Chapter 2

Dark energy solutions

In this chapter we present a class of exact solutions, stationary and non-stationary of the Einstein’s field equations describing non-empty and conformally flat space-times. We study the physical properties of the solutions.

2.1 Introduction

Wang and Wu (1999) have introduced a mass function in the general line-element. Nowadays, it is found that the mass function plays a very important role in generating embedded and non-embedded exact solutions of Einstein’s field equations. The mass function is expressed in the form of a power series of the radial coordinate \( r \) as

\[
M(u, r) = \sum_{n=-\infty}^{+\infty} q_n(u) r^n;
\]

where \( q_n(u) \) are arbitrary functions of retarded time coordinate \( u \). The mass function is being utilized in generating non-rotating embedded Vaidya solution into other spaces by choosing the function \( q_n(u) \) corresponding to the number \( n \). Later on, the utilization of these mass functions has been extended in rotating system and found the role of the number \( n \) in generating rotating
embedded solutions of field equations (Ibohal, 2005). The meaning of the power $n$ in the expansion series (2.1.1) for the known spherically axisymmetric solutions are as follows (Wang and Wu 1999, Ibohal, 2005):

(i) $n = 0$ corresponds to the term containing mass of the vacuum Kerr family solutions such as Schwarzschild, Kerr;

(ii) $n = -1$ is equivalent to the charged term of Kerr family such as Reissner-Nordstrom, Kerr-Newman;

(iii) $n = 1$ furnishes the term of the global monopole solution;

(iv) $n = 3$ provides the de Sitter cosmological models, rotating and non-rotating.

These values of $n$ are conveniently used for generating stationary non-rotating and rotating solutions. It is possible to obtain non-stationary Vaidya-Bonnor black holes, non-rotating (Wang and Wu, 1999) and rotating (Ibohal, 2005) when the terms corresponding to $n = 0$ and $n = -1$ are summed up, since Vaidya-Bonnor metric describes a charged solution – an additional term for the charge. When the above four terms of $n (= -1, 0, 1, 3)$ are together, one can find the charged Vaidya-de Sitter-monopole solution, non-rotating (Wang and Wu, 1999) and rotating (Ibohal, 2005). Non-stationary rotating as well as non-rotating de Sitter cosmological models can also be obtained when we consider only the case $n = 3$ (Ibohal, 2005). In fact the Wang-Wu power series expansion of the mass function turns out to be the most convenient method to generate new (embedded or non-embedded) solutions (Ibohal, 2005) of the field equations if one uses Newman-Penrose (NP) formalism (Newman and Penrose, 1962). From the above identification of index $n$ we observe that the cases $n = 2$ and $-2$ in the mass function (2.1.1) have not been seen considered.
before so far in the scenario of exact solutions of Einstein’s field equations. This is the main aim of the chapter for generating exact solutions of physical interest and to investigate the properties of the energy-momentum tensors describing the matter distribution in the space-time geometry. Here we shall concentrate only the case \( n = 2 \). The other case \( n = -2 \) may be discussed elsewhere.

The chapter is organized as follows: Sections 2.2 and 2.3 deal with the derivation of a class of exact solutions, stationary and non-stationary, of Einstein’s equations. We find that the masses of the solutions proposed here describe the gravitational fields of the space-time geometries and also determine the matter distributions with negative pressures, whose energy equations of state have the value \(-1/2\). However, the energy-momentum tensors of the matter distributions with negative pressures do not describe a perfect fluid, which can be seen in the next sections. These non-perfect fluid distributions are in agreement with the remark of Islam (1985) – “it is not necessarily true that the field is that of a star made of perfect fluid”. We also find that each solution has a coordinate singularity with horizon. Consequently we discuss the areas, entropies and surface gravities on the horizons for the solutions. The existence of the horizons discussed here are also in accord with the cosmological horizon (Gibbon and Hawking, 1977) of de Sitter space with constant \( \Lambda \), which is usually considered to be a common candidate of dark energy with the equation of state parameter \( w = -1 \). This chapter is concluded in Section 2.4 with reasonable remarks and evolution of the solutions with the physical interpretation. Thus we summarize the results of the chapter in the following theorems:

**Theorem 1.** An exact solution (stationary or non-stationary) admitting an energy-momentum tensor having negative pressure with equation of state pa-
rameter $w = -1/2$, is a non-vacuum, conformally flat space-time.

**Theorem 2.** The energy-momentum tensor of the matter distribution in non-vacuum, conformally flat space-time (stationary or non-stationary) violates the strong energy condition leading to a repulsive gravitational force in the geometry.

**Theorem 3.** The time-like vector fields of the matter distributions in the non-vacuum, conformally flat (stationary and non-stationary) space-times are expanding, accelerating and shearing with zero-twist.

**Theorem 4.** The surface gravity at the horizon of a non-vacuum, conformally flat (stationary or non-stationary) space-time is directly proportional to the mass of the solution.

Theorem 1 shows the physical interpretation of the solutions that all components of the Weyl tensors of the space-time metrics vanish indicating conformally flatness of the solutions. The energy-momentum tensors associated with the solutions admit dark energy having the negative pressures and the energy equation of state parameters $w = -1/2$. It is also found that the masses of the solutions not only describe the gravitational fields in the space-time geometries, but also measure the energy densities and the negative pressures in the energy-momentum tensors indicating the non-vacuum status of the solutions. It is the assertion of General Relativity that “the space-time geometry is influenced by the matter distribution” (Wald, 1984). Due to the negative pressure in the energy-momentum tensors, the violation of the strong energy condition is shown in Theorem 2 leading to a repulsive gravitational force in the space-time geometries. Theorem 3 shows the physical interpretation of time-like vector fields of the matter distributions. Theorem 4 indicates the existence of gravity on the horizons depending on their respective masses.

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It is convenient to note the general properties of dark energy (Sahni 2004, Copeland 2006) and (Pabmanabhan 2003, 2008), that (i) the energy-momentum tensor of a space-time geometry possesses negative pressure, (ii) has an equation of state with minus sign, (iii) violates the strong energy condition (iv) the time-like vector field of the observer is expanding as well as accelerating. The commonly accepted dark energy solution having these properties is the de Sitter space-time with cosmological constant $\Lambda$. The energy-momentum tensor of de Sitter dark energy solution satisfies all four properties mentioned above. From Theorem 1 to 3, we observe that the energy-momentum tensors of the solutions present in this paper satisfy all the four properties of dark energy. Hence, we refer the solutions obtained here to as *dark energy solutions*. It is also to mention that the de Sitter dark energy solution is non-vacuum conformally flat space-time, and corresponds to the index number $n = 3$ in the power series (2.1.1). However, the solutions proposed here are obtained from the identification of the index number $n = 2$ in the power series (2.1.1), and are non-vacuum conformally flat space-time, which may be seen in equations (2.8) and (2.23) below.

