CHAPTER 4

SORET DUFOUR DRIVEN INSTABILITY OF AN OLDROYDIAN FLUID IN A HORIZONTAL POROUS MEDIUM

4.1 Introduction
4.2 Physical Problem and Its Mathematical Formulation
   4.2.1 Basic State
   4.2.2 Perturbed State
4.3 Mathematical Analysis and Dispersion Relation
4.4 Case I: Isotropic Porous Medium
4.5 Case II: Anisotropic Porous Medium
   4.5.1 Stationary Convection
   4.5.2 Oscillatory Convection
4.6 Results And Discussion
4.7 Conclusion
4.1 Introduction

In the presence of two diffusive components say, heat and salt, convection depends on solution concentration and thermal stratification as well. This thermosolutal convection appears frequently in oceanography, limnology and engineering. Due to the difference in the diffusion rates of two diffusive agents cross diffusion effects are produced which are observed in many natural phenomena and required in many other situations like in chemical industry for chemical transport in packed bed reactors, in reservoir engineering in connection with thermal recovery process, in hot and salty springs of a sea, in underground spreading of chemical waste and other pollutants, in migration of moisture through air contained in fibrous insulation, in metallurgy, in food processing, in grain storage, in the production of pure medication, in solidification of molten alloy etc.

It was first observed by Ludwig (1856) that when a solution heated from above is maintained for some time in a temperature gradient; the upper part being warmer than the lower part and a difference in concentration is set up. Thus, the flux of salt due to salinity gradient is also affected by temperature gradient. This phenomenon is known as thermophoresis. In 1879, Swiss scientist C. Soret explored it in liquids, therefore, it is also named as Ludwig Soret effect. Themodiffusion is labeled ‘positive’ when particles move from a hot to cold region and ‘negative’ when the reverse is true. Typically, the heavier/larger species in a mixture exhibits positive thermophoretic behavior while the lighter/smaller species exhibit negative behavior. In addition to the sizes of the different types of particles and the steepness of the temperature gradient, the heat conductivity and heat absorption of the particles play an important role. Recently, Duhr and Braun (2006) and Reineck et al. (2010) have suggested that the charge and entropy of the hydration shell of molecules
play a major role for the thermophoresis of biomolecules in aqueous solution. Inverse phenomenon of thermophoresis is called diffusion-thermo, i.e., heat flux is affected by mass concentration in addition of being a function of temperature gradient. This effect was first discovered by Dufour (1873). It was during World War-II that interest in Dufour effect rose when Clusius and Waldmann (1942) got interested in determining thermal expansion factor for gases. The Dufour effect is important in liquid also. In addition, the temperature variations could cause complications and Dufour effect can be used to verify the heat matter Onsagar reciprocal relation. In case of flow of fluid through porous medium, both the fluid and the solid region are responsible for the transfer of heat whereas the solute transfers by diffusion and convection only through the fluid region. Therefore, the properties of the porous medium can affect the transient flow behavior even when the thermal and solutal diffusivities are equal.

Thermosolutal convection in continuous as well as porous medium, with and without Soret-Dufour effects so far are considered to be identical by converting the equations and boundary conditions of Soret-Dufour thermosolutal convection problems into those for thermosolutal convection without these effects with the help of linear transformation [Knobloch (1980)]. But the fact is that Soret effect itself is of great importance in achieving difficult purifications in isomeric substance of various types such as, in mixture between gases with very light molecular weight, such as N₂ or He and of medium molecular weight such as H₂ or air. Soret effect is also important to predict the composition profile of oil fields. Similarly, Eckert and Drake (1972) have described several cases of Dufour effect of considerable magnitude. A lot of literature is available on thermosolutal convection in Newtonian as well as non-Newtonian fluids. However, its counter parts Soret-Dufour effect has received little attention.
The Dufour driven thermosolutal convection was first analyzed by Veronis (1965) while Soret-Dufour driven thermosolutal convection was first analyzed by Tewfik and Yang (1963) and then by Sparrow et al. (1964). Ruddraiah and Malashetty (1986) have investigated the influence of coupled molecular diffusion on double diffusive convection in a porous medium. Soret-Dufour effect on the natural convection in heat and mass transfer in a cavity due to combined horizontal temperature and the concentration gradient has been examined by Weaver and Viskanta (1991). They pointed out that when the differences of the temperature and the concentration were large or the difference of the molecular mass of the two elements in a binary mixture was great, the coupled interaction was significant. They established the important result that the total mass flux through the cavity due to Soret effect can be as much as 10-15% and energy transfer due to Dufour effect can be of appreciable magnitude compared to heat conduction. Malashetty and Gaikwad (2002) have investigated the effect of the cross-coupled diffusion in a system with horizontal temperature and concentration gradients. Postelnicu (2004) has studied the heat and mass transfer by natural convection from a vertical flat plate embedded in an electrically conducting fluid saturated in a porous medium. Kim et al. (2004) have analyzed Soret-Dufour effect on convective instabilities in nano fluids for a Darcy Boussinesq model. The Dufour effect on entropy generation in double diffusive convection has been considered by Magherbi et al. (2007). Gaikwad et al. (2007) have studied the Soret-Dufour effects on the onset of double diffusive convection in a couple stress fluid layers under the linear and non-linear stability models. Narayana and Murthy (2007, 2008) have investigated combined Soret-Dufour effect on free convection heat and mass transfer from a horizontal plate in a Darcy porous medium as well as in doubly stratified Darcy porous medium. Awad et al. (2010) have made the stability analysis, in a highly porous medium saturated with Maxwell fluid using Darcy Brinkman model on double diffusive
convection in the presence of Soret and Dufour effects. **Wang and Tan (2011)** have extended their study of stability analysis of a Maxwell fluid in a porous medium to the Soret-driven double diffusive convection for viscoelastic fluid by using modified Maxwell-Darcy model. **Malashetty et al. (2012)** using the linear and non-linear stability models, have analyzed the double diffusive convection in a couple stress fluid saturated porous layer with Soret effects. **Goyal and Jaimala (2012)** have investigated the stability of thermosolutal convection in a micropolar fluid saturated in a porous layer with Soret-Dufour effect. **Jaimala and Goyal (2012)** have also studied the double diffusive convection of Darcy-Maxwell fluid with Soret and Dufour effects.

