Chapter 4

Mathematical Model Of 2D Cellular Automata

4.1 Introduction

An elegant approach for characterizing various properties of 1D CA, both uniform and hybrid, based on matrix algebraic tools is reported in [Das91, Das90c]. The basic theories underlying the characterization of one dimensional CA has been discussed in chapter 2. However, 2D CA is not a well studied area. Packard et.al, [Packa85b] reported some empirical studies on 2D CA depending on five neighbourhood CA. Chowdhury et.al. [Chowd93b, Chowd94c] extended the theory of 1D CA built around matrix algebra for characterizing 2D CA. However, emphasis was laid on special class of additive 2D CA, known as Restricted Vertical Neighbourhood (RVN) CA. In this class of 2D CA, the vertical dependency of a site (cell) is restricted to either the site on its Top or Bottom, but not both.
CHAPTER 4. MATHEMATICAL MODEL OF 2D CA

In this chapter we have developed an analytical tool to study all the nine nearest neighbourhood 2D CA linear transformations. A general framework has been proposed for the study of the state transition behaviour of this class of 2D CA.

4.2 The Concept

In an 1D CA, all the cells are arranged in a linear array. Of particular importance is the 3- neighbourhood 1D CA, where the next state of a particular cell is assumed to depend on itself and on its two neighbours.

The state $q$ of the $i^{th}$ cell at time $t + 1$ is denoted as

$$q_i(t + 1) = f(q_i(t), q_{i-1}(t), q_{i+1}(t))$$

Where $q_i(t)$ denotes the state of $i^{th}$ cell at time $t$ and $f$ is the next state function.

In a 2D cellular Automata as shown in Fig: 4.1, the cells are arranged in a two-dimensional grid with connections among the neighbouring cells. Consider a 2D CA comprising of $mn$ cells organized as an $m \times n$ array with $m$ rows and $n$ columns. The state of the CA at any time instant can be represented by an $m \times n$ binary matrix. The neighbourhood function specifying the next state of a particular cell of the 2D CA is affected by the current state of itself and eight cells in its nearest neighbourhood.

Mathematically, the next state $q$ of the $(i, j)^{th}$ cell of a 2D CA is given by
where $f$ is the boolean function of 9 variables. To express a transition rule of 2D CA, a specific rule convention proposed by us is noted below.

![Diagram of 2D CA with rule numbers](image)

**Fig: 4.1**

The central box represents the current cell (that is, the cell being considered) and all other boxes represent the eight nearest neighbours of that cell. The number within each box represents the rule number associated with
that particular neighbour of the current cell – that is, if the next state of a cell is dependent only on its present state, it is referred to as rule 1. If the next state of a cell is dependent on the present state of its right neighbour, it is referred to as rule 2. Similarly, if the next state of a cell is dependent on the present state of its top-left cell it is referred to as rule 64 and so on. In case, the next state of a cell depends on the present state of itself and/or its one or more neighbouring cells (including itself), the rule number will be the arithmetic sum of the numbers of the relevant cells. For example,

a) if the next state of a cell depends on the present state of itself and its right neighbour, it is referred to as rule 3 (= 1 + 2).

b) if the next state of a cell depends on the present state of its right, left and bottom-right neighbours, it is referred to as rule 38 (= 2 + 32 + 4).

c) if the next state of a cell depends on the present state of itself and its right, bottom, left and top neighbours, it is referred to as rule 171 (= 1 + 2 + 8 + 32 + 128).

Further suffixing “N” or “P” with a rule no, indicates whether that rule is applied with null boundary or periodic boundary condition respectively. i.e. 4N \implies rule 4 to be applied under Null Boundary Condition.

171p \implies rule 171 to be applied under Periodic Boundary Condition.

To develop a general mathematical model for study of 2D CA state transition behaviour. Let us first go through some examples.

Example 4.1: Let the two dimensional binary information matrix representing the current state of a 2D CA be
[X_t] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad (\text{where } a_{11}, a_{12}, \ldots, a_{33} \text{ take values } 0 \text{ or } 1)

Now, suppose rule 2 is applied to [X_t], under null boundary condition. That is, the next state of each cell is dependent on the present state of its right neighbour. This means \( a_{11} \) will be replaced by \( a_{12} \) and \( a_{12} \) by \( a_{13} \) and \( a_{13} \) by 0, since its right neighbour under null boundary condition is zero and so on. Therefore, the next state \([X_{t+1}]\) of the CA is given by

\begin{equation}
[X_{t+1}] = \begin{bmatrix} a_{12} & a_{13} & 0 \\ a_{22} & a_{23} & 0 \\ a_{32} & a_{33} & 0 \end{bmatrix}
\end{equation}