It is to mention that Schwarzschild solution with $n = 0$ represents a vacuum, non-conformally flat space showing the difference from the non-vacuum, conformally flat solutions discussed here. The presentation of the article is based on mathematical calculation for deriving exact solutions of Einstein’s field equations. In this chapter we utilize the differential form language developed by McIntosh and Hickman (1985) in Newman-Penrose (NP) spin coefficient formalism (Newman and Penrose, 1962) in the $-2$ signature as mathematical tool.
2.2 Stationary dark energy solution

We consider a line-element of a general canonical metric in Eddington-Finklestein coordinate systems \( \{ u, r, \theta, \phi \} \)

\[
ds^2 = \left\{ 1 - \frac{2}{r} M(u, r) \right\} du^2 + 2 du \, dr - r^2 d\Omega^2,
\]
(2.2.1)

where \( d\Omega^2 = d\theta^2 + \sin^2 \theta \, d\phi^2 \) is the line-element on the unit two-sphere, and \( M(u, r) \) is referred to as the mass function and related to the gravitational fields within a given range of radius \( r \). Here \( u \) is the retarded time coordinate.

From the Einstein’s field equations \( R_{ab} - (1/2) g_{ab} R = -K T_{ab} \) associated with the line-element (2.2.1), we find an energy-momentum tensor describing the matter distribution in the gravitational field as

\[
T_{ab} = \mu \ell_a \ell_b + 2 \rho n_a n_b + 2 p m_a \bar{m}_b,
\]
(2.2.2)

where the quantities are found as

\[
\mu = -\frac{2}{K r^2} M(u, r)_u, \quad \rho = \frac{2}{K r^2} M(u, r)_r,
\]
\[
p = -\frac{2}{K r} M(u, r)_{rr}
\]
(2.2.3)

with the universal constant \( K = 8\pi G/c^4 \). Here \( \ell_a, n_a \) and \( m_a \) are given as follows

\[
\ell_a = \delta^1_a, \quad n_a = \frac{1}{2} \left\{ 1 - \frac{2}{r} M(u, r) \right\} \delta^1_a + \delta^2_a, \\
m_a = -\frac{r}{\sqrt{2}} \left\{ \delta^3_a + i \sin \theta \delta^4_a \right\},
\]
(2.2.4)

where \( \ell_a, n_a \) are real null vectors and \( m_a \) is complex having its conjugate \( \bar{m}_a \) with the normalization conditions \( \ell_a n^a = 1 = -m_a \bar{m}_a \) and other inner products are zero.

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From (2.2.3) we observe that there is no straightforward way for solving the non-linear Einstein’s field equations with the mass function \( M(u, r) \) to generate exact solutions of physical interest. In order to have a meaningful physical interpretation of the line-element (2.2.1) one has to consider some certain assumptions on the mass function \( M(u, r) \) as the line-element having the energy-momentum tensor (2.2.2) with the quantities (2.2.3) has no reasonable interpretation to regard it as exact solution of Einstein’s field equations. In order to obtain the physically meaningful line-element, we have to assume the mass function \( M(u, r) \) or rather to restrict it in some forms. For instance, when \( M(u, r) \) sets to a constant \( M \), it is the Schwarzschild solution with \( \mu = \rho = p = 0 \). The Vaidya null radiating solution can be obtained, when one assumes the mass function to be \( M(u, r) = M(u) \), leading to the condition \( \rho = p = 0 \) in (2.2.3). This type of assumption on the mass function \( M(u, r) \) turns out to be the case of \( n = 0 \) in the power series expansion (2.1.1). However, to generate charged solutions like Reissner-Nordstrom and Vaidya-Bonnor, we have to use the cases \( n = 0 \) and \( n = -1 \) together in the power series expansion, that will provides the mass function \( M(u, r) = M(u) - e^2(u)r^{-1} \) for Vaidya-Bonnor solution. When \( M(u) \) and \( e(u) \) become constant with \( M(u, r) = M - e^2r^{-1} \), it gives the Reissner-Nordstrom solution. Therefore, the mass function \( M(u, r) \) can, without loss of generality, be expressed in the powers of \( r \) as in (2.1.1). The above line-element (2.2.1) with the mass function (2.1.1) includes most of the known solutions of Einstein’s field equations, that can be seen with the identifications of the index \( n(= -1, 0, 1, 3) \), depending on the system (rotating or non-rotating) mentioned in the introduction above.

For other examples, the de Sitter solution with cosmological constant \( \Lambda \) is obtained by setting \( q_n(u) = \Lambda / 6 \), when \( n = 3 \) and \( q_n(u) = 0 \), when \( n \neq 3 \).
in (2.1.1), providing the mass function \( M(u, r) = (\Lambda/6)r^3 \) with the energy density \( \rho^* = \Lambda/K \), and the pressure \( p = -\Lambda/K \) in the non-rotating system (Wang and Wu, 1999). Similarly, by assuming \( q_n(u) = m \) when \( n = 0 \), and \( q_n(u) = 0 \) when \( n \neq 0 \), the line-element (2.2.1) is the Schwarzschild solution with constant mass \( m \) in the non-rotating coordinate system. On the other hand, this choice of the power \( n = 0 \) provides the Kerr vacuum solution in a rotating system. This shows the fact that, although there is no straightforward way of solving the field equations with the mass function \( M(u, r) \) in (2.2.1), the utilization of the mass function (2.1.1) in the field equations, seems reasonable in generating exact solutions of physical interest. However, it is found that the case \( n = 2 \) in the power series expansion (2.1.1) has not been considered before in the scenario of exact solutions of Einstein’s field equations. It is hoped that this case \( n = 2 \) may provide exact solutions of physical interest with reasonable interpretation of the matter distributions for both stationary as well as non-stationary space-times.

Here we are looking for new exact solutions of Einstein’s field equations corresponding to the case \( n = 2 \) in the power series expansion (2.1.1). Then we shall investigate the physical meaning of the line-element for this case by studying the properties of the energy-momentum tensor. For this purpose we choose the Wang-Wu function \( q_n(u) \) in (2.1.1) as

\[
q_n(u) = \begin{cases} 
m, & \text{when } n = 2 \\
0, & \text{when } n \neq 2,
\end{cases}
\]

such that the mass function takes the form

\[
M(u, r) \equiv \sum_{n=-\infty}^{+\infty} q_n(u) r^n = mr^2,
\]

where \( m \) is constant and \( u = t - r \) is the retarded time coordinate. Using this mass function in (2.2.1) we find a stationary line-element

\[
ds^2 = (1 - 2mr)du^2 + 2du dr - r^2d\Omega^2,
\]
where the constant $m$ is regarded as the mass of a test particle present in the space-time and is non-zero for the existence of the matter distribution in the geometry. When $u = \text{constant}$, the surface is the future directed null cone. The line-element (2.2.7) describes a stationary solution, and has a coordinate singularity at $r = (2m)^{-1}$ describing a Lorentzian horizon. It is observed that the line-element (2.2.7) is certainly different from (a) Schwarzschild solution ($n = 0$) with $g_{uu} = 1 - 2M/r$ having singularity at $r = 2M$, and $M$ being the Schwarzschild mass; and (b) de Sitter solution ($n = 3$) having a cosmological constant $\Lambda$ with $g_{uu} = 1 - (1/3)r^2\Lambda$ and singularity at $r_{\pm} = \pm(3\Lambda)^{1/2}$.

