CHAPTER 3

Fuzzy Optimization – A Brief Review

3.1. Introduction

The purpose of this chapter is to present a picture of the field of fuzzy optimization. In this field, references may be given to the work of several authors [e.g. Negoita (1981), Kacpryzk & Orlovski (1987), Kandel (1986), Ross (1997), Zimmerman (1976), Ganguly (2013), Song (2013), Ghodousian & Khorram (2012), Chakraborty (2013), Chamani et al. (2013)]. The concepts of fuzzy programming and the solution procedures of fuzzy optimization problems are discussed in Sec. 3.2 and Sec. 3.3 respectively. In Sec. 3.4, we introduce the use of \( \alpha \)-cuts in fuzzy feasible set and in Sec. 3.5, we discuss the problem consisting of multiple objective functions with fuzzy constraints. In Sec. 3.6, we investigate fuzzy linear programming problems. The problems of fuzzy dynamic programming and fuzzy multi-criteria optimization are discussed in Sec. 3.7 and Sec. 3.8 respectively.

3.2. Fuzzy Programming

Fuzzy programming originated in an attempt to extend the applicability of classical programming models. Recalling the classical optimization problem (as defined in Chapter 1), we have

Find \( x^* \in S \) such that \( f(x^*) = \sup \{ f(x) \mid x \in X \} \)

where \( X \subseteq R^n, X \neq \emptyset, f : R^n \rightarrow R, g_i : R^n \rightarrow R, \)

\[ S = \{ x \in X \mid g_i(x) \leq 0, i = 1, 2, \ldots, m \} = \bigcap_{i=1}^{m} \{ x \in X \mid g_i(x) \leq 0 \}, S \neq \emptyset. \]
Here, optimality is viewed as equivalent to efficiency (e.g., linear programming is a technique for efficient choice) which is deceptive in many decision problems where desirable targets are required. In order to make the statement clear, we shall consider the set $X$ as the set of alternatives and the numbers $f(x)$ and $g_i(x)$ as quantifying the effects of choosing an alternative $x \in X$. In many real situations, an alternative $x$ must be selected such that $f(x)$ and $g_i(x)$ are located in some intervals $b = [w_b, W_b]$ and $b_i = [w_{b_i}, W_{b_i}]$ respectively where $b, b_i \subset R$. These intervals are desirable targets. Such procedures can be described as finding a point $x \in X$ [Kandel (1986)] subject to

\begin{align}
 w_b & \leq f(x) \leq W_b \\
 w_{b_i} & \leq g_i(x) \leq W_{b_i}, \quad i = 1, 2, \ldots, m.
\end{align}

where $w$ and $W$ are bounding the interval $b$.

In this problem, optimality means effectiveness in contrast to efficiency. In the classical terms, the first problem (3.2.1) is not always a compatible system and therefore, the initial problem has no solution. Thus, a reformulation is needed for defining a new problem having a solution with properties “as good as possible”. For instance, the quality of an alternative $x$ can be defined by the position of the numbers in the intervals $b$ and $b_i$. These intervals define a preference relation on the real line and in order to express the relation, we can use fuzzy sets. The procedure is as follows:

Let $L$ be a lattice. Then, by using good approximation, one can define fuzzy sets $\tilde{B}, \tilde{B}_i : R \rightarrow L$ such that the pre-order determined by them describes the models of preference from the real process. A necessary condition is

\begin{align*}
 \mu_{\tilde{B}}(x) = 1 & \iff x \in b \\
 \mu_{\tilde{B}_i}(x) = 1 & \iff x \in b_i
\end{align*}

(3.2.3)
Now, consider the fuzzy subsets of $X$:

$$\tilde{f} = \tilde{B} \ast f, \quad \tilde{g}_i = \tilde{B} \ast g_i : X \to L$$  \hspace{1cm} (3.2.4)

where $\ast$ denotes compositions.

