Chapter 5

Characterization of 2D Cellular Automata

5.1 Introduction

In chapter 4, we presented the mathematical model, necessary for understanding the state transition behaviour of 2D Cellular Automata, with both null and periodic boundary conditions. To formulate the state transition function for any 2D additive Cellular Automata, we use Theorem 4.1 through Theorem 4.3, depending upon the neighbourhood dependence and extreme cell connection condition.

Let $X_t$ be the initial state of 2D CA at any time instant $t$, then the next state of the CA, if rule 3N is applied to the CA, at time instant $(t + 1)$, in accordance with Theorem 4.3 is given by

$$[X_{t+1}] = [X_t] \oplus [X_t][T_2]$$

Similarly, the states of the CA at time instant $t + 2, t + 3, \cdots$ is given by
CHAPTER 5. CHARACTERIZATION OF 2D CA

\[ [X_{t+2}] = [X_{t+1}] \oplus [X_{t+1}][T_2] \]

\[ [X_{t+3}] = [X_{t+2}] \oplus [X_{t+2}][T_2] \]

and so on.

If at any time instant \((t+n)\), for \(n \geq 1, 2, 3, \ldots\), \([X_{t+n}] = [X_t]\), the rule is a group rule, otherwise it is a non group rule.

To simplify the characterization of a large number of 2D CA rules we have developed an elegant approach using the characteristic matrix \(T\), discussed in the next section.

5.2 The Characteristic Matrix

For the convenience of analysis, we will convert each of the rules into a transformation denoted by \(T\). We want to look at the transformation \(T\) such that \(T\) operating on the current state \([X]_{mxn}\) (the binary information matrix of the order of \(m \times n\)) generates the next state \([X']_{mxn}\). The convention followed is as noted below.

\[
\begin{bmatrix}
X'_{1} \\
X'_{2} \\
X'_{3} \\
\vdots \\
X'_{m}
\end{bmatrix}
= [T]_{mxm} 
\begin{bmatrix}
X_{1} \\
X_{2} \\
X_{3} \\
\vdots \\
X_{m}
\end{bmatrix}
\]

where \(X_1, X_2, X_3, \ldots, X_m\) are the rows of \([X]_{mxn}\) and \(X'_1, X'_2, X'_3, \ldots, X'_m\) are the rows of the next state \([X']_{mxn}\).

In the following Lemmas, we try to formulate the above \(T\) matrix for a given rule \(R\) applied uniformly over all the 2D CA cells.
Lemma 5.1: The characteristic matrix for any rule $R_N$ with null boundary condition can be represented as

$$T_{RN} = \begin{bmatrix}
D & U & 0 & 0 & \cdots & 0 & 0 & 0 \\
L & D & U & 0 & \cdots & 0 & 0 & 0 \\
0 & L & D & U & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & D & U & 0 \\
0 & 0 & 0 & 0 & \cdots & L & D & U \\
0 & 0 & 0 & 0 & \cdots & 0 & L & D
\end{bmatrix}_{m \times n}$$

where $D$, $L$ & $U$ are one of the following matrices of the order of $n \times n$

$[0], [I], [T_1], [T_2], [I + T_1], [I + T_2], [S] \& [I + S]$.

Proof: For getting the first column of the above matrix, we utilize

$$\begin{bmatrix}
1 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0
\end{bmatrix}_{m \times n}$$

Now applying rule $R$ with null boundary condition, starting from first row, we get the first column of $T_{RN}$. Then, shifting 1 towards right just once, and applying the same rule again we get the second column and so on.

Example 5.1: Let the dimension of the 2D CA be $3 \times 3$. In order to obtain the $T$ matrix corresponding to rule 2 applied over all the cells, with null boundary condition (i.e $T_{2N}$), we follow the following steps:

To obtain the $i^{th}$ column of matrix $T_{2N}$ corresponding to rule 2, we take a $3 \times 3$ binary matrix with all zeros excepting the position $< i/3, i \mod 3 >$
which contain 1. Then, we get the rule 2 applied for each cell, with null boundary condition.

i.e. to obtain the $1^{\text{th}}$ column of $T_{2N}$, we use the following binary matrix

$$
\begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}_{3\times 3}
$$

Now, considering this matrix as state of the 2D CA, we apply rule $2N$. To obtain the $2^{\text{nd}}$ column of $T_{2N}$, we use

$$
\begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}_{3\times 3}
$$

and apply rule $2N$ to each cell. Similarly, to obtain $4^{\text{th}}$ column of $T_{2N}$, we use

$$
\begin{bmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}_{3\times 3}
$$

and apply rule $2N$ to each cell again, and so on we get

$$
T_{2N} = 
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}_{9\times 9}
$$
Lemma 5.2: The characteristic matrix $T$ for any rule $RP$ with periodic boundary condition can be represented as

$$
\begin{pmatrix}
(T_1)_{3\times3} & 0 & 0 \\
0 & (T_1)_{3\times3} & 0 \\
0 & 0 & (T_1)_{3\times3}
\end{pmatrix}_{3\times3}
$$

Proof: Proof is similar to Lemma 5.1. except that the rule is to be applied under periodic boundary conditions.

