Chapter 5

Vaidya black hole in non-stationary de Sitter space

In this chapter we generate an embedded Vaidya black hole in non-stationary de Sitter space, having the limit $-(1/3)\Lambda(u)^{(-1/2)} \leq M(u) \leq +(1/3)\Lambda(u)^{(-1/2)}$ of the Vaidya mass $M(u)$ in the extreme black hole case $9\Lambda(u)M^2(u) = 1$ (Ishwarchandra and Singh, 2013). This limit was not been able to explain with the cosmological constant $\Lambda$ in Mallett (1986) paper and others. The embedded Vaidya-de Sitter black hole with variable $\Lambda(u)$ is an extension of the Schwarzschild-de Sitter black hole having constant $\Lambda$ (Gibbon and Hawking, 1977).

5.1 Introduction

In general relativity the Schwarzschild solution is regarded as a black hole in an asymptotically flat space. The Schwarzschild-de Sitter solution is interpreted as a black hole in an asymptotically de Sitter space with non-zero cosmological constant $\Lambda$ (Gibbons and Hawking, 1977). The Schwarzschild-de Sitter solution is also considered as an embedded black hole that the Schwarzschild solution is embedded into the de Sitter space with cosmological constant $\Lambda$ to
produce the Schwarzschild-de Sitter black hole (Cai, et al. 1998). The Vaidya solution having a variable mass \( m(u) \) with retarded time \( u \) is a non-stationary generalization of Schwarzschild black hole of constant mass \( m \) (Vaidya, 1999). Mallett (1985) has introduced Vaidya-de Sitter solution with constant \( \Lambda \) by making the Schwarzschild mass \( m \) variable with respect to the retarded time \( u \) as \( m(u) \) and studied the nature of the Vaidya-de Sitter space-time (Mallett, 1986).

Here the idea of this chapter is to propose an exact solution of Einstein’s field equations describing Vaidya black hole embedded into the non-stationary de Sitter space to obtain Vaidya-de Sitter black hole with variable \( \Lambda(u) \). This Vaidya-de Sitter solution with variable \( \Lambda(u) \) will have the limit \( m(u) = \pm(1/3)\Lambda(u)^{(-1/2)} \) of the Vaidya mass \( m(u) \) in the extreme black hole case \( 9\Lambda(u)m^2(u) = 1 \), which could not explain with the constant \( \Lambda \) in Mallett (1986). This situation can be seen in the next section of this chapter.

It is well known that the original de Sitter cosmological model is conformally flat \( C_{abcd} = 0 \) space-time with constant curvature \( R_{abcd} = (\Lambda/3)(g_{ac}g_{bd} - g_{ad}g_{bc}) \) (Hawking and Ellis, 1973). It also describes the non-rotating and stationary solution. Therefore, the non-rotating stationary de Sitter model with a cosmological constant \( \Lambda \) is a solution of Einstein’s field equations for an non-empty space \( T_{ab} \neq 0 \) possessing a ratio of pressure \( p = -\Lambda/K \) to the density \( \rho = \Lambda/K \) known as an equation of state parameter \( w = p/\rho = -1 \). This negative parameter \( w = -1 \) is the most important characteristic property of de Sitter space to be considered as a candidate of dark energy solution of Einstein’s field equations (Padmanabhan 2003, Copeland, et al. 2006 and Bousso 2008) and references there in. We have also seen from (2.12) below that the equation of state parameter does not change the form \( w = p/\rho = -1 \) for the non-stationary de Sitter solution with variable \( \Lambda(u) \). The de Sitter
metric has singularity horizons at \( r = \pm \{3\Lambda^{(-1)}(u)\}^{1/2} \) (Ibohal, 2009).

The black hole embedded into de Sitter space plays an important role in classical general relativity that the cosmological constant is found present in the inflationary scenario of the early universe in a stage where the universe is geometrically similar to the original de Sitter space (Guth, 1981). Also embedded black holes can avoid the direct formation of negative mass naked singularities during Hawking’s black hole evaporation process (Ibohal, 2005a). It is also known that the rotating Vaidya-Bonnor black hole with variable mass \( m(u) \) and charge \( e(u) \) is a non-stationary solution. When \( m(u) \) and \( e(u) \) become constants, the rotating Vaidya-Bonnor black hole reduces to the stationary Kerr-Newman black hole. If one wishes to study the physical properties of the gravitational field of a complete non-stationary embedded black hole, e.g. rotating non-stationary Vaidya-Bonnor-de Sitter (not discussed in this note), one needs to derive a new rotating non-stationary de Sitter model with a cosmological term of variable function \( \Lambda(u) \). That is, an observer traveling in a non-stationary space-time must also be able to find a non-stationary cosmological de Sitter space to embed, having a similar space-time structure with time dependent functions. This is the main idea of the paper to derive an embedded Vaidya-de Sitter solution with time dependent mass \( m(u) \) and cosmological variable \( \Lambda(u) \).

The plan of this chapter is as follows: Section 5.2 deals with a brief introduction of the non-stationary de Sitter space-time having variable cosmological function \( \Lambda(u) \) and its physical properties which are different from those of the stationary de Sitter solution with cosmological constant \( \Lambda \). Section 5.3 develops the derivation of an embedded Vaidya-de Sitter black hole with variable \( \Lambda(u) \) based on the power series expansion of the mass function (Wang and Wu, 1999). We find the energy conditions for the energy-momentum tensor of the
Vaidya-de Sitter solution. We also show that the time-like vector field of the observer in the Vaidya-de Sitter space is expanding, accelerating, shearing but non-zero twist. We also derive the temperature proportional to the surface gravity of the space-time. The chapter is concluded in Section 4 with reasonable remarks and evolution of the solutions with the physical interpretation. We prove the energy conservation laws for the energy-momentum tensor in the section (5.3.6) below.