As far as double diffusive convection in anisotropic porous media are concerned there are very few studies available on the cross diffusion effects in Newtonian as well as non-Newtonian fluids. **Patil and Subramanian (1992)** have examined the effect of temperature dependent viscosity on double diffusive convection in an anisotropic porous layer in the presence of Soret coefficient. **Gaikwad et al. (2009)** have studied the problem of double diffusive convection in a horizontal anisotropic porous layer, heated and salted from below in the presence of Soret coefficient by using the linear and non-linear stability models under the Boussinesq approximation. **Srivastava et al. (2012)** have investigated the thermal instability of an electrically conducting Boussinesq fluid in an anisotropic porous layer in the presence of Soret coefficient.

However, effect of anisotropy on double diffusive convection in the presence of cross diffusion effect in Oldroydian fluids despite its importance in many practical problems is not investigated so far. Therefore, the main object of the present Chapter is to study the combined effect of Soret-Dufour coefficients and anisotropy parameter on the onset of double diffusive convection in a viscoelastic (Oldroyd) fluid saturated porous layer.
4.2 Physical Problem and Its Mathematical Formulation

An Oldroydian viscoelastic fluid saturated in an anisotropic porous medium of thickness ‘d’ with permeability \( K \left[ k_x \left( i\hat{i} + j\hat{j} \right) + k_z \left( k\hat{k} \right) \right] \) pervaded by the Soret and Dufour effects, is considered. The fluid layer is bounded by two horizontal isothermal boundaries with a constant temperature difference \( \Delta T \) across the thickness and heated from below. The system is also subjected to solutal convection with lower boundary at concentration \( C_1 \) and the upper boundary at a lower concentration \( C_2 \) such that a constant concentration difference \( \Delta C \) exists across the system.

For the model under consideration the required fundamental equations, under the Boussinesq approximation, for non-Newtonian fluid [Yoon et al. (2004), Alisaev and Mirzadjanzade (1975)] concerning the Darcy’s law, are

\[
\nabla \cdot \mathbf{q} = 0, \quad \ldots(4.1)
\]

\[
\left( 1 + \frac{\partial}{\partial t} \right) \mathbf{q} = -\frac{K}{\mu} \left( 1 + \lambda \frac{\partial}{\partial t} \right) \left[ \nabla p + k_i \rho g \right], \quad \ldots(4.2)
\]

\[
\left( \frac{\partial}{\partial t} + \mathbf{q} \cdot \nabla \right) T = D_{11} \nabla^2 T + D_{12} \nabla^2 C, \quad \ldots(4.3)
\]

\[
\left( \frac{\partial}{\partial t} + \mathbf{q} \cdot \nabla \right) C = D_{21} \nabla^2 T + D_{22} \nabla^2 C \quad \ldots(4.4)
\]

and

\[
\rho = \rho_0 \left[ 1 - \alpha (T - T_i) + \alpha' (C - C_i) \right], \quad \ldots(4.5)
\]

where \( D_{11} \) is thermal diffusivity, \( D_{22} \) is solutal diffusivity, \( D_{12} \) (Dufour coefficient) provides heat flux due to concentration gradient and \( D_{21} \) (Soret coefficient) provides mass flux due to temperature gradient. Other symbols have their same meaning as in Chapter-3. It is assumed that the permeability is
same in both x and y directions in a plane parallel to horizontal boundary, it, however, changes in the direction of gravity.

4.2.1 Basic State

The basic state of the system is given by

\[ \begin{align*} \mathbf{q} &= (0, 0, 0), \quad p = p(z), \quad \rho = \rho(z), \quad T = T(z), \quad \text{and} \quad C = C(z), \end{align*} \]

such that the fundamental equations (4.1)-(4.5) for the basic state to be mathematically consistent, requires that

\[ \begin{align*} \frac{\partial p}{\partial x} &= 0, \\
\frac{\partial p}{\partial y} &= 0, \quad \frac{\partial p}{\partial z} + \rho g &= 0, \end{align*} \]

\[ \nabla^2 T = 0 \Rightarrow T - T_i = -\beta z \]

and

\[ \nabla^2 C = 0 \Rightarrow C - C_i = -\beta' z, \]

where \( \beta = \left( \frac{\Delta T}{d} \right) \) and \( \beta' = \left( \frac{\Delta C}{d} \right) \), both are positive.

4.2.2 Perturbed State

Let \( \mathbf{q}'(u', v', w') \), \( p' \), \( \rho' \), \( \theta' \) and \( \gamma' \) be the infinitesimally small disturbances in velocity, pressure, density, temperature and concentration respectively so that the perturbed state is given by

\[ \begin{align*} \mathbf{q} &= (0, 0, 0) + \mathbf{q}'(u', v', w'), \quad p = p(z) + p', \quad \rho = \rho(z) + \rho', \\
T &= T(z) + \theta' \quad \text{and} \quad C = C(z) + \gamma', \end{align*} \]

where a prime indicates disturbance (also called perturbation).
Chapter 4: Soret Dufour Driven Instability of an Oldroydian Fluid in a Horizontal…

Assuming that the perturbed state given by (4.10) satisfies the fundamental equations (4.1)-(4.5) as does the basic state, the following equations are obtained in fluctuations:

\[
\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0, \quad \ldots (4.11)
\]

\[
\left(1 + \varepsilon \frac{\partial}{\partial t}\right) u' = -\frac{k_x}{\mu} \left(1 + \frac{\partial}{\partial x}\right) \frac{\partial p'}{\partial x}, \quad \ldots (4.12)
\]

\[
\left(1 + \varepsilon \frac{\partial}{\partial t}\right) v' = -\frac{k_y}{\mu} \left(1 + \frac{\partial}{\partial y}\right) \frac{\partial p'}{\partial y}, \quad \ldots (4.13)
\]

\[
\left(1 + \varepsilon \frac{\partial}{\partial t}\right) w' = -\frac{k_z}{\mu} \left(1 + \frac{\partial}{\partial z}\right) \left[ \frac{dp'}{dz} + gp' \right], \quad \ldots (4.14)
\]

\[
\frac{\partial \theta'}{\partial t} - \beta w' = D_1 \nabla^2 \theta' + D_{12} \nabla^2 \gamma', \quad \ldots (4.15)
\]

\[
\frac{\partial \gamma'}{\partial t} - \beta' w' = D_{21} \nabla^2 \theta' + D_{22} \nabla^2 \gamma' \quad \ldots (4.16)
\]

and \( \rho' = -\rho_0 (\alpha \theta' - \alpha' \gamma') \). \quad \ldots (4.17)

4.3 Mathematical Analysis and Dispersion Relation

Under the normal mode analysis, we assume the time-dependent periodic disturbances of the form

\[
[w', p', \theta', \gamma'] = [w(z), p(z), \theta(z), \gamma(z)] e^{(ia_x x + ia_y y + nt)}, \quad \ldots (4.18)
\]

where \( a_x \) and \( a_y \) are the real wave numbers in the x and y directions respectively, \( n \), in general, is complex so that \( n = n_r + i n_i \).