Our aim is to study the physical meaning of the line-element (2.2.7) from various angles through Einstein’s field equations. That we shall find out the characteristic feature of the energy-momentum tensor, which describes the source of the gravitational field of the matter distribution in the space-time geometry. Now by using the mass function (2.2.6) in (2.2.2) and (2.2.3), we find an energy-momentum tensor as

$$T_{ab} = 2 \rho \ell_{(a} n_{b)} + 2 p m_{(a} \bar{m}_{b)}, \quad (2.2.8)$$

where the energy density $\rho$ and the pressure $p$ are given by

$$\rho = \frac{4}{Kr^2} m, \quad p = -\frac{2}{Kr} m, \quad (2.2.9)$$

and $\mu = 0$. Here we have seen that the pressure $p$ has negative sign for the choice of the index $n = 2$ of the power series (2.2.5). This negative pressure shows one of the characteristic features of the space-time (2.2.7). For future utilization, we write the null vector $n_a$ of (2.2.4) as

$$n_a = \frac{1}{2} \left\{1 - 2rm\right\} \delta^1_a + \delta^2_a. \quad (2.2.10)$$

The equation (2.2.9) indicates that the contribution of the gravitational field to the energy-momentum tensor $T_{ab}$ having the negative pressure $p$ is measured
2.2.1 Energy conditions

directly by the mass $m$ of a test particle. From (2.2.7) and (2.2.9) we observe the important role of the mass $m$ that it not only describes the curvature of the geometry in (2.2.7), but also contributes the source of the matter field (2.2.8) present in the space-time. We also find the ratio of the pressure to the energy density as the equation of state for the solution

$$\omega = \frac{p}{\rho} = -\frac{1}{2}. \quad (2.2.11)$$

This negative value of $\omega$ is due to the negative pressure of the matter distribution.

The energy-momentum tensor (2.2.8) obeys the energy conservation laws, given in (2.3.11-2.3.13) in terms of NP spin coefficients:

$$T^{ab};_b = 0, \quad (2.2.12)$$

which shows the fact that the metric of the line-element is a solution of Einstein’s field equations. The components of energy-momentum tensor may be written as:

$$T^u_u = T^r_r = \rho, \quad T^\theta_\theta = T^\phi_\phi = -p \quad (2.2.13)$$

for future use. We find the trace of the energy-momentum tensor $T_{ab} \ (2.2.8)$ as follows

$$T = 2(\rho - p) = \frac{12}{K_T} m. \quad (2.2.14)$$

Here it is found that $\rho - p$ must always be greater than zero for the existence of the solution (2.2.7) with $m \neq 0$, (if $\rho = p$, it implies that $m$ will be vanished). It is to emphasize that the energy-momentum tensor (2.2.8) does not describe a perfect fluid, i.e. for a perfect fluid, one has $T^{(pf)}_{ab} = (\rho + p)u_au_b - pg_{ab}$ with a unit time-like vector $u_a$ and its trace $T^{(pf)} = \rho - 3p$, which is different from the one given in (2.2.14).

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2.2.1 Energy conditions

For the analysis of the energy conditions of the energy momentum tensor, we shall conveniently introduce an orthonormal tetrad with a unit time-like $u^a$ and three unit space-like vector fields $v^a$, $w^a$, $z^a$ using the null tetrad vectors (2.2.4) with (2.2.10) such as

\[
\begin{align*}
    u_a &= \frac{1}{\sqrt{2}}(\ell_a + n_a), \quad v_a = \frac{1}{\sqrt{2}}(\ell_a - n_a), \\
    w_a &= \frac{1}{\sqrt{2}}(m_a + \overline{m}_a), \quad z_a = -\frac{i}{\sqrt{2}}(m_a - \overline{m}_a),
\end{align*}
\]  

(2.2.15)

with the normalization conditions $u_a u^a = 1$, $v_a v^a = w_a w^a = z_a z^a = -1$ and other inner products being zero. Then the metric tensor can be written as

\[
g_{ab} = u_a u_b - v_a v_b - w_a w_b - z_a z_b.
\]  

(2.2.16)

Now we consider a non-space like vector fields for an observer

\[
U_a = \hat{\alpha} u_a + \hat{\beta} v_a + \hat{\gamma} w_a + \hat{\delta} z_a,
\]  

(2.2.17)

where $\hat{\alpha}$, $\hat{\beta}$, $\hat{\gamma}$ and $\hat{\delta}$ are arbitrary constants (Chan et al. 2003), subjected to the condition that

\[
U^a U_a = \hat{\alpha}^2 - \hat{\beta}^2 - \hat{\gamma}^2 - \hat{\delta}^2 \geq 0.
\]  

(2.2.18)

Then the energy-momentum tensor (2.2.8) can be written in terms of the orthonormal tetrad vectors given in (2.2.15) as

\[
T_{ab} = (\rho + p)(u_a u_b - v_a v_b) - p g_{ab}.
\]  

(2.2.19)

Now $T_{ab} U^a U^b$ will represent the energy density as measured by the observer with the tangent vector $U^a$ (2.2.17). It is to emphasize that this $T_{ab}$ is a general energy-momentum tensor of non-rotating stationary space-times. It includes
those of electromagnetic field, monopole, de Sitter cosmological model etc. However, it does not include a perfect fluid. Hence, the energy-momentum tensor \((2.2.8)\) is different from the one of a perfect fluid, and it is convenient to introduce all the energy conditions for \(T_{ab}\) for future use:

(a) **Weak energy condition**: The energy momentum tensor obeys the inequality \(T_{ab} U^a U^b \geq 0\) for any future directed time-like vector \(U^a\) which implies that

\[
\rho \geq 0, \quad \rho + p \geq 0. \tag{2.2.20}
\]

(b) **Strong energy condition**: The Ricci tensor for \(T_{ab}\) satisfies the inequality \(R_{ab} U^a U^b \geq 0\) for any time-like vector \(U^a\), i.e. \(T_{ab} U^a U^b \geq (1/2)T\), which yields

\[
p \geq 0, \quad \rho + p \geq 0. \tag{2.2.21}
\]

(c) **Dominant energy condition**: For any future directed time-like vector \(U^a, T_{ab} U^b\) should be a future directed non-space like (time-like or null) vector field. This condition is equivalent to

\[
\rho^2 \geq 0, \quad \rho^2 - p^2 \geq 0. \tag{2.2.22}
\]

It is noted that the strong energy condition does not imply the weak energy condition.