5.3 Characterization of 2D CA

With the characteristic matrix ( referred to as the $T$ matrix ), it is now easy to formulate the state transition function for additive cellular automata. If $[X_t]$ represents the state of the CA at the $t^{th}$ instant of time, then the states at $(t + 1), (t + 2), (t + 3)$ and so on upto $(t + n)^{th}$ instant of time may be represented as
If the CA under the transformation of operation with $T$, forms a cyclic group, then for all $[X_t]$, there should exist, an integer 'n' such that

$$[T]^n = I \quad \text{(the identity matrix)}$$

and

$$[X_{t+n}] = [T]^n[X_t] = [I][X_t] = [X_t]$$

A CA with such a characteristics is referred to as group CA.

The next result derived in case of 1D CA by Das [Das90c], holds good for 2D CA as well.

**Theorem 5.1:** A 2D CA is a group CA if and only if the determinant of $T$ matrix, corresponding the rule is 1, i.e. $\det[T] = 1$.

**Proof:** We have to prove that all the states or configurations lying on a cycle if and only if $T$ is nonsingular.

We first prove $\implies$

i.e. All configurations lying on a cycle are $T.X, T.X^2, \cdots$.

$\Rightarrow T^m.X = T^n.X$ for some $m, n$ where $m > n$

or, $T^n(T^{m-n} - 1)X = 0$

If $X$ is all zero, then $T.X, T.X^2, \cdots$ can not be on a cycle. We assume $X \neq 0$.

Again $T^n = 0 \Rightarrow T^n.X = 0$ and $T^k.X = 0$ for all $k > n$

which implies no cycle is possible

Thus $T^n \neq 0 \Rightarrow T^{m-n} = I$ or, $T^l = I$ where $l = m - n$ which implies that
T is singular.

Now we prove \( \iff \)

i.e. suppose \( T \) is non-singular and we have to prove that all configurations are lying on a cycle. We consider the set \( T.X, T^2.X, T^3.X \cdots \). Where there exist integers \( m \) and \( n \) such that \( T^m = T^n \) and suppose \( n > m \).

Since \( T \) is non-singular, \( T^{-1} \) exists. Thus operating both sides by \( (T^{-1})^m \) we get

\[ I = T^{n-m} \Rightarrow n = m \] which is a contradiction

Hence the proof.

Next, we give two definitions [Rao92].

**Definition 5.1** The dimension of a vector space \( S \) is the size of any basis of \( S \), denoted by \( d(S) \).

**Definition 5.2** The kernel \( K(f) \) of a linear transformation \( f \) is defined to be \( \{ x \in V_1 : f(x) = 0 \} \). The kernel of \( f \) is a subspace of \( V_1 \) and its dimension is called the nullity of \( f \) or the dimension of kernel of \( f \).

Actually, the nullity of \( f \) equals to \( n - \rho(f) \) where \( n \) is the dimension of \( V_1 \) and \( \rho \) is the rank of the linear transformation.

In the following lemmas some properties of the fundamental matrices are derived, which have been used in subsequent results for characterizing the 2D CA rules.
Lemma 5.3: Rank of fundamental matrices $(T_1)_{n \times n}$ and $(T_2)_{n \times n}$ is $n - 1$.

Proof:

$$T_1 = \begin{bmatrix}
0 & 1 & 0 & 0 & \ldots & 0 & 0 \\
0 & 0 & 1 & 0 & \ldots & 0 & 0 \\
0 & 0 & 0 & 1 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \ldots & 1 & 0 \\
0 & 0 & 0 & 0 & \ldots & 0 & 0
\end{bmatrix}_{n \times n}$$

Shifting $C_1$ (Column 1) to $C_n$ (Column n), we get

$$T_1' = \begin{bmatrix}
1 & 0 & 0 & \ldots & 0 & 0 & 0 \\
0 & 1 & 0 & \ldots & 0 & 0 & 0 \\
0 & 0 & 1 & \ldots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 0 & 1 & 0 \\
0 & 0 & 0 & \ldots & 0 & 0 & 0
\end{bmatrix}_{n \times n}$$

Partitioning $T_1$, we get

$$T_1 = \begin{bmatrix}
(I)_{n-1 \times n-1} & (0)_{n \times 1} \\
(0)_{1 \times (n-1)} & (0)_{1 \times 1}
\end{bmatrix}_{n \times n}$$

Therefore, rank of $T_1 = \text{rank of } (I)_{n-1 \times n-1} = n - 1$

Hence, rank of $T_1 = \rho(T_1) = n - 1$.

Similarly, $\rho(T_2) = n - 1$, can be proved.
Lemma 5.4: Fundamental matrices \((T_{1c})_{n \times n}\) & \((T_{2c})_{n \times n}\) are full rank.

Proof:

\[
T_{1c} = \begin{bmatrix}
0 & 1 & 0 & 0 & \ldots & 0 & 0 \\
0 & 0 & 1 & 0 & \ldots & 0 & 0 \\
0 & 0 & 0 & 1 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \ldots & 0 & 1 \\
1 & 0 & 0 & 0 & \ldots & 0 & 0
\end{bmatrix}_{n \times n}
\]

Shifting \(C_1\) (Column 1) to \(C_n\) (Column n), we get

\[
T_{1c} = \begin{bmatrix}
1 & 0 & 0 & \ldots & 0 & 0 \\
0 & 1 & 0 & \ldots & 0 & 0 \\
0 & 0 & 1 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 0 & 1 \\
0 & 0 & 0 & \ldots & 0 & 0
\end{bmatrix}_{n \times n} = (I)_{n \times n}
\]

Hence rank \(\rho(T_{1c})_{n \times n} = n\).