Substituting (4.18) in equations (4.11)-(4.17), the stability governing equations of the system are given by
\[
Dw = \left( ia_x u + ia_y v \right), \quad \ldots (4.19)
\]

\[
(1 + \varepsilon n) u = -\frac{k_x}{\mu} \left( 1 + \kappa n \right) ia_x p, \quad \ldots (4.20)
\]

\[
(1 + \varepsilon n) v = -\frac{k_x}{\mu} \left( 1 + \kappa n \right) ia_y p, \quad \ldots (4.21)
\]

\[
(1 + \varepsilon n) w = -\frac{k_x}{\mu} \left( 1 + \kappa n \right) \left[ \frac{dp}{dz} - \rho_0 (\alpha \theta - \alpha' \gamma) g \right], \quad \ldots (4.22)
\]

\[
n \theta - \beta w = D_{11} \left(D^2 - a^2\right) \theta + D_{12} \left(D^2 - a^2\right) \gamma, \quad \ldots (4.23)
\]

\[
n \gamma - \beta' w = D_{21} \left(D^2 - a^2\right) \theta + D_{22} \left(D^2 - a^2\right) \gamma, \quad \ldots (4.24)
\]

where \( D = \frac{d}{dz} \) and \( a = \sqrt{a_x^2 + a_y^2} \).

Eliminating various physical quantities from equations (4.19)-(4.22) in favour of \( w \), we obtain

\[
(1 + \varepsilon n) \left(D^2 - K_i a^2\right) w = -\frac{k_x \rho_0 g}{\mu} \left( 1 + \kappa n \right) (\alpha \theta - \alpha' \gamma) a^2 , \quad \ldots (4.25)
\]

where \( K_i = \frac{k_x}{k_z} \).

Employing the following non-dimensional parameters

\[
(x^*, y^*, z^*) = \left( \frac{x}{d}, \frac{y}{d}, \frac{z}{d} \right), \quad a^* = \frac{a}{l/d}, \quad \sigma = \frac{\eta d^2}{D_{11}}, \quad D^* = \frac{D}{l/d}, \quad k_x^* = \frac{k_x}{d^2},
\]

\[
k^*_x = \frac{k_x}{d^2}, \quad w^* = \frac{w}{U_0}, \quad \theta^* = \frac{\theta D_{11}}{\beta d^2 U_0} \quad \text{and} \quad \gamma^* = \frac{\gamma D_{22}}{\beta' d^2 U_0}, \quad \ldots (4.26)
\]

where \( U_0 \) is the dimension of velocity and removing asterisks, equations (4.25), (4.23) and (4.24), in the non-dimensional form respectively become

\[
(1 + \varepsilon \sigma) \left(D^2 - K_i a^2\right) w = -a^2 Ra_{Dx} \left( 1 + \lambda \sigma \right) \left( \theta - N \gamma \right), \quad \ldots (4.27)
\]
\[
\left( D^2 - a^2 - \sigma \right) \theta + \text{Le} \text{Le}_D \left( D^2 - a^2 \right) \gamma = -w 
\]
\[\text{...(4.28)}\]
and
\[
\left( D^2 - a^2 - \sigma \text{Le} \right) \gamma + \text{Le}^{-1} \text{Le}_S \left( D^2 - a^2 \right) \theta = -w ,
\]
\[\text{...(4.29)}\]

where, \( \text{Le} = \frac{D_{11}}{D_{22}} \) (Lewis number)

\( \text{Le}_D = \frac{D_{12} \beta'}{D_{11} \beta} \) (Dufour number)

\( \text{Le}_S = \frac{D_{21} \beta}{D_{22} \beta'} \) (Soret number)

\( \text{Ra}_{Dx} = \frac{k \gamma \alpha \beta d^4}{D_{11} \nu} \) (Darcy Rayleigh number)

\( \lambda = \frac{D_{11}}{d^2} \lambda \) (non-dimensional relaxation parameter)

\( \varepsilon = \frac{D_{11}}{d^2} \varepsilon \) (non-dimensional retardation parameter)

\( N = \frac{\alpha \beta'}{\alpha \beta} \) (buoyancy ratio)

and \( v = \frac{\mu}{\rho_0} \) (kinematic viscosity).

Both the boundaries are taken free from viscous stresses and therefore, the appropriate non-dimensional boundary conditions are

\[
w = D^2 w = 0 \ \text{at} \ z = 0, 1. \ \text{...(4.30)}
\]

Further, the conditions \( \theta = 0, \gamma = 0, \ w = 0 \ \text{at} \ z = 0, 1 \) provide

\[
D^4 w = 0 \ \text{at} \ z = 0, 1. \ \text{...(4.31)}
\]

The eigen value problem given by equations (4.27)-(4.29) with boundary conditions (4.30) and (4.31) involving \( \text{Ra}_{Dx}, \text{Rs}_{Dx}, a, \varepsilon, \lambda, \text{Le}, K_i, \)
Le\textsubscript{D}, Le\textsubscript{s} and $\sigma$ as parameters, is solved upon assuming that amplitudes $w(z), \theta(z)$ and $\gamma(z)$ can be expressed as

$$[w(z), \theta(z), \gamma(z)] = [w_0, \theta_0, \gamma_0] \sin (m\pi z) \text{ for } m = 1, 2, 3 \ldots \quad (4.32)$$

Substituting equation (4.32) into equations (4.27)-(4.29), we obtain

$$w_0 - \left[ (m^2 \pi^2 + a^2) + \sigma \right] \theta_0 - \text{Le} \text{Le}_D \left( m^2 \pi^2 + a^2 \right) \gamma_0 = 0 \quad \ldots (4.34)$$

and

$$w_0 - \text{Le}^{-1} \text{Le}_S \left( m^2 \pi^2 + a^2 \right) \theta_0 - \left[ (m^2 \pi^2 + a^2) + \text{Le} \sigma \right] \gamma_0 = 0, \quad \ldots (4.35)$$

where $Rs_{Dx} = N \text{Le Ra}_{Dx} = \frac{k \rho C' d^4}{D_{22} \nu}$ (solutal Darcy Rayleigh number).