Here, we find that the pressure \(p\) with the minus sign given in \((2.2.9)\) does not satisfy the strong energy condition \((2.2.21)\). This violation of the strong energy condition is due to the negative pressure, and may lead to a repulsive gravitational force of the matter field in the space-time \((2.2.7)\). The violation indicates different properties of the energy-momentum tensor \((2.2.8)\)
from those of the *ordinary matter* fields, like perfect fluid, electromagnetic field etc., having positive pressures. In particular, it is to note that this strong energy condition is satisfied by the energy-momentum tensor of electromagnetic field with $\rho = p = e^2/(Kr^4)$ of Reissner-Nordstrom space-time. However, it is also violated for the energy-momentum tensor of non-rotating stationary de Sitter model, having $\rho = -p = \Lambda/K$. This violation of the strong energy condition with the energy density and the pressure (2.2.9) for the line-element (2.2.7) proves the stationary part of Theorem 2 stated above.

We also find that the gravitational field of the space-time (2.2.7) is *conformally flat* with $C^a_{bcd} = 0$, i.e. all the tetrad components of Weyl tensor are vanished

$$\psi_0 = \psi_1 = \psi_2 = \psi_3 = \psi_4 = 0. \quad (2.2.23)$$

The curvature invariant for the solution (2.2.7) is found as

$$R_{abcd}R^{abcd} = -\frac{32}{r^2} m^2, \quad (2.2.24)$$

which is regular on the ‘singular’ surface $r = (2M)^{-1}$. This invariant diverges only at the origin. This indicates that the origin $r = 0$ is a physical singularity. This shows that the singularity of the solution (2.2.7) at $r = (2m)^{-1}$ is caused due to the coordinate system, just like in Schwarzschild solution with the horizon $r = 2M$ (Stephani, 1985). From (2.2.7), (2.2.9) and (2.2.23), we come to the conclusion of the proof of Theorem 1 stated above for the case of stationary solution.

### 2.2.2 Raychaudhuri equation

Here we shall analyze the nature of the time-like vector fields $u^a = (\sqrt{2})^{-1}(\ell^a + n^a)$ appeared in the energy-momentum tensor (2.2.19) for the stationary solution (2.2.7). The time-like vector $u^a$ is often considered to be the 4-velocity.

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of a fluid. So the 4-velocity vector measures the kinematical properties of a fluid whether the fluid flow is expanding \((\Theta = u^a_{;a} \neq 0)\), accelerating \((\dot{u}_a = u_{ab}u^b \neq 0)\), shearing \(\sigma_{ab} \neq 0\) or non-rotating \((w_{ab} = 0)\). We shall investigate the change of the volume expansion from the Raychaudhuri equation, such that we can understand how the negative pressure of the fluid affects the expansion of the solution. For this purpose we find the covariant derivative of the 4-velocity vector \(u^a\) in terms of null tetrad vectors (2.2.4) with (2.2.10) as follows

\[
\dot{u}_a = \frac{1}{\sqrt{2}} \left\{ m(\ell_a \ell_b - n_a \ell_b) - \frac{1}{r} (1 + 2mr)m(a \bar{m}_b) \right\},
\]

where \(m\) is the mass of the solution. In deriving the above expression (2.2.25) we use the definition of \(u^a\) given in (2.2.15). This expression of \(u_{ab}\) is convenient to calculate the (volume) expansion scalar \(\Theta = u^a_{;a}\) and acceleration vector \(\dot{u}_a = u_{ab}u^b\) as follows

\[
\Theta \equiv u^a_{;a} = \frac{1}{\sqrt{2}} (1 + 3rm)
\]

\[
\dot{u}_a = -\frac{1}{2} m(\ell_a - n_a)
\]

\[
\dot{u}^a_{;a} = -\frac{3m}{2r} (1 - mr).
\]

We find that the vorticity tensor \(w_{ab} = u_{[a;b]} - \dot{u}_{[a}u_{b]}\) is vanished for the solution (2.2.7), and however, the shear tensor \(\sigma_{ab} = u_{(a;b)} - \dot{u}_{(a}u_{b)} - (1/3)\Theta h_{ab}\) exists as

\[
\sigma_{ab} = \frac{1}{6\sqrt{2}r} \left[ (\ell_a \ell_b + n_a n_b) - 2\{\ell(a n_b) + m(a \bar{m}_b)\} \right],
\]

which is orthogonal to \(u^a\) (i.e., \(\sigma_{ab}u^b = 0\). It is found that the mass \(m\) of the solution does not explicitly involve in the expression of \(\sigma_{ab}\) but its involvement can be seen in the null vector \(n_a\) given in (2.2.4). However, the mass \(m\) determines the expansion \(\Theta\) as well as the acceleration \(\dot{u}_a\) as in (2.2.26). We

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find from (2.2.26) that the particle moving in the space-time geometry (2.2.7) follows the non-geodesic path of the time-like vector \( u_{ab} u^b \neq 0 \). This shows the important feature of the solution having the expansion with acceleration. The vanishing of the vorticity tensor \( w_{ab} = 0 \) may be interpreted physically as saying that the matter field of the solution is *twist-free* (non-rotating). This comes to the conclusion of the proof of Theorem 3 for the stationary solution.

Now let us observe the consequence of the Raychaudhuri equation for the stationary solution (2.2.7)

\[
\dot{\Theta} = \dot{u}^a {}_a + 2(w^2 - \sigma^2) - \frac{1}{3} \Theta^2 + R_{ab} u^a u^b \tag{2.2.28}
\]

where \( \dot{\Theta} = \Theta_a u^a \). The shear and vorticity magnitudes are \( 2\sigma^2 = \sigma_{ab} \sigma^{ab} \) and \( 2w^2 = w_{ab} w^{ab} \); \( R_{ab} \) is the Ricci tensor associated with the space-time metric \( g_{ab} \). Then the Raychaudhuri equation for the twist-free time-like vector is found as follows

\[
\dot{\Theta} = -\frac{1}{4r^2} (1 + 2rm), \tag{2.2.29}
\]

which takes the constant value \( \dot{\Theta} = -2m^2 \) on the horizon \( r = (2m)^{-1} \), showing the constancy of the expansion along the time-like vector \( u^a \). The impact of the negative pressure \( p \) given in (2.2.9) is taken care by Ricci tensor \( R_{ab} \) in (2.2.28). The acceleration vector \( \dot{u}_a \) and its scalar \( \dot{u}^a {}_a \) are negative, affecting the expansion rate in (2.2.28). It is found from (2.2.25) that the time-like vector \( u^a \) is not a Killing vector \( L_u g_{ab} = 2u_{(ab)} \neq 0 \), indicating the difference from any static time-like Killing vector \( \xi^a = \delta^a_1 \), \( (L_\xi g_{ab} = 0) \) which has no expansion and shear.

