Cock and Etienne Kerre. First we define and position the notion of IF relations with respect to IFS and fuzzy set theory. Next we consider the elementary operations on IF relations e.g. composition etc. Finally we list some potential properties of IF relations.

Formally, an IF relation $R$ between (not necessarily all distinct) universes $U_1, U_2, \ldots, U_n$ ($n \geq 1$) is an IFS in the cartesian product $U_1 \times \ldots \times U_2 \ldots \times U_n$. Occasionally, we will use the abbreviation IFR. If $U = U_1 \ldots U_n$, $R$ is called an n-ary relation in $U$. Because an IF relation is a special case of an IFS, all operations on IFS's can be applied to IF relations: intersection, union, complement, symmetrical difference etc.

The strength of relations as information models in knowledge based system derives from their fundamental ability to describe observed or predicted "connections", expressed as facts or rules, between selected objects of discourse. Crisp relations like $\subseteq$, $\in$, $=$, $\ldots$ have served well in developing rigorous mathematical frameworks.

The use of fuzzy relations originated from the observation that real life objects can be related to each other to a certain degree (just like elements can belong to a fuzzy set to a certain degree); in this sense they are able to
model vagueness. They are still intolerant of uncertainty, however, since there is no means of attributing reliability or confidence information to the membership degrees. Various frameworks have been developed to deal with this imperfection, amongst others. Here we will concentrate on the IF approach. A possible semantics for IF relations taking inspiration from classical possibility theory was presented in [23-24]. Basically, the idea is to treat an IF relation as an elastic restriction that allows us to discriminate between the more or less plausible values for a variable. For instance, a statement like "John is old" does not allow us to infer John's exact age, yet provides some support in favor of the older ages (allowing that those ages are, to a given extent, possible for him), as well as negative evidence against the younger ones (expressing some certainty or necessity that those ages can't in fact be his). We model this observation by indicating how much the original condition "John is old" needs to be stretched in order for John's age to assume this particular value: we assign two separate $[0, 1]$ valued degrees $\mu_A(u)$ & $\nu_A(u)$ to every age $u$ in the considered domain, the first one indicating the possibility that the john's age assumes this particular value and the second one reflecting our
certainty that differ from the given value \( u \). In classical possibility theory, symmetry between the two indexes is imposed. However, in a sense that from knowledge that it is impossible that John is 25 year old \( \mu(25)=0 \), we immediately derive that it is completely certain that he is not 25 i.e \( \upsilon(25) = 1 \), and more generally from \( \mu(u) = \alpha \) follows \( \upsilon(u) = 1 - \alpha \). Taken together, the various degrees \( \mu(u) \) give rise to a fuzzy set. But what if we cannot be sure that the observer is fully credible?

In other words, we can have varying degrees of trust in an observer, ranging from unconditional in confidence to full creditworthiness, and we should be able to model that trust accordingly; which can be done conveniently by letting the certainty degree \( \upsilon(u) \) range between 0 and 1- \( \mu(u) \). This justifies the use of a more general intuitionistic fuzzy, rather than a fuzzy, relation as a model of describing observations.

**1.12 Intuitionistic fuzzy relation**

Let \( X \) and \( Y \) are two sets. An Intuitionistic fuzzy relation (IFR) \( R \) from \( X \) to \( Y \) is an IFS of \( X \times Y \) characterized by the membership function \( \mu_R \) and non-membership function \( \upsilon_R \). An IFR \( R \) from \( X \) to \( Y \) will be denoted by \( R(X \rightarrow Y) \).
\( R = \{ (x, y), \mu_R (x, y), \nu_R (x, y) \mid x \in X \land y \in Y \} \)

Where \( \mu_R : X \times Y \rightarrow [0,1] \) and \( \nu_R(x) : X \times Y \rightarrow [0,1] \) satisfy the condition

\[ 0 \leq \mu_R (x,y) + \nu_R (x,y) \leq 1 \]

For every \((x,y) \in X \times Y\).

1.12.1 MAX-MIN-MAX Composition

If \( A \) is an IFS of \( X \), the max-min-max composition of the IFR \( R(X \rightarrow Y) \) with \( A \) is an IFS \( B \) of \( Y \) denoted by \( B = R \circ A \) and is defined by the membership function:

\[ \mu_{R \circ A}(y) = \vee [\mu_A(x) \land \mu_R(x,y)] \]

and the non-membership function

\[ \nu_{R \circ A}(y) = \land [\nu_A(x) \lor \nu_R(x,y)] \]

(where \( \land \) =max \( \lor \) =min).

Let \( Q(X \rightarrow Y) \) and \( R(Y \rightarrow Z) \) be two IFRs. The max-min-max composition \( R \circ Q \) is the intuitionistic fuzzy relation from \( X \) to \( Z \), defined by the membership function

\[ \mu_{R \circ Q}(y) = \vee [\mu_Q(x, y) \land \mu_R(y,z)] \]

and the non-membership function

\[ \nu_{R \circ Q}(y) = \land [\nu_Q(x,y) \lor \nu_R(y,z)] \]
(where $\land = \text{max}$; $\lor = \text{min}$).