Then, the resulting fuzzy problem may be defined as:

Find $x$ such that

$$\mu_{\tilde{f}}(x) = \mu_{\tilde{B}}(f(x))$$

$$\mu_{\tilde{g}_i}(x) = \mu_{\tilde{B}_i}(g_i(x)), \quad i = 1, 2, \ldots, m$$  \hspace{1cm} (3.2.5)

is large as possible.

In this way constraints and objectives are viewed as fuzzy subsets of the set of alternatives.

Definition 3.2.1. A non empty set $X$ and a finite number of fuzzy subsets $(\tilde{g}_1, \tilde{g}_2, \ldots, \tilde{g}_m : X \to L)$ is a system of symmetrical fuzzy constraints denoted as $(X, \tilde{g}_1, \tilde{g}_2, \ldots, \tilde{g}_m)$. Thus, a system of inequalities in $\mathbb{R}^n$ is generalized by identifying every inequality with a membership function of the set of points $x \in \mathbb{R}^n$ that verify it.

The notion of a system of fuzzy constraints was a first step towards a generalization of the classical optimization problem. The meaning of a solution for the system of fuzzy constraints may be derived from the following definition.

Definition 3.2.2[Kandel (1986)]. A solution for the system of fuzzy constraints $(X, \tilde{g}_1, \tilde{g}_2, \ldots, \tilde{g}_m)$ is any point which maximizes

$$\tilde{D}(x) = \min \left( \mu_{\tilde{g}_1}(x), \mu_{\tilde{g}_2}(x), \ldots, \mu_{\tilde{g}_m}(x) \right).$$

The formal definition of an optimization in a fuzzy environment can be defined as follows [Kandel (1986)]:

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Definition 3.2.3. An optimization problem in a fuzzy environment is a system of fuzzy constraints together with an objective function $f : X \rightarrow \mathbb{R}$ (multiple functions $f_i : X \rightarrow \mathbb{R}$).

3.3. Solution Procedures of Fuzzy Optimization Problems

In 1970, Zadeh and Bellman proposed a method for solving an optimization problem with the underlying assumption that the fuzzy optimization problem contains an explicitly specified fuzzy set of alternatives that attain a fuzzy goal $\tilde{G} \subseteq X$ besides an explicitly formulated fuzzy set of feasible alternatives $\tilde{C} \subseteq X$ called a fuzzy constrain. The membership function $\mu_G(x)$ of $\tilde{G}$ indicates the degree to which an alternative $x \in X$ satisfies the fuzzy goal $\tilde{G}$. The membership function $\mu_G(x)$ may be defined as:

$$\mu_G(x) = \begin{cases} 1 & \text{for } f(x) \geq \bar{f} \\ \phi(x) & \text{for } \underline{f} \leq f(x) \leq \bar{f} \\ 0 & \text{for } f(x) \leq \underline{f} \end{cases} \tag{3.3.1}$$

where $\underline{f}$ and $\bar{f}$ are lower and upper aspiration levels respectively of objective function $f(x)$. The problem is now stated as:

“Satisfy $\tilde{C}$ and attain $\tilde{G}$”. \tag{3.3.2}

If a fuzzy set $\tilde{D} \subseteq X$ is introduced to solve this problem, (3.3.2) can be written as:

$$\tilde{D} = \tilde{C} \cap \tilde{G} \tag{3.3.3}$$

where “\cap” is an intersection operator corresponding to “and” in (3.3.2).