Similarly, by applying rule 2 under periodic boundary condition, in which the extreme cells are connected to each other, the next state of the CA is

\begin{equation}
[X_{t+1}] = \begin{bmatrix} a_{12} & a_{13} & a_{11} \\ a_{22} & a_{23} & a_{21} \\ a_{32} & a_{33} & a_{31} \end{bmatrix}
\end{equation}

Similarly, applying rule 3N & 3P (where next state of a cell is dependent on the present state of itself and its right neighbour) the next state of the CA, is respectively

\begin{equation}
[X_{t+1}] = \begin{bmatrix} a_{11} \oplus a_{12} & a_{12} \oplus a_{13} & a_{13} \\ a_{21} \oplus a_{22} & a_{22} \oplus a_{23} & a_{23} \\ a_{31} \oplus a_{32} & a_{32} \oplus a_{33} & a_{33} \end{bmatrix}
\end{equation}
Here, the next state of any cell will be obtained by XOR operation of the states of its relevant neighbours associated with the rule.

### 4.3 The Model

The 2D CA behaviour can be analyzed with the help of an elegant mathematical model where we take use of four fundamental matrices, $T_1$ & $T_2$ for null boundary conditions and $T_{1c}$ & $T_{2c}$ for periodic boundary conditions. Where $T_1$ & $T_2$ and $T_{1c}$ & $T_{2c}$ are

\[
T_1 = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 & 0 \\
0 & 0 & 1 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 1 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 0 \\
\end{bmatrix}_{n \times n}, \quad T_2 = \begin{bmatrix}
0 & 0 & 0 & \cdots & 0 & 0 \\
1 & 0 & 0 & \cdots & 0 & 0 \\
0 & 1 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & 0 \\
\end{bmatrix}_{n \times n}
\]

\[
T_{1c} = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 & 0 \\
0 & 0 & 1 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 1 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 0 \\
\end{bmatrix}_{n \times n}, \quad T_{2c} = \begin{bmatrix}
0 & 0 & 0 & \cdots & 0 & 1 \\
1 & 0 & 0 & \cdots & 0 & 0 \\
0 & 1 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & 0 \\
\end{bmatrix}_{n \times n}
\]

Next, we introduce two definitions.
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Definition 4.1: If in a CA, the next state of each cell is dependent on only one of the nine neighbours, the CA is assumed to be configured with a Primary Rule.

Definition 4.2: If in a CA the next state of each cell is dependent on more than nine neighbour (including itself), the CA is assumed to be configured with a Secondary Rule.

The following theorems specify the value of the next state of a 2D CA referred to as \([X_{t+1}]\), given that its current state is \([X_t]\).

Theorem 4.1: The next state transition of all the primary rules (1, 2, 4, 8, 16, 32, 64, 128, 256) under null boundary conditions can be represented as

\[
\begin{align*}
\text{Rule 1} & \implies [X_{t+1}] = [X_t] \\
\text{Rule 2} & \implies [X_{t+1}] = [X_t][T_2] \\
\text{Rule 4} & \implies [X_{t+1}] = [T_1][X_t][T_2] \\
\text{Rule 8} & \implies [X_{t+1}] = [T_1][X_t] \\
\text{Rule 16} & \implies [X_{t+1}] = [T_1][X_t][T_1] \\
\text{Rule 32} & \implies [X_{t+1}] = [X_t][T_1] \\
\text{Rule 64} & \implies [X_{t+1}] = [T_2][X_t][T_1] \\
\text{Rule 128} & \implies [X_{t+1}] = [T_2][X_t] \\
\text{Rule 256} & \implies [X_{t+1}] = [T_3][X_t][T_2]
\end{align*}
\]

Proof: Let

\[
[X_t] = \begin{bmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{bmatrix}
\]

(where \(a_{11}, a_{12}, \ldots, a_{33}\) take values 0 or 1)

be the state of CA at time \(t\).

(i) Post multiplying \([X_t]\) by \(T_2\), we get the state of CA at time \((t + 1)\) as
which is same as rule 2N applied to $X_t$.

(ii) Pre multiplying $X_t$ by $T_1$ and post multiplying the result by $T_2$, we get the state of CA at time instant $(t + 1)$ as

$$
[X_{t+1}] = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix} \begin{bmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{bmatrix} \begin{bmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{bmatrix}
$$

which is same as rule 4N applied to $X_t$.

(iii) Post multiplying $X_t$ by $T_1$, we get the state of CA at time $(t + 1)$ as

$$
[X_{t+1}] = \begin{bmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{bmatrix} \begin{bmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{bmatrix}
$$
which is same as rule $8N$ applied to $X_t$.

Similarly, others can be proved.
Theorem 4.2: The next state transition of all the primary rules, under periodic boundary condition, can be represented as in Theorem 4.1, except that $T_1$ and $T_2$ are to be replaced by $T_{1c}$ and $T_{2c}$ respectively.