Similarly \(\rho(T_{2c})_{n \times n}\) can be proved.

### 5.3.1 Group Rules

CA whose transformation is invertible is called a group CA. For a group CA, the dimension of kernel is 0. That is, the transformation is a full rank one.

Theorem 5.2: All primary rules with periodic boundary conditions are group rules.

Proof: The \(T\) matrix for rule 2P is given by
CHAPTER 5. CHARACTERIZATION OF 2D CA

\[
T_{2p} = \begin{bmatrix}
(T_{1e})_{n \times n} & 0 & 0 & \cdots & 0 \\
0 & (T_{1e})_{n \times n} & 0 & \cdots & 0 \\
0 & 0 & (T_{1e})_{n \times n} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & (T_{1e})_{n \times n}
\end{bmatrix}_{m \times m}
\]

Rank of \((T_{2p}) = m \rho(T_{1e}) = m \times (n) = mn\)

Therefore, dimension of kernel of \(T_{2p} = mn - mn = 0\)

Hence, the rule is a group rule.

Similarly, others can be proved.

**Theorem 5.3:** All rules, with null boundary conditions, whose characteristic matrix is of the form

\[
T_{RN} = \begin{bmatrix}
D & U & 0 & 0 & \cdots & 0 & 0 & 0 \\
L & D & U & 0 & \cdots & 0 & 0 & 0 \\
0 & L & D & U & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & D & U & 0 \\
0 & 0 & 0 & 0 & \cdots & L & D & U \\
0 & 0 & 0 & 0 & \cdots & 0 & L & D
\end{bmatrix}_{mn \times mn}
\]

where \(D = (I)_{n \times n}\), or \((I + T_1)_{n \times n}\) or \((I + T_2)_{n \times n}\)

and \(U \& L = (0)_{n \times n}\), or \((T_1)_{n \times n}\) or \((T_2)_{n \times n}\) or \((I)_{n \times n}\)

such that either \(U = L = 0\) or \(U = 0\) or \(L = 0\), are group rules.

**Proof:** A CA is a group CA iff the corresponding characteristic matrix is non singular.

Case (i) if \(U = L = 0\) the characteristic matrix will be a block diagonal.

Case (ii) if \(U = 0\), the characteristic matrix will be a lower diagonal.
Case (iii) if \( L = 0 \), the characteristic matrix will be an upper diagonal.

In all the cases, the rank of \( T_{RN} = m(\text{rank of } (D)_{n \times n}) \).

If \( D = I \), or \( I + T_1 \) or \( I + T_2 \), rank = \( n \).

hence rank of \( T_{RN} = mn \)

Therefore, dimension of kernel = 0. Hence the rule is a group rule.

2D CA rules with null boundary conditions have been categorized into four classes, based on the characteristic matrix \( T \) as

Class 1: those rules whose \( T \) matrix is a block diagonal (Table 5.1)

Class 2: those rules whose \( T \) matrix is upper diagonal (Table 5.2)

Class 3: those rules whose \( T \) matrix is lower diagonal (Table 5.3)

Class 4: those rules whose \( T \) matrix is other than a block or upper or lower diagonal (Table 5.4).

**Table 5.1: Class 1 group 2D CA**

<table>
<thead>
<tr>
<th>Rule No.</th>
<th>State transition function ([X_{t+1}])</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>([X_t])</td>
</tr>
<tr>
<td>3</td>
<td>([X_t][I + T_2])</td>
</tr>
<tr>
<td>33</td>
<td>([X_t][I + T_1])</td>
</tr>
</tbody>
</table>
Table 5.2: Class 2 group 2D CA