## 5.2 Non-stationary de Sitter space

Here we shall briefly introduce the non-stationary de Sitter space with variable $\Lambda(u)$ and its physical properties (Ibohal, 2009). The derivation of non-stationary de Sitter space is based on the mass function expressed in terms of Wang-Wu function $q_n(u)$ (Wang and Wu, 1999) as

$$M(u, r) \equiv \sum_{n=-\infty}^{+\infty} q_n(u) r^n. \quad (5.2.1)$$

Here the $u$-coordinate is related to the retarded time in flat space-time. The $u$-constant surfaces are null cones open to the future. The $r$-constant is null coordinate. The retarded time coordinate are used to evaluate the radiating (or outgoing) energy momentum tensor around the astronomical body (Chandrasekhar, 1983).

In the context of the derivation of de Sitter space with variable $\Lambda(u)$, we consider the Wang-Wu function as

$$q_n(u) = \begin{cases} 
\Lambda(u)/6, & \text{when } n = 3 \\
0, & \text{when } n \neq 0,
\end{cases} \quad (5.2.2)$$

such that the equation (5.2.1) becomes

$$M(u, r) = \frac{1}{6} r^3 \Lambda(u). \quad (5.2.3)$$

Ng. Ishwarchandra
Then using this mass function in the Schwarzschild-like metric

\[
ds^2 = \left\{ 1 - \frac{2M(u, r)}{r} \right\} du^2 + 2 du \, dr - r^2 d\Omega^2. \tag{5.2.4}\]

with \( d\Omega^2 = d\theta^2 + \sin^2 \theta \, d\phi^2 \), we find a metric, describing a non-stationary de Sitter model with cosmological function \( \Lambda(u) \) in the null coordinate system \((u, r, \theta, \phi)\) as

\[
ds^2 = \left\{ 1 - \frac{1}{3} r^2 \Lambda(u) \right\} du^2 + 2 du \, dr - r^2 d\Omega^2, \tag{5.2.5}\]

Here \( \Lambda(u) \) is an arbitrary non-increasing function of the retarded time coordinate \( u \). From Einstein’s field equations \( R_{ab} - \frac{1}{2} R g_{ab} = -K T_{ab} \), we find that the metric (5.2.5) possesses an energy momentum tensor as

\[
KT_{ab} = -\frac{1}{3} r \Lambda(u) \epsilon_a \epsilon_b + \Lambda(u) g_{ab} \tag{5.2.6}\]

where \( \epsilon_a = \delta_a^1 \) is a null vector and the universal constant \( K = 8\pi G/c^4 \). The trace of the tensor (5.2.6) is given by \( KT = 4\Lambda(u) \). Here it is worth to mention that the energy-momentum tensor (5.2.6) involves a Vaidya-like null radiation term \(-\frac{1}{3} r \Lambda(u) \epsilon_a \epsilon_b\), which arises from the non-stationary state of motion of an observer traveling in the non-stationary de Sitter universe (5.2.5). This non-stationary part of the energy momentum tensor (5.2.6) contributes the nature of null-radiating matters present in (5.2.5) whose energy momentum tensor will vanish when \( r \to 0 \), and has zero trace. However, it still maintains the non-stationary status \( \Lambda(u) \neq \text{constant} \), showing that the space-time of the observer is naturally time dependent even at \( r \to 0 \). Let us denote it as \( T^{\text{NS}}_{ab} \), where ‘NS’ stands for non-stationary as it arises from the non-stationary state of the universe.
5.2 Non-stationary de Sitter space

Using the energy-momentum tensor (5.2.6), we could write Einstein’s field equations as follows

\[ R_{ab} - \frac{1}{2} R g_{ab} + \Lambda(u) g_{ab} = -T_{ab}^{(NS)}, \]  

(5.2.7)

where \( T_{ab}^{(NS)} \) arises from the non-stationary state of motion depending on \( u \)-coordinate and is given by

\[ T_{ab}^{(NS)} = \frac{1}{3} r \Lambda(u), u \ell_a \ell_b, \]  

(5.2.8)

which is zero either at \( r \to 0 \) or when \( \Lambda(u) \) becomes constant \( \Lambda \). It is noted that the right side of the equation (5.2.7) does not involve the universal constant \( K \). From equations (5.2.7) and (5.2.8), we conclude that the derivation of this non-stationary de Sitter cosmological universe (5.2.5) (Ibohal, 2009) is in agreement with the original stationary de Sitter model (Hawking and Ellis, 1973) when \( \Lambda(u) \) takes a constant value.

Now expressing the metric tensor in terms of null tetrad vectors \( \{\ell_a, n_a, m_a, \bar{m}_b\} \) as \( g_{ab} = 2 \ell(a)n(b) - 2m(a)\bar{m}(b) \) (Newman and Penrose, 1962), we can write the energy-momentum tensor (5.2.6) as follows

\[ T_{ab} = \mu \ell_a \ell_b + 2 \rho \ell(a)n(b) + 2 p m(a)\bar{m}(b), \]  

(5.2.9)

where \( \rho \) and \( p \) denote the density and the pressure of the non-stationary de Sitter space respectively, \( \mu \) the null density arisen from the non-stationary state of variable \( \Lambda(u) \), and are obtained as

\[ \rho = -p = \frac{\Lambda(u)}{K}, \quad \mu = -\frac{r}{3K} \Lambda(u), u. \]  

(5.2.10)

The trace of energy-momentum tensor \( T_{ab} \) (5.2.9) is given as

\[ T \equiv g^{ab}T_{ab} = 2(\rho - p) = \frac{4}{K} \Lambda(u). \]  

(5.2.11)
Here it is observed that $\rho - p > 0$ for the non-stationary de Sitter model with $\Lambda(u) > 0$. We have also seen that the trace (5.2.11) is different from the one $T^{(pf)} = \rho - 3p$ of a perfect fluid $T^{(pf)}_{ab} = (\rho + p)u_a u_b - p g_{ab}$ with a unit time-like vector $u_a$. From (5.2.10) we find the equation of state parameter as the ratio of the pressure to the density

$$w = \frac{p}{\rho} = -1 \quad (5.2.12)$$

with the negative pressure associated with the variable $\Lambda(u)$. This shows the fact that the non-stationary de Sitter solution (5.2.5) is in agreement with the cosmological constant $\Lambda$ de Sitter model possessing the equation of state $w = -1$ in the dark energy scenario (Padmanabhan 2003, Copeland, et al. 2006 and Bousso 2008) and references there in, when $\Lambda(u)$ takes a constant value $\Lambda$. The Ricci scalar $\Lambda^* (\equiv \frac{1}{24} g^{ab} R_{ab})$, describing matter field takes the form

$$\Lambda^* = \frac{1}{6} \Lambda(u). \quad (5.2.13)$$

The metric (5.2.5) has an apparent singularity at $r = \pm \{3\Lambda^{-1}(u)\}^{1/2}$. The root $r_+ = 3^{1/2} \Lambda(u)^{-1/2}$ corresponds to a cosmological event horizon.