### 2.2.3 Surface Gravity

The line-element of the stationary solution (2.2.7) has a horizon at \( r = (2m)^{-1} \). In this regard it is to note the fact that the existence of the horizon is in accord

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with the cosmological horizon of de Sitter space with constant $\Lambda$ (Bousso 2008, Gibbon and Hawking 1977), since it is regarded as a common candidate of dark energy with the equation of state parameter $w = -1$. Therefore, it is interesting to observe the role of the mass $m$ of the solution in connection with the area, entropy, surface gravity as well as the temperature on the horizon. So we find the area at the horizon $r = (2m)^{-1}$ as

$$A = \int_0^{2\pi} \int_0^{\pi} (g_{\theta\theta} g_{\phi\phi})^{1/2} d\theta d\phi\bigg|_{r=(2m)^{-1}} = \pi m^{-2},$$

and the entropy, from the entropy-area relation $S = A/4$ (Gibbon and Hawking 1977), as

$$S = \frac{1}{4} \pi m^{-2}. \quad (2.2.30)$$

It indicates that the area and entropy will always exist for the solution with the non-zero mass $m$.

According to Carter (1973) and York (1983), the surface gravity $\kappa$ of a horizon is defined by the relation $n^b \nabla_b n^a = \kappa n^a$, where the null vector $n^a$ in (2.2.10) above is parameterized by the coordinate $u$, such that $d/du = n^b \nabla_b$. Then the surface gravity is expressed in terms of NP spin-coefficient $\gamma$ as follows (Ibohal, 2005)

$$\kappa = n^b \nabla_b n^a \ell_a = - (\gamma + \bar{\gamma}), \quad (2.2.31)$$

where $\gamma = -m/2$. From this we find the surface gravity on the horizon $r = (2m)^{-1}$ as

$$\kappa = m, \quad (2.2.32)$$

which shows that the surface gravity is directly measured by the mass, or in other words, it is directly proportional to the mass of the solution. This
establishes the proof of Theorem 4 in the case of stationary solution. Then the Bekenstein-Hawking temperature for the model at the horizon is found as

\[ T = \frac{\hbar \kappa}{2\pi G k c} = \frac{\hbar m}{2\pi G k c}, \]

(2.2.33)

where \( \hbar \) is the reduced Planck constant, \( c \) the speed of light, \( k \) the Boltzmann constant, and \( G \) the gravitational constant. It indicates that the surface gravity and temperature of the horizon will never become zero for the existence of the stationary non-vacuum conformally flat solution \((m \neq 0)\). When the mass becomes zero \( m = 0 \), the line-element (2.2.7) will be a flat metric, and at this stage the surface gravity as well as the temperature will vanish. It is consistent with the property of flat space-time geometry, where there is no gravity, one cannot determine the surface gravity of the solution. It is noted that the surface gravity \( \kappa_{\text{Sch}} \) of the Schwarzschild black hole is inversely proportional to the Schwarzschild mass \( M \) as \( \kappa_{\text{Sch}} = (4M)^{-1} \) on its horizon \( r = 2M \) (Gron and Hervik, 2007).

### 2.2.4 Size of the mass

It is quite interesting to introduce a possible size of the mass of the solution discussed here. According to Bousso (2008), stars are as distance as billions of light years, so \( r > 10^{60} \) and stars are as old as billions of years, \( t > 10^{60} \). In this length scale, the size of the mass of the solution at the horizon \( r = (2m)^{-1} \) may become

\[ m = \frac{1}{2} r^{-1} < \frac{1}{2} \times 10^{-60}, \]

(2.2.34)

which is bigger than the size of the cosmological constant \( |\Lambda| \leq 3r_{\Lambda}^{-2} \leq 3 \times 10^{-120} \) with the cosmological horizon \( r_{\Lambda} = \sqrt{3/\Lambda} \) (Bousso, 2008). The relation
(2.2.35) may provide an example of a tiny test particle having a small size mass in the Universe.

### 2.2.5 Kerr-Schild ansatz

The line-element (2.2.7) can be expressed in Kerr-Schild ansatz on Minkowski flat background $\eta_{ab}$ as

$$ds^2 = dt^2 - dr^2 - r^2 d\Omega^2 - 2mr du^2$$

under the transformation $t = u + r$. This is the Kerr-Schild ansatz on the flat background $\eta_{ab}$ in spherical coordinate system

$$g_{ab} = \eta_{ab} + 2Q\ell_a\ell_b,$$  \hspace{1cm} (2.2.35)

where $Q = -mr$, $\eta_{ab}$ is the Minkowski flat metric and $\ell_a = \delta_a^w$ is the null vector with respect to $g_{ab}$ and $\eta_{ab}$. This confirms the fact that the stationary solution obtained here is an exact solution of Einstein’s field equations. The line-element (2.2.7) can also be written in the $(t, r, \theta, \phi)$ coordinate system for future use as follows

$$ds^2 = (1 - 2mr)dt^2 - (1 - 2mr)^{-1}dr^2 - r^2 d\Omega^2,$$

under the transformation $dt = du + \{1/(1 - 2mr)\}dr$. This is a very familiar form of the line-element in general relativity, and its determinant is $|g| = -r^4 \sin^2 \theta$. In this coordinate system we can easily observe the singularity at the point $r = (2m)^{-1}$.

### 2.2.6 Physical interpretation of the matter distribution

In order to interpret the physical meaning of the negative pressure of the matter distribution of the solution (2.2.7), we shall compare with other known