Proof: Let

$$[X_t] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

(where $a_{11}, a_{12}, \cdots a_{33}$ take values 0 or 1)

be the state of CA at time $t$.

(i) Post multiplying $[X_t]$ by $T_{2c}$, we get the state of CA at time $(t+1)$ as

$$[X_{t+1}] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} a_{12} & a_{13} & a_{11} \\ a_{22} & a_{23} & a_{21} \\ a_{32} & a_{33} & a_{31} \end{bmatrix}$$

which is same as rule $2P$ applied to $X_t$.

(ii) Pre multiplying $X_t$ by $T_{1c}$ and post multiplying the result by $T_{2c}$, we get the state of CA at time instant $(t+1)$ as

$$[X_{t+1}] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{11} & a_{12} & a_{13} \end{bmatrix}$$
which is same as rule 4P applied to \( X_t \).
Similarly, others can be proved.

**Theorem 4.3:** The next state transition of a CA configured with a secondary rule with Null ( Periodic ) boundary condition, can be represented as modulo 2 sum of the matrices of the concerned primary rules with Null ( Periodic ) boundary condition.

For example;

(i) Rule 3N = Rule 1N + Rule 2N, the next state transition can be represented as

\[
[X_{t+1}] = [X_t] \oplus [X_t][T_2]
\]

(ii) Rule 9P = Rule 8P + Rule 1P, the next state transition can be represented as

\[
[X_{t+1}] = [T_{1c}][X_t] \oplus [X_t]
\]

(iii) Rule 170N = Rule 2N + Rule 8N + Rule 32N + Rule 128N, the next state transition can be represented as

\[
[X_{t+1}] = [X_t][T_2] \oplus [T_1][X_t] \oplus [X_t][T_1] \oplus [T_2][X_t] = [X_t][S] \oplus [S][X_t]
\]
where \([S] = [T_1 \oplus T_2]\).

Proof: Let

\[ [X_t] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \]

be the state of CA at time instant \(t\). Now,

(i) the next state of rule 3N is

\[
[X_{t+1}] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \oplus \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} \oplus a_{12} & a_{12} \oplus a_{13} & a_{13} \\ a_{21} \oplus a_{22} & a_{22} \oplus a_{23} & a_{23} \\ a_{31} \oplus a_{32} & a_{32} \oplus a_{33} & a_{33} \end{bmatrix}
\]

which is same as rule 3N applied to \(X_t\).

(ii) the next state of rule 3P is

\[
[X_{t+1}] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \oplus \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}
\]
which is same as rule 9P applied to $X_t$.

(iii) the next state of rule 170N is

\[
[X_{t+1}] = \begin{bmatrix}
  a_{11} & a_{12} & a_{13} \\
  a_{21} & a_{22} & a_{23} \\
  a_{31} & a_{32} & a_{33}
\end{bmatrix}
\begin{bmatrix}
  0 & 1 & 0 \\
  1 & 0 & 1 \\
  0 & 1 & 0
\end{bmatrix}
\oplus
\begin{bmatrix}
  0 & 1 & 0 \\
  1 & 0 & 1 \\
  0 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
  a_{11} & a_{12} & a_{13} \\
  a_{21} & a_{22} & a_{23} \\
  a_{31} & a_{32} & a_{33}
\end{bmatrix}
\]

\[
= \begin{bmatrix}
  a_{12} & a_{11} \oplus a_{13} & a_{12} \\
  a_{22} & a_{21} \oplus a_{23} & a_{22} \\
  a_{32} & a_{31} \oplus a_{33} & a_{32}
\end{bmatrix}
\oplus
\begin{bmatrix}
  a_{21} & a_{22} & a_{23} \\
  a_{11} \oplus a_{31} & a_{12} \oplus a_{32} & a_{13} \oplus a_{33} \\
  a_{21} & a_{22} & a_{23}
\end{bmatrix}
\]

\[
= \begin{bmatrix}
  a_{12} \oplus a_{21} & a_{11} \oplus a_{13} \oplus a_{22} & a_{12} \oplus a_{23} \\
  a_{11} \oplus a_{22} \oplus a_{33} & a_{21} \oplus a_{23} \oplus a_{12} \oplus a_{32} & a_{22} \oplus a_{13} \oplus a_{33} \\
  a_{32} \oplus a_{21} & a_{31} \oplus a_{33} \oplus a_{22} & a_{31} \oplus a_{23}
\end{bmatrix}
\]

### 4.4 Conclusion

In this chapter we have developed an elegant mathematical model for study of 2D CA state transition, using simple matrix algebra. Based on this mathematical model we have characterized all the nearest nine neighbourhood 2D CA transformations, introduced in the next chapter.