According to Carter (1973) and York (1983), we introduce a scalar $\kappa$ defined by the relation $n^b \nabla_b n^a = \kappa n^a$, where the null vector $n^a$ is parameterized by the coordinate $u$, such that $d/du = n^a \nabla_a$. On the horizon $r = r_+$, the scalar $\kappa$ is referred to the surface gravity of the de Sitter model and is obtained as

$$\kappa = 3^{-1/2} \Lambda(u)^{1/2}.$$

The entropy $S$ on the horizon is related with the area $A$ of the horizon as $S = A/4$ and is obtained as

$$S = 3\pi \Lambda(u)^{-1}.$$
The de Sitter horizon $r = r_+$ has a temperature obtained from the relation

\[ \hat{T} = \frac{\kappa}{2\pi} \Lambda(u)^{1/2}. \]  

(5.2.14)

It is also found that the non-stationary de Sitter space-time is conformally flat ($C_{abcd} = 0$). The Kretschmann scalar for non-rotating de Sitter model (5.2.5) takes the form

\[ K \equiv R_{abcd}R^{abcd} = \frac{8}{3} \Lambda(u)^2, \]  

(5.2.15)

which does not involves any derivative term of $\Lambda(u)$, and will not change its value at $r \to 0$ and $r \to \infty$. The above Kretschmann scalar will become the one of original de Sitter model when $\Lambda(u)$ takes a constant value $\Lambda$.

## 5.3 Vaidya black hole in non-stationary de Sitter space

In this section we propose an exact solution describing the non-rotating Vaidya-de Sitter solution with a non-stationary variable $\Lambda(u)$, which may be treated as the non-stationary Vaidya-de Sitter black hole or the Vaidya black hole on the non-stationary de Sitter background with variable $\Lambda(u)$. It is emphasized that the Vaidya solution embedded into the constant $\Lambda$ de Sitter space is known (Mallett, 1985); however, we have not seen published the Vaidya solution embedded into the variable $\Lambda(u)$ de Sitter space. For deriving the Vaidya-de Sitter solution with $\Lambda(u)$ we consider the Wang-Wu functions $q(u)$ in (5.2.1) as follows:

\[ q_n(u) = \begin{cases} M(u), & \text{when } n = 0 \\ \Lambda(u)/6, & \text{when } n = 3 \\ 0, & \text{when } n \neq 0,3, \end{cases} \]  

(5.3.1)
such that the mass function (5.2.1) takes the form

\[ M(u, r) = M(u) + \frac{1}{6} r^3 \Lambda(u). \] (5.3.2)

Then using this mass function in the metric presented in equation (5.2.4), we obtain a non-stationary metric, describing the Vaidya metric embedded into the non-stationary de Sitter model to produce Vaidya-de Sitter solution with variable cosmological function \( \Lambda(u) \) as

\[ ds^2 = \left\{ 1 - \frac{2M(u)}{r} - \frac{1}{3} r^2 \Lambda(u) \right\} du^2 + 2dudr - r^2d\Omega^2, \] (5.3.3)

where \( M(u) \) is the mass of the Vaidya black hole and \( \Lambda(u) \) denotes the de Sitter cosmological function of retarded time coordinate \( u \). If the Vaidya mass vanishes \( M(u) = 0 \), the line element will recover the non-stationary de Sitter solution (5.2.5) above. When we set the function \( \Lambda(u) \) to be a constant \( \Lambda \), the line element (5.3.3) will become the Vaidya-de Sitter space-time with cosmological constant (Mallett, 1985). If \( M(u) = 0 \), and function \( \Lambda(u) \) sets to be a constant \( \Lambda \), the line element will recover the original de Sitter solution (Hawking and Ellis, 1973). The metric (5.3.3) has a singularity when

\[ g_{uu} = 1 - \frac{2M(u)}{r} - \frac{1}{3} r^2 \Lambda(u) = 0. \]

The complex null vectors for the Vaidya-de Sitter solution can be chosen as follows:

\[ \ell_a = \delta_a^1, \quad n_a = \frac{1}{2r^2} \Delta \delta_a^1 + \delta_a^2, \]
\[ m_a = -\frac{r}{\sqrt{2}} \left\{ \delta_a^3 + i\sin \theta \delta_a^4 \right\}, \] (5.3.4)

where \( \Delta = r^2 - 2rM(u) - \Lambda(u) r^4/3 \). Here \( \ell_a, n_a \) are real null vectors and \( m_a \) is complex with the normalization conditions \( \ell_an^a = 1 = -m_am^a \) and the other inner products of the null vectors are zero.

From Einstein’s field equations \( R_{ab} - (1/2)Rg_{ab} = -KT_{ab} \), we find the energy-momentum tensor for the matter field of the non-stationary space-time...
5.3.1 Decomposition of energy-momentum tensor...