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space-times having negative pressure and the energy equation of state with minus sign. For instance, the cosmological constant $\Lambda$ of the non-rotating de Sitter solution, whose energy-momentum tensor is $T^{(\text{dS})}_{ab} = \Lambda g^{(\text{dS})}_{ab}$, is regarded as a common candidate of dark energy (Bousso 2008; Padmanabhan 2006; Copeland, et al. 2006; Sahni 2004) and references their in, having the negative pressure $p = -\Lambda/K$, and the energy density $\rho = \Lambda/K$ with the equation of state parameter $w = p/\rho = -1$, where $K = 8\pi G/c^4$. $T^{(\text{dS})}_{ab}$ violates the strong energy condition leading to the term – the repulsive (not attractive) cosmological constant $\Lambda$ (Gibbon and Hawking, 1977). This equation of state $w = -1$ is also satisfied for (a) the non-rotating non-stationary de Sitter solution with cosmological function $\Lambda(u)$, whose energy-momentum tensor $T_{ab} = -\frac{1}{3}r^2\Lambda(u)\epsilon_a\epsilon_b + \Lambda(u)g_{ab}$ with $p = -\Lambda(u)/K$, $\rho = \Lambda(u)/K$ and $u$ is the retarded time coordinate (Ibohal, 2009); (b) the rotating non-stationary de Sitter solution $\Lambda(u)$ (Ibohal, 2009) having $p = -r^2\Lambda(u)(r^2 + 2a^2 \cos^2 \theta)/(KR^2R^2)$ and $\rho = r^4\Lambda(u)/(KR^2R^2)$ at the poles $\theta = \pi/2$ and $3\pi/2$, where $R^2 = r^2 + a^2 \cos^2 \theta$ with the non-zero rotational parameter $a$. This includes the case of the rotating stationary de Sitter solution with constant $\Lambda$ (Ibohal, 2005). This indicates that in the study of dark energy problems one needs not concentrate only on the cosmological constant $\Lambda$, and that one can consider the non-constant cosmological function $\Lambda(u)$ in the rotating as well as non-rotating de Sitter space-time geometries with the equation of state parameter $w = -1$ (Ibohal 2005, 2009). It is convenient to mention the equation of state $w$ for ordinary matter field distributions for better understanding the dark energy problem. That we have the equation of state for other ordinary matters having positive sign (i) $w = 1$ for electromagnetic field having $\rho = p = e^2/(KR^2R^2)$ of the Kerr-Newman black hole with the constant electric charge $e$, and for rotating Vaidya-Bonner radiating black hole $\rho = p = e(u)^2/(KR^2R^2)$ with variable
charge $e(u)$ of retarded time coordinate $u$ (Ibohal, 2005), (ii) $w = 1/3$ for radiation field with $\rho - 3p = 0$ (Bousso 2008; Padmanabhan 2006; Copeland et al. 2006; Sahni 2004). This shows the fact that dark energy always has a minus sign in the value of the equation of state parameter; whereas the ordinary matter has a positive sign. The negative sign in the equation of state is an important property for any matter field distribution to be interpreted as a dark energy. Hence, the solution (2.2.7) possesses all the four properties of dark energy mentioned in the introduction, and so we refer it as dark energy solution.

The observations of luminosity-redshift relation for the type Ia supernovas (Perlmutter, et al. 1999; Riess, et al. 2000, 2001) suggest that the missing energy should possess negative pressure $p$ and the equation of state $w = p/\rho$ (Zlatev, et al. 1999). The negative pressure of the dark energy may be the cause of the acceleration of the present Universe. Although the dark energy has been sought in a wide range of physical phenomena depending on the value of the equation of state parameter $w$, (i) a quintessence field $-1 < w < -1/3$, (ii) the cosmological constant $w = -1$, (iii) a phantom field $w < -1$ (Caldwell 2002; Sahni 2004), the nature of the dark energy still remains a complete mystery (Gong, et al. 2007) without any proper space-time geometry, except the assumption of a line-element of a perfect homogeneous and isotropic space-time having a compatible energy-momentum tensor of perfect fluid $T^a_b = \text{diag}\{\rho(t), -p(t), -p(t), -p(t)\}$, with $\rho(t)$ and $p(t)$ being the energy density and pressure of the matter distribution in the Friedmann-Robertson-Walker universe.

From the above scenario of dark energy and with the findings here (i) the negative pressure (2.2.9), (ii) the energy equation of state with minus sign $w = -1/2$ (2.2.11), (iii) the violation of strong energy condition (2.2.21), and (iv)
the accelerating expansion of the time-like vector (2.2.26) for the stationary non-vacuum conformally flat solution, we may regard the matter distribution (2.2.19) as an example of dark energy with negative pressure whose space-time geometry is the line-element (2.2.7).

2.3 Non-stationary dark energy solution

In this section we shall extend the stationary solution (2.2.7) to the non-stationary one. Therefore, in order to extend the stationary to the non-stationary solution, we assume the Wang-Wu function $q_n(u)$ as follows:

$$M(u,r) \equiv \sum_{n=-\infty}^{+\infty} q_n(u) r^n = m(u) r^2 \quad (2.3.1)$$

when $n = 2$. Utilizing this mass function in the general canonical metric (2.2.1) we find a non-stationary line-element as

$$ds^2 = \{1 - 2r m(u)\} du^2 + 2dudr - r^2d\Omega. \quad (2.3.2)$$

where $m(u)$ is considered to be a variable mass of a test particle in the non-stationary system. The above line-element has a coordinate singularity at $r = \{2m(u)\}^{-1}$. Such a generation of a non-stationary line-element from the stationary one (2.2.7) is acceptable in the framework of General Relativity that the non-stationary Vaidya null radiating solution with variable mass $M(u)$ is a generalization of the stationary Schwarzschild vacuum solution with constant mass $M$. Similarly, the non-stationary de Sitter solution (Ibohal 2009) with a cosmological function $\Lambda(u)$ having horizons at $r_\pm = \pm\{3\Lambda(u)\}^{1/2}$ can be obtained from the stationary de Sitter model of constant $\Lambda$. It is convenient to mention that the above line-element (2.3.2) is obtained by choosing the index number $n = 2$ in the power series (2.1.1) as in (2.3.1). Here our aim
is to show the fact that the energy-momentum tensor of this non-stationary line-element (2.3.2) satisfies all the properties of dark energy mentioned above.

Now using the mass function (2.3.1) in (2.2.3) we find the null density $\mu$, the energy density $\rho$ and the pressure $p$ as

$$\mu = -\frac{2}{K} m(u),$$
$$\rho = \frac{4}{K r} m(u),$$
$$p = -\frac{2}{K r} m(u),$$

(2.3.3)

associated with an energy-momentum tensor for the non-stationary matter distribution:

$$T_{ab} = \mu \ell_a \ell_b + (\rho + p)(u_a u_b - v_a v_b) - p g_{ab}.$$  

(2.3.4)

Here $\ell_a$ is the real null vector, $u_a$ a unit time-like $u_a u^a = 1$ and $v_a$ a unit space-like vector $v_a v^a = -1$ defined as in (2.2.15). It is observed that the presence of the null density $\mu$ in $T_{ab}$ is due to the non-constant mass $m(u)$ in the field equations showing the evolution of a non-stationary solution. From (2.3.3) it follows that the energy equation of state parameter $w = p/\rho$ has the value $-1/2$ with minus sign. If one sets the mass function $m(u)$ to a constant $m$, the above energy-momentum tensor will become the one given in (2.2.19) for the stationary solution with the quantities $\rho$ and $p$. The energy-momentum tensor obeys the energy conservation laws $T^{ab}_{;b} = 0$ as shown in (2.3.11)–(2.3.13) below. It shows the fact that the non-stationary solution (2.3.2) is an exact solution of Einstein’s field equations. It is also found that, due to the negative pressure, the $T_{ab}$ violates the strong-energy condition,

$$\frac{1}{2} \mu + p \geq 0, \quad \frac{1}{2} \mu + \rho + p \geq 0.$$  

(2.3.5)

This violation indicates that the gravitational force of the non-stationary model is repulsive as in the case of stationary one (2.2.7) above which completes the proof of Theorem 2. However, $T_{ab}$ (2.3.4) with the negative pressure