Equations (4.33)-(4.35) constitute a system of three linear homogeneous algebraic equations in three variables $w_0, \theta_0$ and $\gamma_0$ and for a non-trivial solution, we must necessarily have

$$\begin{vmatrix}
(1+\varepsilon \sigma)(m^2 \pi^2 + K_i a^2) & -(1+\lambda \sigma) a^2 R_{Dx} & (1+\lambda \sigma) a^2 R_{Sx} \\
1 & -\left[ (m^2 \pi^2 + a^2) + \sigma \right] & -\text{Le} \text{Le}_D \left( m^2 \pi^2 + a^2 \right) \\
1 & -\text{Le}^{-1} \text{Le}_S \left( m^2 \pi^2 + a^2 \right) & -\left[ (m^2 \pi^2 + a^2) + \text{Le} \sigma \right]
\end{vmatrix} = 0 \quad \ldots (4.36)$$

On simplification, we get the following cubic equation in $\sigma$:

$$A_4 \sigma^3 + A_3 \sigma^2 + A_2 \sigma + A_1 = 0, \quad \ldots (4.37)$$

where $A_4 = \varepsilon \text{Le} \left( m^2 \pi^2 + K_i a^2 \right), \quad \ldots (4.38)$
Chapter 4: Soret Dufour Driven Instability of an Oldroydian Fluid in a Horizontal…

\[ A_2 = \left( m^2 \pi^2 + K, a^2 \right) \left\{ Le + \varepsilon (1 + Le) \left( m^2 \pi^2 + a^2 \right) \right\} - \lambda a^2 \left( Ra_{Dx} Le - Rs_{Dx} \right), \]

\[ A_3 = \left( m^2 \pi^2 + a^2 \right) \left( m^2 \pi^2 + K, a^2 \right) \left\{ (1 + Le) + \varepsilon (1 - Le, Le_s) \left( m^2 \pi^2 + a^2 \right) \right\} \]

\[ + \lambda a^2 \left( m^2 \pi^2 + a^2 \right) \left[ Ra_{Dx} \left( Le Le_D - 1 \right) + R_{sDx} \left( 1 - Le^{-1} Le_s \right) \right] - a^2 \left( Ra_{Dx} Le - R_{sDx} \right), \]

\[ A_4 = \left( m^2 \pi^2 + a^2 \right)^2 \left( m^2 \pi^2 + K, a^2 \right) \left( 1 - Le, Le_s \right) \]

\[ + a^2 \left( m^2 \pi^2 + a^2 \right) \left[ Ra_{Dx} \left( Le Le_D - 1 \right) + R_{sDx} \left( 1 - Le^{-1} Le_s \right) \right]. \]

Here, \( A_1 \) is positive definite whereas the coefficients \( A_2, A_3, \) and \( A_4 \) can be positive or negative depending upon the values of various physical parameters.

We shall now discuss the following two cases separately:

Case I: Isotropic Porous Medium

Case II: Anisotropic Porous Medium

4.4 Case I: Isotropic Porous Medium

For isotropic case, \( K_1 = 1 \) and for the lowest value of \( m \), that is, for \( m = 1 \), equation (4.36) yields

\[ Ra_D = \frac{\Lambda \delta^2 \left[ (\sigma + \delta^2) (\delta^2 + \sigma Le) - Le, Le_s \delta^4 \right]}{a^2 \left[ \delta^2 \left( 1 - Le Le_D \right) + \sigma Le \right]} = \frac{R_{aS} \left[ \delta^2 \left( Le - Le_s \right) + \sigma Le \right]}{\left[ \delta^2 \left( 1 - Le Le_D \right) + \sigma Le \right]^2}, \]

where, \( \Lambda = \frac{1 + \varepsilon \sigma}{1 + \lambda \sigma}, \) \( Ra_D = \left( \frac{Kg \alpha \beta d^4}{D_{11} \nu} \right), \) \( Ra_s = \frac{R_{sD}}{Le} = \left( \frac{Kg \alpha \beta' d^4}{D_{11} \nu} \right) \)

\( \delta^2 = \left( \pi^2 + a^2 \right) \) and \( k_s = k_z = K. \)
In the absence of Soret and Dufour numbers \((\text{Le}_S = 0, \text{Le}_D = 0)\), equation (4.42) reduces to

\[
\text{Ra}_D = \left(\frac{\sigma + \delta^2}{a^2} \right) \left[ \Lambda \delta^2 + \frac{a^2 \text{Ra}_S}{\sigma + \delta^2 \text{Le}^{-1}} \right]
\]

\[\ldots (4.43)\]

and further, in the absence of salt \((\text{Ra}_S = 0)\), equation (4.43) is identical with that of \textit{Yoon et al. (2004)} for isotropic porous material.

4.5 Case II: Anisotropic Porous Medium

Equation (4.36) for the lowest value of \(m\), that is, for \(m = 1\), yields

\[
\text{Ra}_T = \frac{\Lambda \delta^2 \left[(\sigma + \delta^2)(\delta^2 + \sigma \text{Le}) - \text{Le}_D \text{Le}_S \delta^4\right]}{a^2 \left[\delta^2 (1 - \text{Le}_D \text{Le}_S) + \sigma \text{Le}\right]} + \frac{\text{Ra}_D \left[\delta^2 (\text{Le} - \text{Le}_S) + \sigma \text{Le}\right]}{\delta^2 (1 - \text{Le}_D \text{Le}_S) + \sigma \text{Le}},
\]

\[\ldots (4.44)\]

where, \(\Lambda = \frac{1 + \varepsilon \sigma}{1 + \lambda \sigma}\), \(\text{Ra}_T = \frac{\text{Ra}_D \text{K}_1}{\text{K}_1 \text{K}_1} = \left(\frac{k_z g \alpha \beta d^4}{\text{D}_1 \nu}\right), \text{Ra}_S = \frac{\text{Ra}_S \text{K}_1}{\text{K}_1 \text{Le}} = \left(\frac{k_z g \alpha \beta' d^4}{\text{D}_1 \nu}\right)\)

\(\xi^{-1} = \frac{k_z}{k_z}, \delta_i^2 = (\xi^{-1} \pi^2 + a^2)\) and \(\delta^2 = (\pi^2 + a^2)\).