The fuzzy set $\tilde{D}$ is known as a fuzzy decision in the literature. In terms of membership functions, (3.3.3) can be written as:

$$\mu_{\tilde{D}}(x) = \mu_{\tilde{C}}(x) \land \mu_{\tilde{G}}(x) \text{ for each } x \in X \tag{3.3.4}$$

where “\land” is the minimum operator. Then the problem is to choose an alternative $x^* \in \tilde{D}$ to the maximum extent, that is, to choose $x^* \in X$ such that:

$$\mu_{\tilde{D}}(x^*) = \max_{x \in \tilde{D}} \mu_{\tilde{D}}(x) \tag{3.3.5}$$
For determining $x^* \in X$ in (3.3.5), some well known methods of mathematical programming can be used [see e.g. Tanaka et al. (1974), Negoita & Ralescu (1977) etc.]. In the case of multiple fuzzy goals $\tilde{G}_1, \tilde{G}_2, \ldots, \tilde{G}_n \subseteq X$ and fuzzy constraints $\tilde{C}_1, \tilde{C}_2, \ldots, \tilde{C}_m \subseteq X$, (3.3.4) becomes

$$\mu_D(x) = \mu_{\tilde{G}_1}(x) \land \ldots \land \mu_{\tilde{G}_n}(x) \land \mu_{\tilde{C}_1}(x) \land \ldots \land \mu_{\tilde{C}_m}(x)$$

for each $x \in X$. (3.3.6)

Then, we seek $x^* \in X$ such that (3.3.5) is satisfied.

When fuzzy constraint $\tilde{C}$ is defined in $X$, $\tilde{C} \subseteq X$ and the fuzzy goal $\tilde{G}$ in $Y$, $\tilde{G} \subseteq Y$ and a function $Y = w(x)$ is known, then, we can define a fuzzy goal $\tilde{G}' \subseteq X$ induced by $\tilde{G} \subseteq Y$. In this case, (3.3.4) becomes

$$\mu_D(x) = \mu_{\tilde{G}}(w(x)) \land \mu_{\tilde{C}}(x)$$

for each $x \in X$ and (3.3.5) remains the same.

3.4. Use of $\alpha$ - Cuts of the Fuzzy Feasible Set

Recalling Def. 3.2.2, we consider a fuzzy optimization problem. Let the problem be defined as

$$f(x) \rightarrow \max_{x \in \tilde{C}} \quad (3.4.1)$$

where $f : X \rightarrow R$ is an objective function and $\tilde{C} \subseteq X$ is a fuzzy constraint characterized by its membership function $\mu_{\tilde{C}} : X \rightarrow [0,1]$.

Using $\alpha$ - cuts of $\tilde{C}$ i.e. $\tilde{C}_{\alpha} = \{ x \in X \mid \mu_{\tilde{C}}(x) \geq \alpha \}$, $\alpha \in [0,1]$, we introduce the set(non fuzzy):

$$N(\alpha) = \left\{ x \in X \mid f(x) = \sup_{z \in \tilde{C}_{\alpha}} f(z) \right\}$$

for each $\alpha$ and $\tilde{C}_{\alpha} \neq \phi$. 

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Then, a solution to the problem (3.4.1) will be a fuzzy set $\tilde{S}_j \subseteq X$ characterized by the membership function:

$$
\mu_{\tilde{S}_j}(x) = \begin{cases} 
\sup_{x \in N(\alpha)} \alpha & \text{for } x \in \cup_{\alpha > 0} N(\alpha) \\
0 & \text{otherwise}
\end{cases}
$$

$$
= \begin{cases} 
\mu_c(x) & \text{for } x \in \cup_{\alpha > 0} N(\alpha) \\
0 & \text{otherwise}
\end{cases}
$$

(3.4.2)

In this case, solution exists if and only if there exists $\alpha > 0$ such that $N(\alpha) \neq \emptyset$.

Next the fuzzy maximal value of $f(x)$ over $\mu_c(x)$ is defined [Kacpryk & Orlovski (1987)] as

$$
\mu_f(r) = \sup_{x \in f^{-1}(r)} \mu_c(x) = \sup_{x \in f^{-1}(r)} \sup_{x \in N(\alpha)} \alpha \text{ for each } r \in R.
$$

(3.4.3)