(5.3.3) as

\[ T_{ab} = \mu \ell_a \ell_b + 2 \rho \ell_a n_b + 2 p m_a \bar{m}_b, \quad (5.3.5) \]

where the coefficients \( \rho, p \) and \( \mu \) denote the density, the pressure and the null density, respectively and are found as

\[
\begin{align*}
\rho &= - p = \frac{\Lambda(u)}{K}, \\
\mu &= - \frac{1}{Kr^2} M(u)_a - \frac{r}{3K} \Lambda(u)_u. 
\end{align*}
\] (5.3.6)

The trace of energy-momentum tensor \( T_{ab} \) (5.3.5) is given as

\[ T \equiv g^{ab} T_{ab} = 2(\rho - p) = \frac{4}{K} \Lambda(u). \] (5.3.7)

Here it is observed that \( \rho - p > 0 \) for the non-stationary de Sitter model and the trace \( T \) does not involve the Vaidya mass \( m(u) \), showing the null fluid distribution of the space-time (5.3.3). The Ricci scalar \( \Lambda^* \) \( (\equiv (1/24)g^{ab} R_{ab}) \), describing matter field takes the form \( \Lambda^* = (1/6)\Lambda(u) \). The energy-momentum tensor (5.3.5) satisfies the energy conservation equation

\[ T^{ab}_{\;\;\;\;b} = 0. \] (5.3.8)

The proof of this conservation law is established in (5.3.38)–(5.3.40) below, showing that the embedded Vaidya-de Sitter space-time (5.3.3) with variable \( \Lambda(u) \) is a solution of Einstein’s field equations.

5.3.1 Decomposition of energy-momentum tensor of Vaidya-de Sitter space

As the physical properties of a space-time geometry are, in general relativity determined by the nature of the matter distribution in the space, it is convenient to express the energy-momentum tensor (5.3.5) in such a way that one
5.3.1 Decomposition of energy-momentum tensor...

should be able to understand it easily. Thus, the total energy-momentum
tensor (EMT) for the solution (5.3.3) may, without loss of generality, be written
in the following decomposition form as:

\[ T_{ab} = T^{(V)}_{ab} + T^{(NS)}_{ab} + T^{(dS)}_{ab}, \]  

(5.3.9)

where the \( T^{(V)}_{ab} \) stands for the Vaidya null radiating fluid, \( T^{(NS)}_{ab} \) the non-
stationary contribution of de Sitter field \( \Lambda(u) \) associated with the derivative
term \( \Lambda(u)_u \) and \( T^{(dS)}_{ab} \) the cosmological de Sitter matter are given, respectively

\[ T^{(V)}_{ab} = \mu^{(V)} \ell_a \ell_b, \]  

(5.3.10)

\[ T^{(NS)}_{ab} = \mu^{(NS)} \ell_a \ell_b, \]  

(5.3.11)

\[ T^{(dS)}_{ab} = 2 \rho^{(dS)} \ell_{(a} \bar{m}_{b)} + 2 p^{(dS)} \ell_{(a} \bar{m}_{b)}, \]  

(5.3.12)

with the coefficients

\[ \mu^{(V)} = -\frac{1}{Kr^2} M(u)_u, \quad \mu^{(NS)} = -\frac{r}{3K} \Lambda(u)_u \]

\[ \rho^{(dS)} = -p^{(dS)} = \frac{\Lambda(u)}{K}, \]  

(5.3.13)

where \( \mu^{(V)} \) is the null density for the Vaidya null fluid \( T^{(V)}_{ab} \), \( \mu^{(NS)} \) the non-
stationary null density associated with the derivative of \( \Lambda(u) \), \( \rho^{(dS)} \) and \( p^{(dS)} \)
are the density and the pressure of de Sitter matter. When the function \( \Lambda(u) \)
becomes constant, it will provide \( T^{(NS)}_{ab} = 0 \), then the space-time will be that of
the Vaidya-de Sitter with constant \( \Lambda \). If \( M(u) = 0, \Lambda(u) \neq \text{constant} \) we have
\( T^{(V)}_{ab} = 0 \), then the remaining space-time (5.3.3) will be the non-rotating non-
stationary de Sitter model with variable \( \Lambda(u) \) (Ibohal, 2009). At that stage
the EMTs \( T^{(NS)}_{ab} \) and \( T^{(dS)}_{ab} \) will exist indicating the non-stationary de Sitter
matter distribution; and for constant \( \Lambda \), \( T^{(NS)}_{ab} = 0 \), then the total energy-
momentum tensor (5.3.9) will reduce to \( T_{ab} = T^{(dS)}_{ab} = \Lambda g_{ab} \) for the well-known
de Sitter model with constant \( \Lambda \).
For the metric (5.3.3) the energy-momentum tensor (5.3.5) can be written in the form of Guth’s modification of $T_{ab} \rightarrow T_{ab} + \Lambda g_{ab}$ for early inflation of the Vaidya black hole (Guth, 1981) as

$$T_{ab} = T_{ab}^{(V)} + T_{ab}^{(NS)} + \Lambda(u)g_{ab}, \quad (5.3.14)$$

where $g_{ab}$ is the non-rotating de Sitter metric tensor. This energy-momentum tensor represents the matter distribution of radiating Vaidya black hole in the non-stationary de Sitter background space with cosmological variable $\Lambda(u)$. The dynamical evolution of the non-stationary de sitter background is shown by the present of $T_{ab}^{(NS)}$ which is zero at $r \rightarrow 0$ without disturbing the non-stationary status of $\Lambda(u)$.

### 5.3.2 Kerr-Schild ansatze of Vaidya-de Sitter solution with $\Lambda(u)$

Here we shall clarify the nature of the embedded solution in the form of Kerr-Schild ansatze in different backgrounds. The Vaidya-de Sitter metric can be expressed in Kerr-Schild ansatz

$$g_{ab}^{(VdS)} = g_{ab}^{(dS)} + 2Q(u,r)\ell_a\ell_b \quad (5.3.15)$$

where $Q(u,r) = -M(u) r^{-1}$. Here, $g_{ab}^{(dS)}$ is the non-stationary de Sitter metric given in (5.2.5) and $\ell_a$ is geodesic, shear free, expanding and zero twist null vector for both $g_{ab}^{(dS)}$ as well as $g_{ab}^{(VdS)}$. The above Kerr-Schild form can also be recast on the Vaidya background as