In the absence of Soret and Dufour numbers \((\text{Le}_S = 0, \text{Le}_D = 0)\), equation (4.44) reduces to

\[
\text{Ra}_T = \left(\frac{\sigma + \delta^2}{a^2} \right) \left[ \Lambda \delta_i^2 + \frac{a^2 \text{Ra}_S}{\sigma + \delta_i^2 \text{Le}^{-1}} \right].
\]

\[\ldots (4.45)\]

It is important to note that the same result was obtained by \textit{Malashetty et al. (2009)} in the absence of rotation.
4.5.1 Stationary Convection

For stationary convection at marginal state, that is, for \( \sigma = 0 \), equation (4.44) reduces to

\[
Ra_T^{\text{stat}} = \frac{1}{(1 - \text{Le}_D \text{Le}_s)} \left[ (1 - \text{Le}_D \text{Le}_s) \left( \frac{\pi^2}{a^2} + 1 \right) \left( \frac{\pi^2}{\zeta} + a^2 \right) + (\text{Le} - \text{Le}_s) Ra_s \right].
\]

\[
\text{...(4.46)}
\]

The minimization of \( Ra_T \) with respect to wave number ‘a’ yields the critical Darcy Rayleigh number for stationary convection as

\[
Ra_{T,\text{min}}^{\text{stat}} = \frac{1}{(1 - \text{Le}_D \text{Le}_s)} \left[ \frac{\pi^2}{\zeta} (1 - \text{Le}_D \text{Le}_s) \left( 1 + \sqrt{\zeta} \right)^2 + (\text{Le} - \text{Le}_s) Ra_s \right], \quad \text{...(4.47)}
\]

and the corresponding critical wave number is given by

\[
a_c^2 = \frac{\pi^2}{\sqrt{\zeta}}. \quad \text{...(4.48)}
\]

We observe that the critical Darcy Rayleigh number for stationary mode depends upon anisotropy parameter \( \zeta \), Soret number \( \text{Le}_s \), Dufour number \( \text{Le}_D \), Lewis number \( \text{Le} \) and solutal Darcy Rayleigh number \( Ra_s \) and is independent of relaxation parameter \( \lambda \) and retardation parameter \( \epsilon \).

In the absence of Soret and Dufour numbers (\( \text{Le}_s = 0, \text{Le}_D = 0 \)), equation (4.47) reduces

\[
Ra_{T,\text{min}}^{\text{stat}} = \left[ \text{Le} Ra_s + \frac{\pi^2}{\zeta} \left( 1 + \sqrt{\zeta} \right)^2 \right]. \quad \text{...(4.49)}
\]

Further for isotropic porous material, i.e., \( \zeta = 1 \) equation (4.49) yields the critical Darcy Rayleigh number as
Chapter 4: Soret Dufour Driven Instability of an Oldroydian Fluid in a Horizontal…

\[ \text{Ra}_{T,\text{min}} = \text{Le} \cdot \text{Ra}_s + 4\pi^2 \quad \text{when} \quad a_c = \pi. \] ...(4.50)

In the absence of solute (\( \text{Ra}_s = 0 \)), the classical results, \( \text{Ra}_T = 4\pi^2 \) and \( a_c = \pi \) [Horton and Rogers (1945), Lapwood (1948), Combarnous and Bories (1975), Cheng (1978) and Yoon et al. (2004)] are recovered.

4.5.2 Oscillatory Convection

Let the marginal state be oscillatory so that \( \sigma_r = 0 \) and \( \sigma_i \neq 0 \), then from equation (4.44), we obtain

\[ \text{Ra}_T = \Delta_1 + i\sigma_i \Delta_2, \] ...(4.51)

where \( \Delta_1 = \frac{\delta^2 \left(1 + \sigma_r^2 \lambda \varepsilon\right) \delta^2 \left(1 - \text{Le}_D \text{Le}_s\right)(1 - \text{Le}_D) + \delta^2 \sigma_r^2 \text{Le}^2 (1 + \text{Le}_D)}{a^2 \left\{ \delta^2 (1 - \text{Le}_D)^2 + \sigma_r^2 \text{Le}^2 \right\}(1 + \lambda^2 \sigma_i^2)} \)

\[ - \frac{\delta^2 \sigma_r^2 (\varepsilon - \lambda) \left\{ \delta^4 (1 + \text{Le})(1 - \text{Le}_D) - \delta^4 \text{Le}(1 - \text{Le}_D \text{Le}_s) + \sigma_r^2 \text{Le}^3 \right\}}{a^2 \left\{ \delta^4 (1 - \text{Le}_D)^2 + \sigma_r^2 \text{Le}^2 \right\}(1 + \lambda^2 \sigma_i^2)} \]

\[ + \frac{\text{Ra}_s \left\{ \delta^4 (\text{Le} - \text{Le}_s)(1 - \text{Le}_D) + \text{Le}^2 \sigma_i^2 \right\}}{\left\{ \delta^4 (1 - \text{Le}_D)^2 + \sigma_r^2 \text{Le}^2 \right\}} \]

\[ \Delta_2 = \frac{\delta^2 \left(1 + \sigma_r^2 \lambda \varepsilon\right) \delta^2 \left(1 + \text{Le}\right)(1 - \text{Le}_D) - \delta^2 \text{Le}(1 - \text{Le}_D \text{Le}_s) + \sigma_r^2 \text{Le}^2}{a^2 \left\{ \delta^2 (1 - \text{Le}_D)^2 + \sigma_r^2 \text{Le}^2 \right\}(1 + \lambda^2 \sigma_i^2)} \)

\[ + \frac{\delta^2 \left(\varepsilon - \lambda\right) \left\{ \delta^4 (1 - \text{Le}_D \text{Le}_s)(1 - \text{Le}_D) - \delta^4 \sigma_r^2 \text{Le}(1 - \text{Le}_D \text{Le}_s) + \delta^4 \sigma'_i \text{Le}(1 + \text{Le}) \right\}}{a^2 \left\{ \delta^4 (1 - \text{Le}_D)^2 + \sigma_r^2 \text{Le}^2 \right\}(1 + \lambda^2 \sigma_i^2)} \]