Proposition: If $R$ and $S$ are two IFRs on $X \times Y$ and $Y \times Z$, respectively, then

(i) $(R^{-1})^{-1} = R$

(ii) $(S \circ R)^{-1} = R^{-1} \circ S^{-1}$

1.12.2 Some properties of Intuitionistic Fuzzy Relation

**Reflexive**

A IF relation $R (X \times Y)$ is said to be reflexive if

$$\forall x_1, x_2 \in X, \mu_R (x, x) = 1$$

**Symmetric:** iff $x_1, x_2 \in X$,

$$\mu_R (x_1, x_2) = \mu_R (x_2, x_1) \text{ and}$$

$$\nu_R (x_1, x_2) = \nu_R (x_2, x_1)$$

**Transitive:** iff $R^2$ is subset of $R$

Where $R^2 = R \circ R$

**Transitive Closure of an IFR:**

The transitive closure of an IFR $R$ on $X \times X$ is $R$ defined by

$$R = R^1 \cup R^2 \cup R^3 \cup \ldots$$
\( \alpha \)-Cut of an IFS:

Let \( A \) be an IFS of set \( E \). For \( \alpha \) belongs to \([0,1]\), then \( \alpha \) cut of \( A \) is the crisp set \( A \), defined by \( A_\alpha = \{ x : x \in E, \text{ either } \mu_A(x1) \geq \alpha \text{ or } \nu_A \leq 1 - \alpha \} \).

It may be noted that the condition \( \mu_A(x1) \geq \alpha \) either \( \nu_A \leq 1 - \alpha \) but not conversely.

**Intuitionistic fuzzy Tolerance Relation and Intuitionistic Fuzzy Equivalence Relation**

An intuitionistic fuzzy relation \( R \) on the Cartesian product \( (X \times X) \), is called:

i) An intuitionistic fuzzy tolerance relation if \( R \) is reflexive and symmetric.

ii) An intuitionistic fuzzy similarity (or intuitionistic equivalence) relation if \( R \) is reflexive, symmetric and transitive.

**\( \alpha \) similar Elements:**

If \( T \) be an intuitionistic fuzzy tolerance relation on \( X \), then given an \( \alpha \) belongs to \([0,1]\), two elements \( x, y \) belongs to \( X \) are \( \alpha \)-similar (denoted by \( x \ T_\alpha \ y \)) if and only if \( \nu_T(x,y) \leq 1 - \alpha \).
α-Tolerate Elements and Crisp relation $T^\alpha$: 

If $T$ be an intuitionistic fuzzy tolerance relation on $X$, then given an $\alpha$ belongs to $[0, 1]$, two elements $x, z$ belongs $X$ are $\alpha$ tolerate (denoted by $x T^\alpha_z$) if and only either $x T^\alpha_z$ or there exists a sequence $y_1, y_2, \ldots, y_r$ belongs to $X$ such that $x T^\alpha_{y_1} T^\alpha_{y_2} \ldots T^\alpha_{y_r} T^\alpha_z$.

**Lemma**

If $T$ be an intuitionistic fuzzy tolerance relation on $X$ then $T^\alpha$ is an equivalence relation. For any $\alpha$ belongs to $[0,1]$, $T^\alpha$ partitions $X$ into disjoint equivalence classes. If $T$ is an intuitionistic fuzzy similarity relation on $X$ then $T^\alpha$ is an equivalence relation for any $\alpha$ belongs to any $[0,1]$.

**Lemma**

Let $T$ be an intuitionistic fuzzy similarity relation on $X$ and $\alpha$ belongs to $[0, 1]$ be fixed. A set $Y$ is a subset of $X$ is an equivalence class in the partition determined by $T_\alpha$ with respect to $T$ if and only if $Y$ is a maximal sunset obtained by merging elements from $X$ that satisfy
Max[ν_T(x, y)] ≤ 1 - α x, y belongs to Y.

**Lemma**

If T is an intuitionistic fuzzy similarity relation on X, then for \( α \in [0, 1] \), \( T_α \) and \( T'^α \) generate identical equivalence classes.

**Lemma**

The transitive closure \( T \) of an intuitionistic fuzzy tolerance relation T is the minimal intuitionistic fuzzy similarity relation containing T.

1.12.3 Applications of Intuitionistic Fuzzy Relational Calculus

IF relations as explained, have laid the foundations of a kind of calculus that is flexible and less restricted as the interpretation of IFR's on the semantically level allows us to reason with statements that involve imprecision. Approximate reasoning, then, is a domain of research that attempts to implement this calculus in the solution of everyday problems that cannot be handled adequately by precise techniques because they are either too complex or do not require the precision of an exact, crisp
method. Nowadays, people start to accept such “fuzzy systems” as flexible and convenient tools to solve a numerous of ill-defined but otherwise (for humans) straightforward tasks example controlling fluid levels in a reactor, automatically lens focusing in cameras, adjusting an aircraft's navigation to the change of winds, etc. The next step is to try and meet more challenging requirements (e.g. aspects of logical consistency; incorporation of varying facets of imprecision) in order to implement a successful artificial reasoning unit. In terms of trade-off between efficiency and expressiveness, IF relations proved very well (compared to fuzzy relations with or without certainty factors on one hand, and second-order fuzzy relations on the other hand), and therefore seem to fit the challenges well.

1.13 Conclusion

This chapter gives brief introduction of Fuzzy set theory along with some basic operations applicable on it. We have also summarized different types of membership function and membership function determination methods.
In the second part of this chapter Intuitionistic fuzzy set theory (IFS) is defined with detailed explanation of different operation applicable on it. We have also focused on Intuitionistic Fuzzy Relational Calculus as a framework for processing imprecise data represented as IF relations. We have also summarized different operation and properties of IF relation e.g. Max–Min–Max composition of IF relation. At the end of this section application of IFR calculus is discussed for proper understanding.