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satisfies the *weak* energy condition (i) $\mu/2 + \rho \geq 0$, (ii) $\mu/2 + \rho + p \geq 0$, and *dominant* energy condition (i) $\mu\rho + \rho^2 \geq 0$, (ii) $-\mu\rho + \rho^2 - p^2 \geq 0$. Here we observe the difference between the strong energy conditions of the stationary (2.2.21) and that of the non-stationary (2.3.5) with the null density $\mu$. We also find that the non-stationary solution is *conformally* flat with $C_{abcd} = 0$, i.e.
\[ \psi_0 = \psi_1 = \psi_2 = \psi_3 = \psi_4 = 0. \] (2.3.6)

This shows that the solution (2.3.2) with a mass function $m(u)$ has the same characteristic property of conformally flatness (2.2.23) of stationary solution with constant mass. Equations (2.3.3), (2.3.4) and (2.3.6) provide the proof of the non-stationary part of Theorem 1. It is also noted that the structure equation of the Reimann curvature invariant for variable mass (2.3.2) has a similar form of that of the solution (2.2.7) in (2.2.24) as
\[ R_{abcd}R^{abcd} = -\frac{32}{r^2} m^2(u), \] (2.3.7)
which is regular on the ‘singular’ surface $r = \{2m(u)\}^{-1}$. This invariant is divergent only at the origin $r = 0$, which is a physical singularity.

We also have the expansion scalar $\Theta$ and the acceleration vector $\dot{u}_a$ with its scalar $\dot{u}_a^a$ for the time-like vector $u_a$ appeared in (2.3.4)
\[ \Theta \equiv u_a^a = \frac{1}{\sqrt{2} r} \{1 + 3rm(u)\} \]
\[ \dot{u}_a = -\frac{1}{2} m(u) \{\ell_a - n_a\} \]
\[ \dot{u}_a^a = \frac{1}{2} m(u, a) - \frac{3m(u)}{2r} \{1 - m(u)r\}. \] (2.3.8)

However, the shear tensor $\sigma_{ab}$ remains unchanged as in the stationary solution (2.2.24) and the rotation tensor $w_{ab}$ is vanished (zero-twist). This follows the proof of Theorem 3 of the non-stationary case. From the Raychaudhuri

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equation, we have the rate of change of the expansion scalar as follows

\[ \dot{\Theta} = \frac{1}{2} m(u)_u - \frac{1}{4r^2} \{ 1 + 2rm(u) \}. \]  

(2.3.9)

The surface gravity of the non-stationary solution at the horizon \( r = \{2m(u)\}^{-1} \) takes the form

\[ \kappa = m(u) \]  

(2.3.10)

which shows the proof of Theorem 4 of non-stationary part. We have also seen the evolution of the non-stationary solution with the mass function \( m(u) \) in (2.3.3), (2.3.8) and (2.3.9). From (2.3.3), (2.3.5) and (2.3.8) we may regard the line-element (2.3.2) as a non-stationary dark energy solution admitting an energy-momentum tensor (2.3.4) with negative pressure and having an equation of state parameter \( w = p/\rho = -1/2 \).

### 2.3.1 Energy conservation equations

In this subsection we shall show the fact that the energy-momentum tensor (2.3.4) satisfies the energy conservation equations \( T^{ab}{}_{;b} = 0 \). These equations are four in number, which, by using the NP complex spin coefficients (Newman and Penrose, 1962), can equivalently be expressed in three equations – two real and one complex. Hence, we find the following

\[ D\rho = (\rho + p)(\rho^* + \bar{\rho}^*), \]  

(2.3.11)

\[ \delta p = \mu \kappa^* + (\rho + p)(\tau - \bar{\pi}), \]  

(2.3.12)

\[ D\mu + \nabla\rho = \mu \{ (\rho^* + \bar{\rho}^*) - 2(\epsilon + \bar{\epsilon}) \} \]

\[ - (\rho + p)(\mu^* + \bar{\mu}^*), \]  

(2.3.13)

where \( \kappa^*, \rho^*, \mu^*, \tau, \pi, \) etc. are spin coefficients, and \( D, \nabla \) and \( \delta \) are the intrinsic derivative operators. These (2.3.11-13) are general equations for an energy-momentum tensor of the type (2.3.4).
Now, in order to verify the conservation equations (2.3.11-13) for the components of $T_{ab}$ with the quantities $\mu, \rho, p$ given in (2.3.3) we present the NP spin coefficients for the non-stationary metric (2.3.2):

$$\kappa^* = \sigma = \lambda = \epsilon = \pi = \tau = \nu = 0,$$

$$\rho^* = -\frac{1}{r}, \quad \mu^* = -\frac{1}{2r}\{1 - 2rm(u)\},$$

$$\beta = -\alpha = \frac{1}{2\sqrt{2r}} \cot \theta, \quad \gamma = \frac{1}{2} m(u).$$

(2.3.14)

The intrinsic derivative operators are given as follows:

$$D \equiv \ell^a \partial_a = \partial_r,$$

$$\nabla \equiv n^a \partial_a = \partial_u - \frac{1}{2}\{1 - 2rm(u)\}\partial_r,$$

$$\delta \equiv m^a \partial_a = \frac{1}{2\sqrt{2r}} \left\{ \partial_\theta + \frac{i}{\sin \theta} \partial_\phi \right\}.$$

(2.3.15)

where $\ell_a, n_a$ and $m_a$ are the null tetrad vectors associated with the solution (2.3.2). The equations (2.3.11) and (2.3.12) are satisfied by using (2.3.14) and (2.3.15). By virtue of (2.3.3), (2.3.14) and (2.3.15), we find the left side of (2.3.13) as

$$D\mu + \nabla \rho = \frac{4}{Kr} m(u)_{,u} - \frac{2}{Kr^2} m(u)\{1 - 2rm(u)\}.$$  (2.3.16)

This can be shown equal to the right side of (2.3.13) after using (2.3.3) and (2.3.14). It leads to the conclusion of the verification that the energy-momentum tensor (2.3.4) satisfies the energy conservation equations $T_{ab}^{;b} = 0$. This indicates that the non-stationary line element (2.3.2) is an exact solution of Einstein’s equations. It is also to mention that when $m(u)$ sets to a constant $m$ for the stationary solution (2.2.7), the energy-momentum tensor (2.2.8) satisfies the energy conservation equations (2.2.12).
2.4 Conclusion

In this paper we develop a class of exact (stationary and non-stationary) solutions of Einstein’s field equations describing non-vacuum and conformally flat space-times, whose energy-momentum tensors possess dark energy with the negative pressure and the equation of state parameter \( w = -1/2 \). The most exotic property of these solutions is that the metrics describe both the background space-time structure and the dynamical aspects of the gravitational field in the form of the energy-momentum tensors. That is to say that the masses of the solutions play the role of both the curvature of the space-time (non-flat) as well as the source of the energy-momentum tensor with \( T_{ab} \neq 0 \) (non-vacuum) measuring the energy density and the negative pressure. This indicates that when we set the masses of the solutions to be zero, the space-times will become the flat Minkowski space with vacuum structure \( T_{ab} = 0 \). In the case of Schwarzschild solution, the mass plays only the role of curvature of the space-time and cannot determine the energy-momentum tensor. That is why the Schwarzschild solution is a curved non-flat, vacuum space-time with \( T_{ab} = 0 \). Here the solutions (2.2.7) and (2.3.2) are curved non-flat and non-vacuum space-times with \( T_{ab} \neq 0 \). It is also to emphasize that the solutions discussed here provide examples of conformally flat space-times, while other examples of conformally flat solutions are the non-rotating de Sitter models with cosmological constant \( \Lambda \) (Hawking and Ellis, 1973), cosmological function \( \Lambda(u) \) (Ibohal, 2009) and the Robertson-Walker metric (Stephani, 1985).