$$g_{ab}^{(VdS)} = g_{ab}^{(V)} + 2Q(u,r)\ell_a\ell_b \quad (5.3.16)$$

where $Q(u,r) = -\Lambda(u)r^2/6$. These two Kerr-Schild forms (5.3.15) and (5.3.16) support the fact that the non-stationary Vaidya-de Sitter space-time (5.3.3)
with variable $\Lambda(u)$ is a solution of Einstein’s field equations. They establish the structure of embedded black hole that either “the null radiating Vaidya black hole is embedded into the non-stationary de Sitter cosmological space to obtain Vaidya-de Sitter black hole” or the non-stationary de Sitter universe is embedded into the Vaidya black hole to obtain the de Sitter-Vaidya black hole – both nomenclatures (Vaidya-de Sitter and de Sitter-Vaidya) possess geometrically the same physical structure. This is the most exotic characteristic feature of embedded solutions that we cannot physically predict which space started first to embed into another. When the cosmological function $\Lambda(u)$ takes a constant value $\Lambda$, the line element (5.3.3) will reduces to non-rotating Vaidya-de Sitter black hole (Mallett, 1985). If we set $M(u)$ and $\Lambda(u)$ both constant, the metric (5.3.3) becomes the stationary Schwarzschild-de Sitter black hole (Gibbon and Hawking, 1977). This means that the Kerr-Schild ansatze (5.3.15) and (5.3.16) will be those of embedded Schwarzschild-de Sitter black hole with constant $m$ and $\Lambda$. However, in the context of combination of exact solutions, it is to mention the fact that the two metrics $g_{ab}^{(V)}$ and $g_{ab}^{(dS)}$ cannot be added to obtain $g_{ab}^{(VdS)}$ as

$$g_{ab}^{(VdS)} \neq \frac{1}{2} \left\{ g_{ab}^{(V)} + g_{ab}^{(dS)} \right\}. \quad (5.3.17)$$

This indicates that it is not possible to derive a new embedded solution by adding two physically known solutions in general relativity.

### 5.3.3 Surface gravity of Vaidya-de Sitter solution with $\Lambda(u)$

The metric (5.3.3) will describe a cosmological black hole with the horizons at the values of $r$ for which the polynomial equation $\Delta = r^2 - 2rm(u) - \Lambda(u)r^4/3 = 0$ has three roots $r_1$, $r_2$, and $r_3(= \bar{r}_2)$. The explicit roots are
given as

\begin{align*}
  r_1 &= -\frac{1}{(3Q)^{\frac{1}{3}}} - \frac{1}{\Lambda(u)}(3Q)^{\frac{1}{3}} \\
  r_2 &= \frac{1}{2\Lambda(u)} [(1 + i\sqrt{3})\Lambda(u)(3Q)^{-\frac{1}{3}} + (1 - i\sqrt{3})(3Q)^{\frac{1}{3}}] \\
  r_3 &= \frac{1}{2\Lambda(u)} [(1 - i\sqrt{3})\Lambda(u)(3Q)^{-\frac{1}{3}} + (1 + i\sqrt{3})(3Q)^{\frac{1}{3}}]
\end{align*}

(5.3.18)

where

\[ Q = \Lambda^2(u) \{ M(u) + (1/3)\Lambda(u)^{(-1/2)} \times \sqrt{9\Lambda(u)m^2(u) - 1} \}. \]  

These roots satisfy the following relation

\[ (r - r_1)(r - r_2)(r - r_3) = \frac{-3}{\Lambda(u)} \left\{ r - 2M(u) - \frac{\Lambda(u)}{3} r^3 \right\}. \]  

Here we are interested only the real root \( r_1 \) which may describe the horizon of the Vaidya-de Sitter cosmological black hole, as the complex roots \( r_2, r_3 \) have less physical interpretation.

The surface gravity \( \kappa \) of a horizon is defined by the relation \( n^b \nabla_b n^a = \kappa n^a \), where the null vector \( n^a \) in (5.3.5) above is parameterized by the coordinate \( u \), such that \( d/du = n^b \nabla_b \) (Carter, 1973 and York, 1983). \( \nabla_b \) is the covariant derivative. The surface gravities at \( r = r_i (i = 1, 2, 3) \) can be obtained from \( \kappa = -2\gamma \) as

\[ \kappa = -\frac{1}{r^3} \left\{ r - M(u) - \frac{\Lambda(u)r^3}{6} \right\} \bigg|_{r=r_i}, \]  

(5.3.20)

where the \( \gamma \) is the spin coefficient given by

\[ \gamma = \frac{1}{2r^2} \left\{ m(u) - \frac{1}{3} r^3 \Lambda(u) \right\}. \]

and accordingly the temperature of the horizon \( r = r_1 \) can be obtained from the Bekenstein-Hawking relation

\[ T = \kappa/2\pi. \]  

(5.3.21)
An area of the horizon of black hole can be obtained as follows (Chandrasekhar, 1983):

\[ A = \int_0^\pi \int_0^{2\pi} \sqrt{g_{\theta\theta}g_{\phi\phi}} \, d\theta \, d\phi \bigg|_{\Delta^* = 0} \]  

(5.3.22)

depending on the values of the roots of \( \Delta^* = 0 \). The entropy proportional to the area \( S = A/4 \) of the horizon \( r = r_1 \) is found as,

\[ S = \pi r_1^2 \bigg|_{r = r_1}. \]  

(5.3.23)

Here in (5.3.19) we consider a case of extreme Vaidya-de Sitter black hole having the mass function \( M(u) = \pm(1/3)\Lambda(u)^{-1/2} \), if \( \Lambda(u) > 0 \). This implies that the real root \( r_1 \) take the values \( r_1 = -2\Lambda(u)^{-1/2} \), and the two complex roots are coincided \( r_2 = r_3 = \Lambda(u)^{-1/2} \). The surface gravity on the cosmological black hole horizon \( r = r_1 \) takes the form

\[ \kappa = -\frac{1}{r^2} \left\{ r - M(u) - \frac{2\Lambda(u)r^3}{3} \right\} = -(3/4)\Lambda(u)^{-1/2}. \]

However, it is vanished at \( r = r_2 = r_3 \). Then we obtain the Hawking’s temperature of the cosmological black hole horizon at \( r = r_1 \) from the relation \( T = \kappa/2\pi \) as

\[ T = -\frac{3}{8\pi} \Lambda(u)^{-\frac{1}{2}}. \]  

(5.3.24)