\[ + \frac{\text{Ra}_s \delta^2 \text{Le}(1 - \text{Le}_D \text{Le}_s) - \text{Le} + \text{Le}_s}{\left\{ \delta^4 (1 - \text{Le}_D)^2 + \sigma_r^2 \text{Le}^2 \right\}}. \]

\[ \ldots(4.52) \]

and

\[ \Delta_2 = \frac{\delta^2 \left(1 + \sigma_r^2 \lambda \varepsilon\right) \delta^2 \left(1 + \text{Le}\right)(1 - \text{Le}_D) - \delta^2 \text{Le}(1 - \text{Le}_D \text{Le}_s) + \sigma_r^2 \text{Le}^2}{a^2 \left\{ \delta^2 (1 - \text{Le}_D)^2 + \sigma_r^2 \text{Le}^2 \right\}(1 + \lambda^2 \sigma_i^2)} \)

\[ + \frac{\delta^2 \left(\varepsilon - \lambda\right) \left\{ \delta^4 (1 - \text{Le}_D \text{Le}_s)(1 - \text{Le}_D) - \delta^4 \sigma_r^2 \text{Le}(1 - \text{Le}_D \text{Le}_s) + \delta^4 \sigma'_i \text{Le}(1 + \text{Le}) \right\}}{a^2 \left\{ \delta^4 (1 - \text{Le}_D)^2 + \sigma_r^2 \text{Le}^2 \right\}(1 + \lambda^2 \sigma_i^2)} \]

\[ + \frac{\text{Ra}_s \delta^2 \text{Le}(1 - \text{Le}_D \text{Le}_s) - \text{Le} + \text{Le}_s}{\left\{ \delta^4 (1 - \text{Le}_D)^2 + \sigma_r^2 \text{Le}^2 \right\}}. \]

\[ \ldots(4.53) \]
For oscillatory onset, the consistency of equation (4.53) requires that 
\[ \Delta_x = 0 \] 
and this gives a dispersion relation of the form 
\[ B_2x^2 + B_1x + B_0 = 0, \] 
\[ \ldots(4.54) \]
where \( x = \sigma_i^2 \),
\[ B_2 = \lambda\varepsilon\delta_i^2, \] 
\[ \ldots(4.55) \]
\[ B_1 = \delta_i^3 \left[ 1 + \lambda\varepsilon\delta_i^4 L_e^{-2}(1 + L_e)(1 - L_e L_{aD}) - \lambda\varepsilon\delta_i^4 L_e^{-3}(1 - L_e L_{aD}) \right] + \delta_i^3(\varepsilon - \lambda)(1 + L_e) \] 
\[ + a^2\delta_i^2\lambda^2 L_e^{-1}R_s(1 - L_e L_d - L_e + L_s) \] 
\[ \ldots(4.56) \]
and 
\[ B_0 = \delta_i^4\delta_i^3 \left[ L_e^{-2}(1 + L_e)(1 - L_e L_{aD}) + \delta_i^2 L_e^{-3}(\varepsilon - \lambda)(1 - L_e L_{aD})(1 - L_e L_d) \right] 
- L_e^{-1}(1 - L_s L_{aD}) + a^2\delta_i^2 L_e^{-1}R_s(1 - L_e L_{aD} - L_e + L_s), \] 
\[ \ldots(4.57) \]
further, Rayleigh number for oscillatory convection in an anisotropic medium is obtained as
\[ \text{Ra}_T^{\text{osc}} = \frac{\delta_i^2}{a^2L_6(1 + \lambda\varepsilon\sigma_i^2)^2} \left\{ (1 + \lambda\varepsilon\sigma_i^2)(\delta_i^4 L_2 L_4 + \delta_i^2 L_5^2 \sigma_i^2) 
- \sigma_i^2(\varepsilon - \lambda)(\delta_i^4 L_4 L_1 - \delta_i^4 L_5 L_2) + L_2^2 \sigma_i^2 \right\} \] 
\[ + \frac{R_s(\delta_i^4 L_5 L_4 + L_6^2 \sigma_i^2)}{L_5}, \] 
\[ \ldots(4.58) \]
where \( L_1 = 1 - L_e L_{aD} \), \( L_2 = 1 - L_e L_s \), \( L_3 = 1 + L_e \), \( L_4 = 1 + L_e \), \( L_5 = L_e - L_s \) and \( L_6 = \delta_i^2 L_5^2 + L_6^2 \sigma_i^2 \).

### 4.6 Results and Discussion

The neutral stability curves in \((a, \text{Ra}_D)\) and \((a, \text{Ra}_T)\) planes for various parameter values are shown in Figs. 4.1-4.6. These figures provide neutral
stability curves for oscillatory modes in terms of critical Rayleigh number $Ra_D$ or $Ra_T$, below (above) which the system is stable (unstable) under the linear stability criteria. Neutral stability curves for stationary convection are also included in these figures from a comparison point of view.

Figs. 4.1 (a) and (b) show the neutral stability curves for different values of Soret number $Le_S$ for stationary and oscillatory modes. It is found that the Soret number enhances the onset of instability in both stationary and oscillatory modes. The critical wave number moves towards the left and the critical Rayleigh number moves towards the smaller values as Soret number increases. It is also observed that the system becomes unstable first for oscillatory modes and then for stationary modes for a given value of Soret number. This ensures a destabilizing character of Soret number $Le_S$. A comparison of Fig. 4.1(a) (isotropic case) and Fig. 4.1(b) (anisotropic case with $\xi=0.5$) clearly shows that anisotropy postpones instability in both stationary and oscillatory convections provided that the horizontal permeability is smaller than the vertical permeability.

The effect of Dufour number $Le_D$ on the stationary and oscillatory Rayleigh numbers for fixed values of other parameters is shown in Figs. 4.2(a) and (b). The effect of Dufour number is shown to inhibit the onset of neutral stationary and oscillatory convections. The critical wave number is found to shift towards the right as Dufour number increases. This shows that the range of stable wave numbers increases as Dufour number $Le_D$ increases. Further, larger the values of $Le_D$, larger are the values of $Ra_{stat}^T$ and $Ra_{osc}^T$. This ensures a stabilizing character of Dufour number $Le_D$. A comparison of Fig. 4.2 (a) (isotropic case) with Fig. 4.2(b) (anisotropic case with $\xi=0.5$) shows that the medium anisotropy postpones instability.
The effect of anisotropy of the medium is shown in Fig. 4.3. It shows that as the anisotropy parameter $\xi$ increases, the critical Rayleigh number and the critical wave number for both oscillatory and stationary modes decrease implying, thereby, that with the increase in the anisotropy parameter $\xi$, the system becomes more unstable and the range of unstable wave numbers increases. Fig. 4.3 also shows a sharp decline in the values of the critical Rayleigh number as $\xi$ increases form 0.1 to 0.5 for both the stationary and oscillatory modes and beyond this value the critical values of $Ra_{T,\text{stat}}$ and $Ra_{T,\text{osc}}$ decrease slowly. Thus, the anisotropy of the medium enhances the instability of the system. This ensures a destabilizing character of the anisotropy parameter $\xi$. Further, the oscillatory convection becomes unstable first as compared to the stationary convection.