<table>
<thead>
<tr>
<th>Rule No.</th>
<th>State transition function $[X_{t+1}]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>$[X_i] + [T_1][X_i][T_2]$</td>
</tr>
<tr>
<td>7</td>
<td>$[X_i] + [I + T_1][X_i][T_2]$</td>
</tr>
<tr>
<td>9</td>
<td>$[I + T_1][X_i]$</td>
</tr>
<tr>
<td>11</td>
<td>$[X_i][I + T_2] + [T_1][X_i]$</td>
</tr>
<tr>
<td>13</td>
<td>$[X_i] + [T_1][X_i][I + T_2]$</td>
</tr>
<tr>
<td>15</td>
<td>$[I + T_1][X_i][I + T_2]$</td>
</tr>
<tr>
<td>17</td>
<td>$[X_i] + [T_1][X_i][T_1]$</td>
</tr>
<tr>
<td>19</td>
<td>$[X_i][I + T_2] + [T_1][X_i][T_1]$</td>
</tr>
<tr>
<td>21</td>
<td>$[X_i] + [T_1][X_i][S]$</td>
</tr>
<tr>
<td>23</td>
<td>$[X_i][I + T_2] + [T_1][X_i][S]$</td>
</tr>
<tr>
<td>25</td>
<td>$[X_i] + [T_1][X_i][I + T_1]$</td>
</tr>
<tr>
<td>27</td>
<td>$[X_i][I + T_2] + [T_1][X_i][I + T_1]$</td>
</tr>
<tr>
<td>29</td>
<td>$[X_i] + [T_1][X_i][I + S]$</td>
</tr>
<tr>
<td>31</td>
<td>$[X_i][I + T_2] + [T_1][X_i][I + S]$</td>
</tr>
<tr>
<td>37</td>
<td>$[X_i][I + T_1] + [T_1][X_i][T_2]$</td>
</tr>
<tr>
<td>41</td>
<td>$[I + T_1][X_i] + [X_i][T_1]$</td>
</tr>
<tr>
<td>45</td>
<td>$[I + T_1][X_i] + [T_1][X_i][I + T_2]$</td>
</tr>
<tr>
<td>49</td>
<td>$[X_i] + [I + T_1][X_i][T_1]$</td>
</tr>
<tr>
<td>53</td>
<td>$[X_i][I + T_1] + [T_1][X_i][S]$</td>
</tr>
<tr>
<td>57</td>
<td>$[I + T_1][X_i][I + T_1]$</td>
</tr>
<tr>
<td>61</td>
<td>$[X_i][I + T_1] + [T_1][X_i][S + I]$</td>
</tr>
<tr>
<td>Rule No.</td>
<td>State transition function $[X_{t+1}]$</td>
</tr>
<tr>
<td>---------</td>
<td>--------------------------------------</td>
</tr>
<tr>
<td>65</td>
<td>$[X_t] + [T_3][X_t][T_1]$</td>
</tr>
<tr>
<td>67</td>
<td>$[X_t][I + T_2] + [T_2][X_t][T_1]$</td>
</tr>
<tr>
<td>97</td>
<td>$[X_t] + [I + T_2][X_t][T_1]$</td>
</tr>
<tr>
<td>129</td>
<td>$[I + T_2][X_t]$</td>
</tr>
<tr>
<td>131</td>
<td>$[I + T_2][X_t] + [X_t][T_2]$</td>
</tr>
<tr>
<td>161</td>
<td>$[X_t][I + T_1] + [T_2][X_t]$</td>
</tr>
<tr>
<td>193</td>
<td>$[X_t] + [T_2][X_t][I + T_1]$</td>
</tr>
<tr>
<td>195</td>
<td>$[X_t][I + T_2] + [T_2][X_t][I + T_1]$</td>
</tr>
<tr>
<td>225</td>
<td>$[I + T_2][X_t][I + T_1]$</td>
</tr>
<tr>
<td>257</td>
<td>$[X_t] + [T_2][X_t][T_2]$</td>
</tr>
<tr>
<td>259</td>
<td>$[X_t] + [I + T_2][X_t][T_2]$</td>
</tr>
<tr>
<td>289</td>
<td>$[X_t] + [I + T_1] + [T_2][X_t][T_2]$</td>
</tr>
<tr>
<td>321</td>
<td>$[X_t] + [T_2][X_t][S]$</td>
</tr>
<tr>
<td>323</td>
<td>$[X_t][I + T_2] + [T_2][X_t][S]$</td>
</tr>
<tr>
<td>353</td>
<td>$[X_t][I + T_1] + [T_2][X_t][S]$</td>
</tr>
<tr>
<td>385</td>
<td>$[X_t] + [T_2][X_t][I + T_2]$</td>
</tr>
<tr>
<td>387</td>
<td>$[I + T_2][X_t][I + T_2]$</td>
</tr>
<tr>
<td>417</td>
<td>$[X_t][I + T_1] + [T_2][X_t][I + T_2]$</td>
</tr>
<tr>
<td>449</td>
<td>$[X_t] + [T_2][X_t][I + S]$</td>
</tr>
<tr>
<td>451</td>
<td>$[X_t][I + T_2] + [T_2][X_t][I + S]$</td>
</tr>
<tr>
<td>481</td>
<td>$[X_t][I + T_1] + [T_2][X_t][I + S]$</td>
</tr>
</tbody>
</table>
Table 5.4: Class 4 group 2D CA

<table>
<thead>
<tr>
<th>Rule No.</th>
<th>State transition function ([X_{t+1}])</th>
</tr>
</thead>
<tbody>
<tr>
<td>73</td>
<td>([I + T_1][X_1] + [T_2][X_1][T_1])</td>
</tr>
<tr>
<td>81</td>
<td>([S][X_1][T_1] + [X_1])</td>
</tr>
<tr>
<td>89</td>
<td>([X_1] + [T_1][X_1] + [S][X_1][T_1])</td>
</tr>
<tr>
<td>105</td>
<td>([I + T_1][X_1] + [I + T_2][X_1][T_1])</td>
</tr>
<tr>
<td>113</td>
<td>([X_1] + [I + S][X_1][T_1])</td>
</tr>
<tr>
<td>121</td>
<td>([I + T_1][X_1] + [I + S][X_1][T_1])</td>
</tr>
<tr>
<td>133</td>
<td>([I + T_2][X_1] + [T_1][X_1][T_2])</td>
</tr>
<tr>
<td>135</td>
<td>([I + T_2][X_1] + [I + T_1][X_1][T_2])</td>
</tr>
<tr>
<td>145</td>
<td>([I + T_2][X_1] + [T_1][X_1][T_1])</td>
</tr>
<tr>
<td>177</td>
<td>([I + T_2][X_1] + [I + T_1][X_1][T_2])</td>
</tr>
<tr>
<td>209</td>
<td>([I + T_2][X_1] + [S][X_1][T_2])</td>
</tr>
<tr>
<td>241</td>
<td>([X_1] + [I + T_1][X_1][T_1] + [T_2][X_1][I + T_1])</td>
</tr>
<tr>
<td>261</td>
<td>([X_1] + [S][X_1][T_2])</td>
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<td>263</td>
<td>([X_1] + [I + S][X_1][T_2])</td>
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<td>265</td>
<td>([I + T_1][X_1] + [T_2][X_1][T_2])</td>
</tr>
<tr>
<td>267</td>
<td>([I + T_1][X_1] + [I + T_2][X_1][T_2])</td>
</tr>
<tr>
<td>269</td>
<td>([I + T_1][X_1] + [S][X_1][T_2])</td>
</tr>
<tr>
<td>271</td>
<td>([I + T_1][X_1] + [I + S][X_1][T_2])</td>
</tr>
<tr>
<td>389</td>
<td>([I + T_2][X_1] + [S][X_1][T_2])</td>
</tr>
<tr>
<td>391</td>
<td>([I + T_2][X_1] + [I + S][X_1][T_2])</td>
</tr>
</tbody>
</table>
5.3.2 Non Group rules