The temperature associated with the real root \( r_1 \) in (5.3.18) will never vanish as long as the de Sitter variable \( \Lambda(u) \) exists in the space-time geometry of the Vaidya-de Sitter black hole. The condition \(-(1/3)\Lambda(u)^{(-1/2)} \leq M(u) \leq +(1/3)\Lambda(u)^{(-1/2)} \) of the Vaidya mass \( M(u) \) is the non-stationary generalization of the condition \(-(1/3)\Lambda^{(-1/2)} \leq m \leq +(1/3)\Lambda^{(-1/2)} \) of the Schwarzschild-de Sitter black hole with constant mass and \( \Lambda \) (Gibbon and Hawking 1977).
5.3.4 Energy conditions of Vaidya-de Sitter solution with $\Lambda(u)$

We observe that the energy-momentum tensor (5.2.9) for non-stationary de Sitter solution and that (5.3.5) of Vaidya-de Sitter have the same form. It will be convenient to discuss the energy conditions in details for future utilization. For the study of the energy conditions, we shall conveniently introduce an orthonormal tetrad with a unit time-like $u^a$ and three unit space-like vector fields $v^a, w^a, z^a$ using the null tetrad vectors (5.3.4) such as

\begin{align*}
  u_a &= \frac{1}{\sqrt{2}}(\ell_a + n_a), \quad v_a = \frac{1}{\sqrt{2}}(\ell_a - n_a), \\
  w_a &= \frac{1}{\sqrt{2}}(m_a + \bar{m}_a), \quad z_a = -\frac{i}{\sqrt{2}}(m_a - \bar{m}_a), \\
  \quad (5.3.25)
\end{align*}

with the normalization conditions $u_a u^a = 1, v_a v^a = w_a w^a = z_a z^a = -1$ and other inner products being zero. Then the metric tensor can be written as

\begin{equation}
  g_{ab} = u_a u_b - v_a v_b - w_a w_b - z_a z_b. \quad (5.3.26)
\end{equation}

Now we consider a non-space like vector field for an observer in the space-time (5.3.3)

\begin{equation}
  U_a = \hat{\alpha}u_a + \hat{\beta}v_a + \hat{\gamma}w_a + \hat{\delta}z_a, \quad (5.3.27)
\end{equation}

where $\hat{\alpha}, \hat{\beta}, \hat{\gamma}$ and $\hat{\delta}$ are arbitrary constants with the condition (Cai, et al 2003) that

\begin{equation}
  U^a U_a = \hat{\alpha}^2 - \hat{\beta}^2 - \hat{\gamma}^2 - \hat{\delta}^2 \geq 0. \quad (5.3.28)
\end{equation}

Then the energy-momentum tensor (5.3.5) can be written in terms of the orthonormal tetrad vectors given in (5.2.15) as

\begin{equation}
  T_{ab} = \mu \ell_a \ell_b + (\rho + p)(u_a u_b - v_a v_b) - p g_{ab}. \quad (5.3.29)
\end{equation}
Now $T_{ab} U^a U^b$ will represent the energy density as measured by the observer with the tangent vector $U^a$ (5.3.25). It is to emphasize that this $T_{ab}$ is a general energy-momentum tensor of non-rotating stationary space-times. It includes those of electromagnetic field, monopole, de Sitter cosmological model etc. However, it does not include a perfect fluid with unit time-like vector $u_a$.

Hence, the energy-momentum tensor (5.3.27) is different from the one of a perfect fluid $T^{(pf)}_{ab} = (\rho + p) u_a u_b - p g_{ab}$ with unit time-like vector $u_a$, and it is convenient to introduce all the energy conditions for $T_{ab}$ for future use:

(a) **Weak energy condition**: The energy momentum tensor obeys the inequality $T_{ab} U^a U^b \geq 0$ for any future directed time-like vector $U^a$ which implies that

$$\mu \geq 0, \quad \rho \geq 0, \quad \rho + p \geq 0.$$ (5.3.30)

(b) **Strong energy condition**: The Ricci tensor for $T_{ab}$ satisfies the inequality $R_{ab} U^a U^b \geq 0$ for any time-like vector $U^a$, i.e. $T_{ab} U^a U^b \geq (1/2) T$, which yields

$$\mu \geq 0, \quad p \geq 0, \quad \rho + p \geq 0.$$ (5.3.31)

(c) **Dominant energy condition**: For any future directed time-like vector $U^a$, $T_{ab} U^b$ should be a future directed non-space like (time-like or null) vector field. This condition is equivalent to

$$\mu \geq 0, \quad \rho^2 \geq 0, \quad \rho^2 - p^2 \geq 0.$$ (5.3.32)

It is noted that the strong energy condition does not imply the weak energy condition.

Here, we find that the pressure $p$ with the minus sign given in (5.3.13) does not satisfy the strong energy condition (5.3.29). This violation of the
strong energy condition is due to the negative pressure, and may lead to a repulsive gravitational force of the matter field associated with $\Lambda(u)$ in the space-time (5.3.3). The violation indicates different properties of the energy-momentum tensor (5.3.5) or (5.3.9) with (5.3.13) from those of the ordinary matter fields, like perfect fluid, electromagnetic field etc., having positive pressures. In particular, it is to note that this strong energy condition is satisfied by the energy-momentum tensor of electromagnetic field with $\rho = p = e^2/(Kr^4)$ of Reissner-Nordstrom space-time.