We find that the time-like vector of the source is expanding \( \Theta \neq 0 \), accelerating \( u_a \neq 0 \) (2.2.26) as well as shearing \( \sigma_{ab} \neq 0 \) (2.2.27), but non-rotating \( w_{ab} = 0 \). This means that the stationary observer of the solution does not follow the time-like geodesic path as \( u_{a;b}u^b \neq 0 \). Similarly, a non-stationary
observer in (2.3.2) follows the non-geodesic path (2.3.8). We also find that the energy-momentum tensors for the solutions violate the strong energy conditions. The violation of strong energy conditions is due to the negative pressure of the matter fields content in the space-time geometries, which can be seen in (2.2.21) and (2.3.5) above, and is not an assumption to obtain the solutions, like other dark energy models mentioned in (Sahni, 2004). This violation indicates that the gravitational fields of the solutions are repulsive as pointed out in (Tipler 1978) leading to the accelerated expansions of the universe. The expansion of the space-time with acceleration is in agreement with the observational data (Perlmutter, et al. 1999; Riess, et al., 2000, 2001). It is also noted that the strong energy conditions for the stationary solution (2.2.21) associated with the stress-energy momentum tensor (2.2.19) and that of the non-stationary one (2.3.3) are different from that of the perfect fluid ($\rho \geq 0, \rho + 3p \geq 0$). This indicates that the strong energy condition is mainly depended upon the structure equation of a particular energy-momentum tensor.

It is emphasized the fact that our approach in the development of the solutions here is necessarily based on the identification of the index $n = 2$ in the Wang-Wu mass function without any extra assumption. This identification of the power index $n = 2$ in the mass function (2.1.1) has considered here for the first time, and not been seen discussed before in the scenario of exact solutions of Einstein’s field equations. It is also noted that we do not consider the Friedman-Robertson-Walker metric, filled with perfect fluid, which is assumed to be the standard approach for the investigation of dark energy problem as mentioned earlier (Bousso 2008; Padmanabhan 2006; Copeland, et al. 2006; Sahni 2004). That is why the energy-momentum tensors associated with the solution (2.2.7) and (2.3.2) do not describe a perfect fluid. This fact can be observed from the trace $T = 2(\rho - p)$ of the $T_{ab}$ for both stationary and non-
stationary solutions. This non-perfect fluid distribution of the solutions is also in accord with Islam’s suggestion that it is not necessarily true that stars are made of perfect fluid (Islam, 1985).

From the study of the above solutions, we find that the energy densities are only contributed from the masses of the matters. It is the fact that without the mass of the solutions, one cannot measure the energy density and the pressure of the energy-momentum tensors in order to obtain the energy equation of state $w = -1/2$. This means that the pressures and the energy densities associated with the energy-momentum tensors (2.2.19) and (2.3.4) are measured by the masses that produce the gravitational field in the space-time geometries of the solutions. Hence, we may conclude that the stationary and non-stationary solutions may explain the essential part of Mach’s principle – “The matter distribution influences the space-time geometry” (d’Inverno, 2005). It is emphasized that the equations of state parameters $w = -1/2$ for the matter distributions (2.2.19) and (2.3.4) are belonged to the range $-1 < w < 0$ focused for the best fit with cosmological observations in (Caldwell, 1998) and references therein.

The interesting feature of the solutions with $n = 2$ proposed here is that the space-time metrics are non-asymptotically flat when $r \to \infty$. However, they are flat at the origin $r \to 0$. Geometrically, it is acceptable that any curved space is locally flat. The idea of non-asymptotic flatness is in agreement with the Carter’s suggestion (1969, 1970) in generating exact solutions that “it is not necessary to assume asymptotic flatness nor make the assumption that there are no other Killing vectors than $\xi = \partial_t$ and $\eta = \partial_\phi$” (Islam, 1985). Other examples of non-asymptotic space-times are (a) the de Sitter solution with cosmological constant (Hawking and Ellis, 1973) and (b) rotating Kerr-NUT solution.

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2.4 Conclusion

The metrics appear singular when \( r = 0 \) and \( r = (2m)^{-1} \) for stationary and \( r = \{2m(u)\}^{-1} \) for non-stationary at a particular value of \( u \). These values of \( r \) have special importance; that at the origin \( r = 0 \), there is a physical singularity where the curvature invariants diverge as shown in (2.2.24) and (2.3.7); at \( r = (2m)^{-1} \) for stationary and \( r = \{2m(u)\}^{-1} \) for non-stationary, the curvature invariants are well behaved and finite. Accordingly, we find areas, entropies as well as surface gravities at the horizons. It is found that the surface gravities given in (2.2.33) and (2.3.10) are directly proportional to their masses of the solutions. This indicates that the existence of the masses imply the existence of their surface gravities and the temperatures on the horizons. The existence of horizons is also in accord with the cosmological horizon of de Sitter space with constant \( \Lambda \) (Gibbon and Hawking, 1977), which is considered to be a common candidate of dark energy with the parameter \( w = -1 \) (Bousso 2008; Padmanabhan 2006; Copeland, et al. 2006; Sahni 2004). This parameter of equation of state is also true for both the cosmological constant \( \Lambda \) as well as the cosmological function \( \Lambda(u) \) of the rotating and non-rotating de Sitter solutions (Ilbohal, 2009). According to the length scale \( r > 10^{60} \) suggested by Bousso (2008), we find the approximate sizes of the masses less than \((1/2) \times 10^{-60}\), which are bigger than the size of the cosmological constant \(|\Lambda| \leq 3 \times 10^{-120}\) with the horizon \( r_\Lambda = \sqrt{3/\Lambda} \). It is noted that to the best of the authors knowledge, the solutions are not been seen discussed before. We hope that the exact solutions (2.2.7) and (2.3.2) may provide examples of stationary and non-stationary space-times admitting energy-momentum tensors of dark energy having negative pressures with the equation of state parameters \( w = -1/2 \) in the accelerated expanding space-time geometries.