The state transition diagrams of non-group CA basically exhibit three types of behaviour:

1. those for which all the state transitions finally lead into zero graveyard state, irrespective of the number of rows and columns \((m \times n)\) of the CA (see Fig 5.1 a),

2. those for which all state transitions finally lead into some cycle, irrespective of the number of rows and columns of the CA (see Fig 5.1 b) and

3. those for which states lie on some cycle, provided that a particular set of rows and columns is chosen (see Fig 5.1 c)

![Fig: 5.1(a)](image)

To classify, such a huge number of non-group CA, with different \(m \& n\), is totally impractical. However, in order to have exhaustive study of 2D CA behaviour we have analyzed a large number of non group 2D CA rules. We next analyze some of these rules which have been employed in subsequent chapters to develop few end applications.
Theorem 5.4: All primary rules other than rule 1N with null boundary condition are non group rules.

Proof: (i) The $T$ matrix for rule 2N is given by

$$T_{2N} = \begin{bmatrix}
(T_1)_{n \times n} & 0 & 0 & \cdots & 0 \\
0 & (T_1)_{n \times n} & 0 & \cdots & 0 \\
0 & 0 & (T_1)_{n \times n} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & (T_1)_{n \times n}
\end{bmatrix}_{m \times m}$$

Rank of $T_{2N} = m[\rho(T_1)] = mn - 1 = mn - m$

Therefore, dimension of kernel of $T_{2N} = mn - mn + m = m$
Hence the rule is a non group.

(ii) The $T$ matrix for rule 8N is given by

$$T_{8N} = \begin{bmatrix}
0 & (I)_{n	imes n} & 0 & \cdots & 0 & 0 \\
0 & 0 & (I)_{n	imes n} & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & (I)_{n	imes n} \\
0 & 0 & 0 & \cdots & 0 & 0
\end{bmatrix}_{m \times m}$$

$$T_{34N} = \begin{bmatrix}
(I)_{n	imes n} & 0 & 0 & \cdots & 0 & 0 \\
0 & (I)_{n	imes n} & 0 & \cdots & 0 & 0 \\
0 & 0 & (I)_{n	imes n} & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & (I)_{n	imes n} & 0 \\
0 & 0 & 0 & \cdots & 0 & 0
\end{bmatrix}_{m \times m}$$

Rank of $T_{8N} = (m - 1)[\rho(I)] = (m - 1)n = mn - n$

Therefore, dimension of kernel of $T_{8N} = mn - mn + n = m$

Hence the rule is a non group.

Similarly, others can be proved.

**Theorem 5.5:** The dimension of kernel of $T_{34N}$ is $m$ if $n$ is odd and 0 if $n$ is even.

**Proof:** From Lemma 4.1, the $T$ matrix for rule 34N is given by

$$T_{34N} = \begin{bmatrix}
S_{n\times n} & 0 & 0 & \cdots & 0 \\
0 & S_{n\times n} & 0 & \cdots & 0 \\
0 & 0 & S_{n\times n} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & S_{n\times n}
\end{bmatrix}_{m \times m}$$
or we can write in kronecker product form as

\[ T_{34N} = (I_m) \otimes (S_n) \]

therefore,

\[ \text{rank of } (T_{34N}) = \text{rank of } (I_m) \cdot \text{rank of } (S_n) \]

or \( \rho(T_{34N}) = \rho(I_m) \cdot \rho(S_n) \)

\[ = m \times \rho(S_n) \]

since \( \rho(S_n) = \begin{cases} n & \text{if } n \text{ is even} \\ n - 1 & \text{if } n \text{ is odd} \end{cases} \)

so \( \rho(T_{34N}) = \begin{cases} mn & \text{if } n \text{ is even} \\ m(n - 1) & \text{if } n \text{ is odd} \end{cases} \)

hence, dimension of kernel of \( T_{34N} \) = \begin{cases} 0 & \text{if } n \text{ is even} \\ m & \text{if } n \text{ is odd} \end{cases} \)

This result implies that if \( n \) is taken even, the number of predecessors of a state is \( 2^d = 2^0 = 1 \) (where \( d \) is the dimension of the kernel), hence lies on a cycle.