5.3.5 Properties of the time-like vector of Vaidya-de Sitter solution with $\Lambda(u)$

The motion of a matter distribution is determined by the nature of the time-like vector field $u_a = (1/\sqrt{2})(\ell_a + n_a)$ associated with energy-momentum tensor (5.3.29). The 4-velocity vector measures the kinematical properties of a fluid – expansion ($\Theta = u^a_{;a}$), acceleration ($\dot{u}_a = u_{ab}u^b$), shear $\sigma_{ab}$ and twist ($w_{ab}$). It is found that the fluid flow of the Vaidya-de Sitter model having variable $\Lambda(u)$ is expanding ($\Theta = u^a_{;a} \neq 0$), accelerating ($\dot{u}_a = u_{ab}u^b \neq 0$), shearing $\sigma_{ab} \neq 0$ and zero twist ($w_{ab} = 0$).

\[
\begin{align*}
\Theta &= \frac{1}{\sqrt{2}r^2} \left\{ r + M(u) + \frac{2}{3} r^3 \Lambda(u) \right\} \\
\dot{u}_a &= \frac{1}{\sqrt{2}r^2} \left\{ M(u) - \frac{1}{3} r^3 \Lambda(u) \right\} v_a \\
\sigma_{ab} &= -\frac{1}{3\sqrt{2}r^2} \left\{ r + 4M(u) - \frac{1}{3} r^3 \Lambda(u) \right\} \times \left( v_a v_b - m_{(a} m_{b)} \right)
\end{align*}
\]

where $v_a = \frac{1}{\sqrt{2}}(\ell_a - n_a)$ is a space-like vector field $v^a v_a = -1$. From these we observe that both the mass $M(u)$ and the variable $\Lambda(u)$ are appeared in all the three equations. When $M(u) = 0$, the remaining equations will be for non-stationary de Sitter space, whereas, if $\Lambda(u) = 0$, these will be for Vaidya radiating black hole. This means that the time-like observer in the Vaidya
space will have a four velocity vector field which is expanding, accelerating, shearing but zero-twist. This is also true for the observer in non-stationary de Sitter space, when $M(u) = 0$.

The non-rotating Vaidya-de Sitter metric (5.3.3) describes a non-stationary embedded spherically symmetric solution whose Weyl curvature tensor is type D

$$\psi_2 \equiv -C_{pqrs}n^p n^q n^r n^s = -M(u)r^{-3}$$

(5.3.36)
in Petrov classification possessing a geodesic, shear free, expanding and zero-twist null vector $\ell_a$ given in (5.3.5), as other Weyl scalars are vanished $\psi_0 = \psi_1 = \psi_3 = \psi_4 = 0$. The Kretschmann scalar for non-rotating Vaidya-de Sitter model (5.3.3) takes the form

$$R_{abcd}R^{abcd} = \frac{48}{r^6}m^2(u) + \frac{8}{3}\Lambda^2(u),$$

(5.3.37)

which does not involves any derivative term of $M(u)$ and $\Lambda(u)$, and will not change its form when the mass $m(u)$ and the variable $\Lambda(u)$ take the constant values. With the constant values $m$ and $\Lambda$, the Kretschmann scalar will reduce to that of Schwarzschild-de Sitter solution. This invariant (5.3.37) does not diverge at the origin $r \to 0$.

5.3.6 Energy conservation equations of Vaidya-de Sitter solution with $\Lambda(u)$

In this appendix we shall show the fact that the energy-momentum tensor (5.3.5) for the embedded Vaidya-de Sitter solution satisfies the energy conservation equations $T^{ab};_b = 0$. These four equations can equivalently be transcribed in the NP complex spin coefficients Newman and Penrose (1962) – two real and one complex. This transcription can be obtained from Ibohal (2009)
5.3.6 Energy conservation equations of Vaidya-de Sitter...

for a non-rotating space-time geometry as follows:

\[ D\rho = (\rho + p) (\rho^* + \bar{\rho}^*), \]  
\[ \delta p = \mu \kappa^* + (\rho + p)(\tau - \bar{\pi}), \]  
\[ D\mu + \nabla \rho = \mu \{(\rho^* + \bar{\rho}^*) - 2(\epsilon + \bar{\epsilon})\} \]
\[ - (\rho + p)(\mu^* + \bar{\mu}^*), \]  

where \( \kappa^*, \rho^*, \mu^*, \tau, \pi, \) etc. are spin coefficients, and \( D, \nabla \) and \( \delta \) are the intrinsic derivative operators. The quantities \( \mu, \rho \) and \( p \) are given in (5.3.6). In fact these (5.3.38)-(5.3.40) are general equations for an energy-momentum tensor of the type (5.3.5).

Now, for verification of the conservation equations (5.3.38)-(5.3.40) with the quantities \( \mu, \rho, p \) associated with the energy-momentum tensor (5.3.5), the NP spin coefficients for the non-stationary metric (5.3.3) can be obtained from those of the general rotating metric Ibohal (2005) and are given as:

\[ \kappa^* = \sigma = \lambda = \epsilon = \pi = \tau = \nu = 0, \]
\[ \rho^* = -\frac{1}{r}, \quad \beta = -\alpha = \frac{1}{2\sqrt{2}r} \cot \theta, \]
\[ \mu^* = -\frac{1}{2r} \left\{ 1 - \frac{1}{r} M(u) - \frac{1}{3} r^2 \Lambda(u) \right\}, \]  
\[ \gamma = \frac{1}{2r^2} \left\{ M(u) - \frac{1}{3} r^3 \Lambda(u) \right\}. \]  

The intrinsic derivative operators are given as follows:

\[ D \equiv \ell^a \partial_a = \partial_r, \]
\[ \nabla \equiv n^a \partial_a = \partial_u - \frac{1}{2} \left\{ 1 - \frac{2}{r} M(u) - \frac{r^2}{3} \Lambda(u) \right\} \partial_r \]
\[ \delta \equiv m^a \partial_a = \frac{1}{\sqrt{2}r} \left\{ \partial_{\theta} + \frac{i}{\sin \theta} \partial_{\phi} \right\}. \]

where \( \ell_a, n_a \) and \( m_a \) are the null tetrad vectors associated with the solution (5.3.3). Using the equations (5.3.6), (5.3.41) and (5.3.42), it can be shown that

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all the three equations (5.3.38)-(5.3.40) are satisfied for the Vaidya-de Sitter black hole with variable $\Lambda(u)$, establishing the energy conservation equations $\mathcal{T}^b_{ab} = 0$. This certainly confirms the fact that the embedded Vaidya-de Sitter black hole (5.3.3) derived above is an exact solution of Einstein’s field equations.