Figs. 4.4 (a) and (b) show the effect of Lewis number $Le$, defined as the ratio of thermal and solutal diffusivities, on the stationary and oscillatory modes. The figures show that Lewis number inhibits the onset of instability. As $Le$ increases, $Ra_{T,\text{stat}}$ increases at a faster rate as compared to $Ra_{T,\text{osc}}$, the range of stable wave numbers increases and larger the value of $Le$, larger are the values of $Ra_{T,\text{stat}}$ and $Ra_{T,\text{osc}}$. This ensures a stabilizing character of Lewis number $Le$. It is clear from these figures that, the role of medium anisotropy with $\xi<1$ is stabilizing as compared to isotropic case ($\xi=1$).

The stationary and oscillatory neutral curves for different values of the relaxation parameter $\lambda$ are shown in Figs. 4.5 (a) and (b). It is observed that the effect of the relaxation parameter $\lambda$ is to enhance the onset of neutral stationary and oscillatory convections. The critical wave number is found to shift towards the left and the critical Rayleigh numbers $Ra_{T,\text{stat}}$ and $Ra_{T,\text{osc}}$ move towards the smaller values as the relaxation parameter increases. Thus larger the value of
the relaxation parameter \( \lambda \), smaller are the values of critical Rayleigh numbers \( R_{\text{a,stat}} \) and \( R_{\text{a,osc}} \), implying, thereby, that \( \lambda \) enhances instability. The shifting of the critical wave number towards the left implies that the range of unstable wave numbers increases as \( \lambda \) increases. Thus the relaxation parameter \( \lambda \) has a destabilizing effect. The figures also show that the system becomes unstable first for the oscillatory convection and not for the stationary convection. The role of medium anisotropy is found to be stabilizing.

Figs. 4.6(a) and (b) show the effect of retardation parameter \( \epsilon \) on the stationary and oscillatory Rayleigh numbers for fixed values of other parameters. The effect of \( \epsilon \) is shown to inhibit the onset of stationary as well as oscillatory convection. The critical wave number is found to shift towards right as \( \epsilon \) increases. In other words, the range of stable wave numbers increases as the retardation parameter \( \epsilon \) increases. Further, larger the value of \( \epsilon \), larger are the values of \( R_{\text{a,stat}} \) and \( R_{\text{a,osc}} \). Thus the retardation parameter has a stabilizing effect. Further, the oscillatory convection becomes unstable first as compared to the stationary convection. The medium anisotropy predicts the same stabilizing character with anisotropic case (\( \xi=0.5 \)) as compared to isotropic case (\( \xi=1 \)).

Neutral stability curves for \( R_{\text{a,D,c}} \) and \( R_{\text{T,c}} \) versus \( R_{\text{a,S}} \) are drawn in Figs. 4.7(a) and (b) for stationary and oscillatory convections both for fixed values of other parameters. As \( R_{\text{a,S}} \) increases, \( R_{\text{T,c}} \) for the stationary convection, increases at a faster rate as compared to that for the oscillatory convection. It is observed that \( R_{\text{T,c}}^{\text{osc}} < R_{\text{T,c}}^{\text{stat}} \) for a given value of \( R_{\text{a,S}} \) implying, thereby, that the oscillatory modes become unstable first, in other words, the marginal state is oscillatory. The solutal Rayleigh number \( R_{\text{a,S}} \), as expected on physical grounds, has a stabilizing character.
Figs. 4.8(a) and (b) exhibit neutral stability curve for $Ra_{D,c}$ and $Ra_{T,c}$ versus $Le_D$ and similar observations are made as from Figs. 4.7(a) and (b) except that the oscillatory modes do not exist for values of $Le_D$ below $Le_D^*$ and in this range of parameter $Le_D$, the marginal state is stationary.

Figs. 4.9(a) and (b) show the variations of $Ra_{D,c}$ and $Ra_{T,c}$ with Soret number $Le_S$. It is clear from these figures that the marginal state is oscillatory and $Ra_{T,c}$ decreases as $Le_S$ increases exhibiting a destabilizing role of Soret number.

The effect of medium anisotropy $\xi$ on the variations of the critical Rayleigh number $Ra_{T,c}$ is shown in Fig. 4.10. It is clear that the marginal state is oscillatory for a given value of $\xi$ beyond $\xi=1$, and $Ra_{T,c}$ decreases as $\xi$ increases, exhibiting a destabilizing role of medium anisotropy.

The variations of $Ra_{D,c}$ and $Ra_{T,c}$ with Lewis number $Le$ are shown in Figs. 4.11(a) and (b). The observations are similar to those made from Fig. 4.7 and Fig. 4.8 except that oscillatory modes do not exist for some range of wave numbers when $0.01<Le<0.42$ and the marginal state is stationary in this range for wave numbers. [This phenomenon is exhibited in Figs 4.4 more clearly]

Figs. 4.12 (a) and (b) show the variations of $Ra_{D,c}$ and $Ra_{T,c}$ with the relaxation parameter $\lambda$. It is clear from these figures that the marginal state is oscillatory for a given value of $\lambda$ and $Ra_{T,c}$ decreases as $\lambda$ increases which exhibits a destabilizing character of $\lambda$. 