**Theorem 5.6:** The dimension of kernel of \( T_{136N} \) is \( n \) if \( m \) is odd and 0 if \( m \) is even.

**Proof:** From Lemma 5.1, the \( T \) matrix for rule 136N is given by
or we can write in kronecker product form as

\[
T_{136N} = (S_m) \otimes (I_n)
\]

to 65

therefore,

\[
\text{rank of } (T_{136N}) = \text{rank of } (S_m) \cdot \text{rank of } (I_n)
\]

or \( \rho(T_{136N}) = \rho(S_m) \cdot \rho(I_n) \)

\[
= \rho(S_m) \times n
\]

since \( \rho(S_m) = \begin{cases} 
  m & \text{if } m \text{ is even} \\
  m - 1 & \text{if } m \text{ is odd}
\end{cases} \)

so \( \rho(T_{136N}) = \begin{cases} 
  mn & \text{if } m \text{ is even} \\
  (m - 1)n & \text{if } m \text{ is odd}
\end{cases} \)

hence, dimension of kernel of \( T_{136N} \) = \begin{cases} 
  0 & \text{if } m \text{ is even} \\
  n & \text{if } m \text{ is odd}
\end{cases} \)

Next we analyze rule 170N.
According to Lemma 5.1, the $T_{170N}$ can be represented by

$$T_{170N} = \begin{bmatrix}
S_n & I_n & 0 & \ldots & 0 & 0 \\
I_n & S_n & I_n & \ldots & 0 & 0 \\
0 & I_n & S_n & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & I_n & S_{n}
\end{bmatrix}_{m \times m}$$

**Theorem 5.7:** The characteristic polynomial for $S_n$ can be expressed by the following recurrence

$$p_n(\lambda) = \lambda p_{n-1}(\lambda) + p_{n-2}(\lambda)$$

where $p_0(\lambda) = 1$ & $p_1(\lambda) = \lambda$

**Proof:**

$$p_n(\lambda) = |S_n + \lambda I_n| = \begin{vmatrix}
\lambda & 1 & 0 & \ldots & 0 \\
1 & \lambda & 1 & \ldots & 0 \\
0 & 1 & \lambda & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & \lambda
\end{vmatrix}_{n \times n}$$

$$= \lambda \begin{vmatrix}
1 & 0 & \ldots & 0 \\
0 & 1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & \lambda
\end{vmatrix}_{(n-1) \times (n-1)} + \begin{vmatrix}
1 & 1 & 0 & \ldots & 0 \\
0 & \lambda & 1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & \lambda
\end{vmatrix}_{(n-1)(n-1)}$$

$$= \lambda p_{n-1}(\lambda) + p_{n-2}(\lambda).$$
Lemma 5.5: If $A$ is any matrix of the form

$$
\begin{bmatrix}
I & 0 & 0 & \cdots & 0 & 0 & 0 \\
S & I & 0 & \cdots & 0 & 0 & 0 \\
I & S & I & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & I & S & I
\end{bmatrix}
$$

then $A^{-1} =$

$$
\begin{bmatrix}
I & 0 & 0 & \cdots & 0 \\
S & I & 0 & \cdots & 0 \\
I + S^2 & S & I & \cdots & 0 \\
S^3 & I + S^2 & S & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
p_{m-2}(S) & p_{m-3}(S) & p_{m-4}(S) & \cdots & p_0(S) = I
\end{bmatrix}
$$

Proof: Now

$$A_{m=1} = I \implies A_1^{-1} = I$$

$$A_2 = \begin{bmatrix} I & 0 \\ S & I \end{bmatrix} \implies A_2^{-1} = \begin{bmatrix} I & 0 \\ S & I \end{bmatrix}$$

$$A_3 = \begin{bmatrix} I & 0 & 0 \\ S & I & 0 \\ I & S & I \end{bmatrix} = \begin{bmatrix} A_2 & 0 \\ \end{bmatrix} C_2 \begin{bmatrix} I \\ S \\ I \end{bmatrix}, \text{ where } C_2 = \begin{bmatrix} I & S \end{bmatrix}$$

$$\implies A_3^{-1} = \begin{bmatrix} A_2^{-1} & 0 \\ -C_2A_2^{-1} & I^{-1} \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ S & I & 0 \\ \begin{bmatrix} I & 0 \end{bmatrix} (I & S) (S & I) I \end{bmatrix}$$
Similarly if \( C_3 = (0 \ 1 \ S) \) then

\[
A_4^{-1} = \begin{bmatrix}
A_3^{-1} & 0 \\
-C_3A_3^{-1} & I
\end{bmatrix} = \begin{bmatrix}
I & 0 & 0 & 0 \\
S & I & 0 & 0 \\
I + S^2 & S & I & 0 \\
S^3 & I + S^2 & S & I
\end{bmatrix}
\]

hence for \( m=k \)

\[
A_k^{-1}(k, 1) = IA_{k-1}^{-1}(k - 2, 1) + SA_{k-1}^{-1}(k - 1, 1)
\]

\[
A_k^{-1}(k, 2) = IA_{k-1}^{-1}(k - 2, 2) + SA_{k-1}^{-1}(k - 1, 2)
\]

using theorem 5.7

\[
p_k = p_{k-2} + Sp_{k-1}
\]

hence the result.