5.4 Conclusion of Chapter 5

We present an exact solution of Einstein field equations for the radiating Vaidya black hole embedded in the non-stationary de Sitter background with variable $\Lambda(u)$ (Ibohal, 2009). This solution may be regarded as a generalization of Vaidya-de Sitter solution with constant $\Lambda$ (Mallett, 1985). The Vaidya-de Sitter solution with mass $M(u)$ and constant $\Lambda$ is also a generalized form of the Schwarzschild-de Sitter model with constant mass $m$ and constant $\Lambda$. The Schwarzschild-de Sitter solution is interpreted a black hole in asymptotically de Sitter space (Gibbon and Hawking, 1977). Thus, in this regard, our solution Vaidya-de Sitter with variable $\Lambda(u)$ (5.3.3) may be interpreted as a black hole in asymptotically non-stationary de Sitter space. We have presented the explicit expressions of the three roots of the cubic polynomial $\Delta = 0$, that enable us to explain the nature of the roots when the cosmological black hole becomes the extremal one. From these expressions we observe that in the extreme Vaidya-de Sitter black hole, the Vaidya mass $M(u)$ has the limit $-3^{-1}\Lambda(u)^{(-1/2)} \leq M(u) \leq +3^{-1}\Lambda(u)^{(-1/2)}$ for $\Lambda(u) > 0$ as mentioned above. This range of the mass limit is a straightforward generalization of Schwarzschild-de Sitter case $-3^{-1}\Lambda^{(-1/2)} \leq m \leq +3^{-1}\Lambda^{(-1/2)}$ associated with constant mass $m$ and $\Lambda(> 0)$.

We have shown the dynamical evolution of the non-stationary de Sitter background in the form of energy-momentum tensor (5.3.9), which represents

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the Guth’s modification of momentum tensor as (5.3.14) in the inflationary de Sitter space with a variable $\Lambda(u)$. The characteristic property of the dynamical evolution tensor $T^{(\text{NS})}_{ab}$ will be zero when $r \to 0$. This establishes the fact that the derivation of the embedded Vaidya-de Sitter with variable $\Lambda(u)$ is not the direct replacement of the constant $\Lambda$ by a variable $\Lambda(u)$ in order to obtain the non-stationary de Sitter background of Vaidya black hole. As shown above the derivation of the embedded solution (5.3.3) is based on the power series expansion of the mass function expressed in terms of Wang-Wu function $q_n(u)$ (5.2.1) and with the choice of $q_n(u)$ in (5.3.1). It is also shown in (5.3.38)–(5.3.40) that the energy-momentum tensor for the solution satisfies the energy conservation laws. This confirms that the embedded Vaidya-de Sitter solution having variable $\Lambda(u)$ derived here is clearly an exact solution of Einstein’s field equations.

The energy conditions discussed in (5.3.30)-(5.3.32) are general conditions for any non-stationary space-time having the energy-momentum tensor of the type (5.2.9) for non-stationary de Sitter solution (5.2.5) or that (5.3.5) for the Vaidya-de Sitter solution (5.3.3). These energy conditions are also included those of the energy-momentum tensor of the Vaidya-Bonnor-de Sitter with constant $\Lambda$ (Ibohal, 2005) and those of non-stationary dark energy solution (Ibohal, et al. 2011). From these energy conditions we have seen the violation of the strong energy condition due to the negative pressure, and leading to a repulsive gravitational force of the matter field associated with $\Lambda(u)$ in the space-time (5.3.3). It is observed that the variable $\Lambda(u)$ does not involve in the Weyl scalar $\psi_2$, which shows the conformally flat property of de Sitter space with variable $\Lambda(u)$ even in the embedded Vaidya-de Sitter solution (5.3.3). Also there is no involvement of Vaidya mass $M(u)$ in the expressions of the pressure $p$ and the density $\rho$ (5.3.6) keeping the equation of state parameter...
5.4 Conclusion of Chapter 5

\[ w = \frac{p}{\rho} = -1 \] unaffected for the embedded solution. We have also seen that the 4-velocity vector \( u_a \) of the fluid flow of the Vaidya-de Sitter solution having variable \( \Lambda(u) \) is expanding (\( \Theta = u^a_{;a} \neq 0 \)), accelerating (\( \dot{u}_a = u_{a;b}\,u^b \neq 0 \)), shearing \( \sigma_{ab} \neq 0 \) and zero twist (\( w_{ab} = 0 \)). It is emphasized that the embedded Vaidya-de Sitter solution (5.3.3) discussed here includes the following solutions in particular:

(i) non-stationary de Sitter solution \( \Lambda(u) \neq 0, \, T_{ab}^{(NS)} = T_{ab}^{(dS)} \neq 0 \) when \( m(u) = 0, \, T_{ab}^{(V)} = 0 \) (Ibohal, 2009);

(ii) Vaidya solution \( m(u) \neq 0, \, T_{ab}^{(V)} \neq 0 \) when \( \Lambda(u) = 0, \, T_{ab}^{(NS)} = T_{ab}^{(dS)} = 0 \) (Vaidya, 1999);

(iii) Vaidya-de Sitter \( m(u) \neq 0, \, T_{ab}^{(V)} = T_{ab}^{(dS)} \neq 0 \) when \( \Lambda(u) \) becomes constant and \( T_{ab}^{(NS)} = 0 \), (Mallett, 1985);

(v) Schwarzschild-de Sitter \( T_{ab}^{(dS)} \neq 0 \) when \( M(u) \) and \( \Lambda(u) \) are constant with \( T_{ab}^{(V)} = T_{ab}^{(NS)} = 0 \) (Gibbon and Hawking, 1977);

(v) de Sitter \( T_{ab}^{(dS)} \neq 0 \) when \( m(u) = 0 \) and \( \Lambda(u) \) is constant with \( T_{ab}^{(V)} = T_{ab}^{(NS)} = 0 \) (Hawking and Ellis, 1973).

It is emphasized that the Vaidya solution embedded into the constant \( \Lambda \) de Sitter space is well known; however, we have not seen published the Vaidya solution embedded into the variable \( \Lambda(u) \) de Sitter model as Vaidya-de Sitter solution having \( \Lambda(u) \). It is expected that all known results with constant \( \Lambda \) may be extended with this variable \( \Lambda(u) \) of the Vaidya-de Sitter space-time.