Here, a compromised is needed because the choice of a single $x$ as a final solution is not simply based on taking the $x$ with the highest $\mu_{\tilde{S}_j}(x)$ but also on the value of $f(x)$ corresponding to that $x$. For example, for $x = x_*$ such that $f(x_*) = r_*$, the greater $r_*$, the smaller value of $\mu_c(x)$. Problem (3.4.1) also can be solved based on the concept of the pareto optimum. The pareto optimum set can be defined as follows:

Definition 3.4.1. Consider the two functions $f(x)$ and $\mu_c(x)$. Define the subset $P$ of $X$ such that $x \in P$. If there exists no $y \in X$ for which either —

$$
f(y) > f(x) \text{ and } \mu_c(y) \geq \mu_c(x) \text{ or } f(y) \geq f(x) \text{ and } \mu_c(y) > \mu_c(x),
$$

then, $P$ is called the set of pareto maximal elements for the two functions $f(x)$ and $\mu_c(x)$.
Now, the solution of (3.4.1) can be defined as a fuzzy set \( \hat{S}_2 \subseteq X \) such that

\[
\mu_{\hat{S}_2}(x) = \begin{cases} 
\mu_c(x) & \text{for } x \in P \\
0 & \text{otherwise}
\end{cases}
\]  

(3.4.4)

This solution gives the same fuzzy maximal value of \( f(x) \) over \( \mu_c(x) \) as the solution given in (3.4.2).

3.5. The Case of Multiple Objective Functions and Fuzzy Constraints

We consider a fuzzy optimization problem having \( n \) objective functions \( f_i : X \to R, \ i=1,2...,n \) and \( m \) fuzzy constraints \( \mu_{\hat{c}_1}(x), \mu_{\hat{c}_2}(x),...\mu_{\hat{c}_m}(x) \). We seek \( x^* \in X \) such that

\[
\left( f_1(x), f_2(x),...,f_n(x) \right) \to \max_y
\]

subject to \( \mu_{\hat{c}_1}(x), \mu_{\hat{c}_2}(x),...\mu_{\hat{c}_m}(x) \)  

(3.5.1)

where \( \max \) denotes maximization in the sense of pareto.

We assume the objective functions are not real valued but take on fuzzy values \( \tilde{F}_i(x) \subseteq X \). The real valued functions are here evidently special cases. Then, the membership value of a value of \( f_i(x) = r \) in \( \tilde{F}_i(x) \) is \( \mu_{\tilde{F}_i(x)}(r) \). Thus, we can define \( n \) fuzzy non-strict preference relations \( \tilde{P}_i \) over the set of alternatives \( X \) i.e. \( \tilde{P}_i : X \times X \to [0,1] \) given by:

\[
\tilde{P}_i(x_1,x_2) = \sup_{z \in y} \left( \mu_{\tilde{F}_i(x_1)}(z) \land \mu_{\tilde{F}_i(x_2)}(y) \right), i=1,2,...,n.
\]  

(3.5.2)

where \( f(x_1) = z, f(x_2) = y \) and " \land " is the minimum operator.
To compare the alternatives using these \( n \) fuzzy preference relations, we define a fuzzy strict preference relation \( \tilde{P}^i: X \times X \to [0,1] \) corresponding to \( \tilde{P}_i \) as
\[
\tilde{P}^i (x_1, x_2) = \max \left( 0, \tilde{P}_i (x_1, x_2) - \tilde{P}_i (x_2, x_1) \right)
\]
\[
= 0 \vee \left( \tilde{P}_i (x_1, x_2) - \tilde{P}_i (x_2, x_1) \right)
\]
Then, \( \tilde{P}^* = \bigwedge_{i=1}^n \tilde{P}^i (x_1, x_2) \) (3.5.3)
is the degree to which \( x_1 \) is strictly preferred to \( x_2 \).