**Theorem 5.8:** Finding the dimension of the kernel for \( T_{170} \) \( \dim(\ker(T_{170})) \) is equivalent to finding the dimension of the kernel of the matrix \( p_m(S_n) \).
Proof: Dimension of the kernel $T_{170} = \text{dimension of the kernel}$

\[
\begin{bmatrix}
A & B \\
\uparrow & \uparrow \\
I & 0 & 0 & \cdots & 0 & | & S \\
S & I & 0 & \cdots & 0 & | & I \\
I & S & I & \cdots & 0 & | & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & | & \vdots \\
0 & 0 & 0 & \cdots & I & | & 0 \\
\end{bmatrix}
\]

\[
= \dim \left[ \ker \begin{pmatrix} A & B \\ C & D \end{pmatrix} \right]
\]

\[
= \dim(\ker[A]) + \dim(\ker[D + CA^{-1}B])
\]

\[
= \dim(\ker[CA^{-1}B])
\]

where

\[
A = \begin{pmatrix}
I & 0 & 0 & \cdots & 0 \\
S & I & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & I \\
\end{pmatrix}_{(mn-n) \times (mn-n)}
\]

\[
B = \begin{pmatrix}
S \\
I \\
\vdots \\
\end{pmatrix}_{(mn-n) \times n}
\]

\[
C = \begin{pmatrix}
0 & 0 & \cdots & I & S \\
\end{pmatrix}_{m \times (mn-n)}
\]

\[
D = (0)_{n \times n}
\]
CHAPTER 5. CHARACTERIZATION OF 2D CA

Now, by mere calculation,

\[ \dim(\ker((CA^{-1}B)_{m=1})) = \dim(\ker(S_n)) \quad \text{and} \quad p_1(\lambda) = \lambda (\text{Theorem 4.7}) \]

\[ \dim(\ker((CA^{-1}B)_{m=2})) = \dim(\ker(I + S_n^2)) \quad \text{and} \quad p_2(\lambda) = \lambda^2 + 1 \]

\[ \dim(\ker((CA^{-1}B)_{m=3})) = \dim(\ker(S_n^3)) \quad \text{and} \quad p_3(\lambda) = \lambda^3 \]

\[ \dim(\ker((CA^{-1}B)_{m=4})) = \dim(\ker(I + S_n^2 + S_n^4)) \quad p_4(\lambda) = 1 + \lambda^2 + \lambda^4 \]

\[ \dim(\ker((CA^{-1}B)_{m=5})) = \dim(\ker(S_n^5)) \quad p_5(\lambda) = \lambda + \lambda^5. \]

We now proceed to prove the recurrence

\[
(CA^{-1}B)_m = S_n((CA^{-1}B)_{m-1}) + (CA^{-1}B)_{m-2} = P_m(S_n) \]

\[ = S_n(S_n(CA^{-1}B)_{m-2} + (CA^{-1}B)_{m-3}) \]

\[ = +(S_n(CA^{-1}B)_{m-3} + (CA^{-1}B)_{m-4}) \]

\[ = S_n^2 \cdot (CA^{-1}B)_{m-2} + (CA^{-1}B)_{m-4} \quad (5.1) \]

Now, using Theorem 5.7,

\[ p_{m-1}(\lambda) = \lambda p_{m-2}(\lambda) + p_{m-3}(\lambda) \quad (5.2) \]

\[ &\& p_{m-2}(\lambda) = \lambda p_{m-3}(\lambda) + p_{m-4}(\lambda) \quad (5.3) \]

Now, multiplying eqn (5.2) by \( \lambda \) & adding with eqn (5.3) we get:

\[ \lambda p_{m-1}(\lambda) + p_{m-2}(\lambda) = \lambda^2 p_{m-2}(\lambda) + p_{m-4}(\lambda) \quad (5.4) \]

comparing with eqn (5.1) & eqn (5.4) where \( \lambda = S_n \),

\[
((CA^{-1}B)_{n \times n})_m = P_m(S_n)
\]

Thus \( \dim(\ker((CA^{-1}B)_m) = \dim(\ker P_m(S_n)). \)

If we interchange \( m \) & \( n \), in a similar way we can establish that

\[
((CA^{-1}B)_{m \times m})_n = p_n(S_m). \]

So, just by mere observation \((CA^{-1}B)\) matrix is the same as \( p_m(\lambda) \). Where \( \lambda = S_n \). Now, using Theorem 5.7

\[ p_m(S_n) = S_n p_{m-1}(S_n) + p_{m-2}(S_n) \]

\[ = p_1(S_n) p_{m-1}(S_n) + p_{m-2}(S_n) p_0(S_n) \quad (5.5) \]
Theorem 5.9: The recurrence:

\[ p_m(\lambda) = p_n(\lambda)p_{m-n}(\lambda) + p_{m-n-1}(\lambda)p_{n-1}(\lambda) \]  \hspace{1cm} (5.6)

for \( m \geq n \) where eqn (5.5) above holds for \( n = 1 \)

Proof: Let \( m = n = 1 \).

\[ p_0(\lambda)p_1(\lambda) + p_{-1}(\lambda)p_0(\lambda) = \lambda = p_1(\lambda) \]

Let \( m = 2, n = 1 \).

\[ p_1(\lambda)p_1(\lambda) + p_0(\lambda)p_0(\lambda) = (\lambda)^2 + 1 = p_2(\lambda) \]

Let \( m = 2, n = 2 \).

\[ p_0(\lambda)p_2(\lambda) + p_{-1}(\lambda)p_1(\lambda) = p_2(\lambda) \]

Let \( m = 3, n = 1 \).

\[ p_2(\lambda)p_1(\lambda) + p_1(\lambda)p_0(\lambda) = \lambda p_2(\lambda) + p_1(\lambda) = p_3(\lambda) \]

Let \( m = 3, n = 2 \).

\[ p_1(\lambda)p_2(\lambda) + p_0(\lambda)p_1(\lambda) = (\lambda)^2 = p_3(\lambda) \]

Let \( m = 3, n = 3 \).

\[ p_0(\lambda)p_3(\lambda) + p_{-1}(\lambda)p_2(\lambda) = p_3(\lambda) \]

Let the above recurrence is true up to some particular \( m - 1 \).