102
4.7 Conclusion

The effect of Soret number, Dufour number, anisotropic parameter, Lewis number, relaxation parameter and retardation parameter on the onset of double diffusive convection of an Oldroydian fluid saturated in an anisotropic porous medium, which is heated and soluted from below, is investigated under the linear stability analysis. The effect of Soret number, Dufour number, anisotropic parameter, Lewis number, relaxation parameter and retardation parameter on the critical Darcy Rayleigh number for stationary and oscillatory convections is shown graphically. Soret number and relaxation parameter have a destabilizing effect on the system whereas the effect of Dufour number, Lewis number, solutal Rayleigh number and retardation parameter is stabilizing. The anisotropy parameter $\xi$ has a stabilizing character for $\xi < 1$ whereas its character for $\xi > 1$ is reversed.
Chapter 4: Soret Dufour Driven Instability of an Oldroydian Fluid in a Horizontal…

Fig. 4.1(a): Stationary and oscillatory neutral stability curves in \((a, \text{Ra}_D)\) plane for different values of Soret number \(\text{Le}_S\) in isotropic medium

Fig. 4.1(b): Stationary and oscillatory neutral stability curves in \((a, \text{Ra}_T)\) plane for different values of Soret number \(\text{Le}_S\) in anisotropic medium
Chapter 4: Soret Dufour Driven Instability of an Oldroydian Fluid in a Horizontal…

Fig. 4.2(a): Stationary and oscillatory neutral stability curves in $(a, Ra_D)$ plane for different values of Dufour number $Le_D$ in isotropic medium

Fig. 4.2(b): Stationary and oscillatory neutral stability curves in $(a, Ra_T)$ plane for different values of Dufour number $Le_D$ in anisotropic medium
Fig. 4.3: Stationary and oscillatory neutral stability curves in $(a, Ra_T)$ plane for different values of anisotropy parameter $\zeta$.

- $Le_S = 0.05$, $Le_D = 0.2$,
- $Ra_S = 100$, $Le = 0.5$, $\lambda = 0.4$, $\epsilon = 0.3$
Fig. 4.4(a): Stationary and oscillatory neutral stability curves in \((a, Ra_D)\) plane for different values of Lewis number \(Le\) in isotropic medium.

Fig. 4.4(b): Stationary and oscillatory neutral stability curves in \((a, Ra_T)\) plane for different values of Lewis number \(Le\) in anisotropic medium.
Fig. 4.5(a): Stationary and oscillatory neutral stability curves in $(a, Ra_D)$ plane for different values of relaxation parameter $\lambda$ in isotropic medium.

Fig. 4.5(b): Stationary and oscillatory neutral stability curves in $(a, Ra_T)$ plane for different values of relaxation parameter $\lambda$ in anisotropic medium.
Fig. 4.6(a): Stationary and oscillatory neutral stability curves in \((a, Ra_D)\) plane for different values of retardation parameter \(\varepsilon\) in isotropic medium.

Fig. 4.6(b): Stationary and oscillatory neutral stability curves in \((a, Ra_T)\) plane for different values of retardation parameter \(\varepsilon\) in anisotropic medium.
Chapter 4: Soret Dufour Driven Instability of an Oldroydian Fluid in a Horizontal…

Fig. 4.7(a): Variations of critical Rayleigh numbers ($Ra_{D,c}^{osc}$ & $Ra_{D,c}^{stat}$) with $Ra_s$ in isotropic medium

Fig. 4.7(b): Variations of critical Rayleigh numbers ($Ra_{T,c}^{osc}$ & $Ra_{T,c}^{stat}$) with $Ra_s$ in anisotropic medium
Fig. 4.8(a): Variations of critical Rayleigh numbers \( (\text{Ra}_{D,c}^\text{osc} \& \text{Ra}_{D,c}^\text{stat}) \) with \( \text{Le}_D \) in isotropic medium

\[
\begin{align*}
\text{Le}_S &= 0.05, \quad \text{Ra}_S = 100, \quad \text{Le} = 0.5, \quad \lambda = 0.4, \quad \varepsilon = 0.3
\end{align*}
\]

Fig. 4.8(b): Variations of critical Rayleigh numbers \( (\text{Ra}_{T,c}^\text{osc} \& \text{Ra}_{T,c}^\text{stat}) \) with \( \text{Le}_D \) in anisotropic medium

\[
\begin{align*}
\text{Le}_S &= 0.05, \quad \text{Ra}_S = 100, \quad \text{Le} = 0.5, \\
\xi &= 0.5, \quad \lambda = 0.4, \quad \varepsilon = 0.3
\end{align*}
\]
Fig. 4.9(a): Variations of critical Rayleigh numbers \( \left( Ra_{D,c}^{osc} \text{ and } Ra_{D,c}^{stat} \right) \) with \( Le_S \) in isotropic medium

Fig. 4.9(b): Variations of critical Rayleigh numbers \( \left( Ra_{T,c}^{osc} \text{ and } Ra_{T,c}^{stat} \right) \) with \( Le_S \) in anisotropic medium
Fig. 4.10: Variations of critical Rayleigh numbers \( (\text{Ra}_{T,c}^{\text{osc}} \& \text{Ra}_{T,c}^{\text{stat}}) \) with \( \xi \)

\[
\begin{align*}
\text{Le}_S &= 0.05, \text{Ra}_S = 100, \text{Le} = 0.5, \\
\text{Le}_D &= 0.2, \lambda = 0.4, \varepsilon = 0.3
\end{align*}
\]
Fig. 4.11(a): Variations of critical Rayleigh numbers \( \left( \text{Ra}^{\text{osc}}_{D,c} \text{ and } \text{Ra}^{\text{stat}}_{D,c} \right) \) with \( \text{Le} \) in isotropic medium

\[
\text{Le}_S = 0.05, \; \text{Ra}_S = 100, \; \text{Le}_D = 0.2, \; \lambda = 0.4, \; \varepsilon = 0.3
\]

Fig. 4.11(b): Variations of critical Rayleigh numbers \( \left( \text{Ra}^{\text{osc}}_{T,c} \text{ and } \text{Ra}^{\text{stat}}_{T,c} \right) \) with \( \text{Le} \) in anisotropic medium

\[
\text{Le}_S = 0.05, \; \text{Ra}_S = 100, \; \xi = 0.5, \\
\text{Le}_D = 0.2, \; \lambda = 0.4, \; \varepsilon = 0.3
\]
Fig. 4.12(a): Variations of critical Rayleigh numbers \( (Ra_{D,c}^{osc} \text{ & } Ra_{D,c}^{stat}) \) with \( \lambda \) in isotropic medium.

Fig. 4.12(b): Variations of critical Rayleigh numbers \( (Ra_{T,c}^{osc} \text{ & } Ra_{T,c}^{stat}) \) with \( \lambda \) in anisotropic medium.