By theorem 5.7:

\[ p_{m+1}(\lambda) = \lambda p_m(\lambda) + p_{m-1}(\lambda) \]  \hspace{1cm} (5.7)

\[ p_{m-n}(\lambda) = \{\lambda p_{m-n-1}(\lambda) + p_{m-n-2}(\lambda)\} \]  \hspace{1cm} (5.8)

\[ p_{m-n-1}(\lambda) = \{\lambda p_{m-n-2}(\lambda) + p_{m-n-3}(\lambda)\} \]  \hspace{1cm} (5.9)
Multiplying eqn (5.8) by $p_n(\lambda)$ & eqn (5.9) by $p_{n-1}(\lambda)$ & using eqn (5.7) we get

$$p_{m-n}(\lambda)p_n(\lambda) + p_{m-n-1}(\lambda)p_{n-1}(\lambda) = \lambda\{p_{m-n-1}(\lambda)p_n(\lambda) + p_{m-n-2}(\lambda)p_{n-1}(\lambda)\} + \{p_{m-n-2}(\lambda)p_n(\lambda) + p_{m-n-3}(\lambda)p_{n-1}(\lambda)\} = \lambda p_{m-1}(\lambda) + p_{m-2}(\lambda) = p_m(\lambda)$$

Thus, we have shown that if the recurrence eqn (5.6) is true for $m = m - 1$ and $m = m - 2$ it is true for $m = m$

**Corollary 5.1:** The dimensions of Kernel of $T_{170}, p_m(S_n)$ and $p_n(S_m)$ are same.

**Proof:** From previous result,

$$\text{dim}(\ker(T_{170})) = \text{dim}(\ker((CA^{-1}B)_{n\times n}))_m = \text{dim}(\ker p_m(S_n)) \quad (5.10)$$

In a similar way, interchanging $m$ & $n$, one can get

$$\text{dim}(\ker(T'_{170})) = \text{dim}(\ker((CA^{-1}B)_{m\times m}))_n = \text{dim}(\ker p_n(S_m)) \quad (5.11)$$

Now to see that eqn (5.10) & eqn (5.11) are same, we observe

$$T_{170} = (S_{n\times n}X_{n\times m} + X_{n\times m}S_{m\times m}) \quad (5.12)$$

Here interchanging $n$ & $m$

$$T'_{170} = S_{m\times m}X_{m\times n} + X_{m\times n}S_{n\times n} \quad (5.13)$$

Now if we take transpose on eqn (5.13) & utilising the fact that $S$ is a symmetric matrix, we get $T_{ortho} = X_{m\times n}S_{n\times n} + S_{m\times m}X_{m\times n}$ which is nothing but eqn (5.12), hence the result.
Corollary 5.2: The matrix \( \{p_{n-1}(S_n)\}_{n \times n} \) has dimension of kernel 0.

Proof: By Corollary 5.1,

\[
\dim (\ker (p_{n-1}(S_n))) = \dim (\ker (p_n(S_{n-1})))
\]

Now,

\[
p_n(S_{n-1}) = S_{n-1}p_{n-1}(S_{n-1}) + p_{n-2}(S_{n-1})
\]

\[
= p_{n-2}(S_{n-1}) \cdots \cdots \text{(By Caley Hamilton theorem)}
\]

So, expanding continuously when \( n \) is even = \( p_0(S_1) \)

and expanding continuously when \( n \) is odd = \( p_1(S_2) \)

For both values of \( n \), \( p_n(S_{n-1}) \) is invertible.

As a consequence of these results, we can prove the following result which was first arrived at by Sutner [Sutne90] in connection with \( \sigma \). The same result was derived by Barua et.al. [Barua96] who viewed rule 170N as a superposition of 1D vertical & horizontal dependency transformations.

Theorem 5.10: The dimension of the kernel \( T_{170N} = \gcd(m + 1, n + 1) - 1 \)

The above results relate to the algebraic properties of the corresponding transformations - that is, for a given \( m \) and \( n \), the number of predecessors for any reachable state in the state transition diagram (STD) can easily be calculated. To complete the characterization, we need to consider the minimal polynomial for the corresponding transformation. Let us illustrate this with the help of an example.

Example 5.2: Consider a 2 \( \times \) 3 2D CA configured as rule 170N. The corresponding \( T \) matrix is
On diagonalization [Datta91], we find the factor polynomials are \((x^2 + 1)\) and \((x^4 + 1)\). Using Elspas’s theorem [Elspa59], we find that the cyclic structure is \([4(1), 6(2), 12(4)]\). So, the state transition diagram corresponding to this 2D CA contains 4 cycles of length 1, 6 cycles of length 2 and 12 cycles of length 4. (see Fig 5.2)

\[
\begin{bmatrix}
0 & 1 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 0
\end{bmatrix}_{6x6}
\]

\[
= \begin{bmatrix}
(S)_{3x3} & (I)_{3x3} \\
(I)_{3x3} & (S)_{3x3}
\end{bmatrix}
\]

5.4 Conclusion

In this chapter, we presented the characterization of 2D CA into group and non-group rules, with the help of the characteristic matrix. Classification of group rules and analysis of some of the non-group rules have been reported.

In the next chapter, we present the algorithms for replacement of some basic graphics transforms, based on the theory of 2D CA discussed so far.