Next we introduce a fuzzy subset of non-dominated alternatives
\[
\mu_{\text{ND}} (x) = 1 - \sup_{y \in X} \tilde{P}^* (y, x)
\]
\[
= 1 - \sup_{y \in X} \bigwedge_{i=1}^n \tilde{P}^i (y, x)
\]
\[
= 1 - \sup_{y \in X} \left( \tilde{P}_i (y, x) - \tilde{P}_i (x, y) \right) \quad (3.5.4)
\]
The value of \( \mu_{\text{ND}} (x) \) is a non-dominance degree of alternative \( x \). If \( \mu_{\text{ND}} (x) \geq \alpha \), then, \( x \) may be strictly dominated by some other alternatives to a degree smaller than \( 1 - \alpha \).

We also require to define a second element, the degree of feasibility of alternative \( x \) with respect to the fuzzy constraints \( \tilde{C}_1, \tilde{C}_2, ..., \tilde{C}_m \). This can be done as follows:

We define the degree of feasibility \( \mu_{\text{FS}} \) of the fuzzy constraints as
\[
\mu_{\text{FS}} (x) = \mu_{\tilde{C}_1} (x) \wedge \mu_{\tilde{C}_2} (x) \wedge ... \wedge \mu_{\tilde{C}_m} (x) \quad (3.5.5)
\]
The solution of the optimization problem (3.5.1) is now meant to find an alternative \( x^* \in X \) for which
\[
\mu_{\text{ND}} (x^*) \geq \alpha \quad \text{and} \quad \mu_{\text{FS}} (x^*) \geq \beta \quad (3.5.6)
\]
where \( \alpha \in [0,1] \) is a desired degree of non-dominance and \( \beta \in [0,1] \) is a desired degree of feasibility. It is also noted that a compromise between \( \alpha \) and \( \beta \) is sought.
There are several other approaches among which most of them also being based on some degree of dominance. In these regards, we should mention e.g. Takeda & Nishida (1980), Leung (1983) and Carlsson (1982).

In the following, we present some basic approaches to introduce fuzziness into the general mathematical programming problems. Emphasis is on fuzzy linear programming as its non-fuzzy counterpart is of particular relevance from the practical view of point.

3.6. Fuzzy Linear Programming

The first attempt to fuzzify a crisp linear programming problem (LP):

\[
\begin{align*}
\text{Maximize } f(x) &= c^T x \\
\text{subject to } Ax &\leq b, \ x \geq 0 \\
&\quad c, x \in \mathbb{R}^n, \ b \in \mathbb{R}^m, \ A \in \mathbb{R}^{m \times n}
\end{align*}
\]  

(3.6.1)

is due to Zimmermann (1975). The setting of fuzzy linear programming (FLP) corresponding to (3.6.1) is as follows:

Assuming that the decision maker (DM) can establish an aspiration level \( z \) for the value of the objective function that he wants to achieve, each of the constraints is modelled as a fuzzy set. Then, (3.6.1) is modelled as a fuzzy LP in the following way:

Find \( x \) such that

\[
\begin{align*}
c^T x &\geq z \\
Ax &\leq b \\
x &\geq 0
\end{align*}
\]

or

\[
\begin{align*}
-c^T x &\leq -z \\
Ax &\leq b \\
x &\geq 0
\end{align*}
\]

(3.6.2)

where \( \leq \) denotes the fuzzified version of \( \leq \) and has the linguistic interpretation “essentially smaller than or equal” and \( \geq \) denotes the fuzzified version of \( \geq \) and has the linguistic interpretation “essentially greater than or equal”.

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The problem (3.6.2) is symmetric with respect to objective function and constraints and can be written as

$$\text{Find } x$$

such that

$$Bx \leq d$$

$$x \geq 0$$

$$(3.6.3)$$

$$B = \begin{pmatrix} -c^T \\ A \end{pmatrix} \text{ and } d = \begin{pmatrix} -z \\ b \end{pmatrix}$$

By using subjectively chosen constants $p_i$'s of admissible violation of the constraints and objective function, each of the $(m+1)$ constraints $B_i x (i^{th}$ constraint of $B)$ of (3.6.3) can be represented by a fuzzy set. A simplest type of linear membership function over the “tolerance interval” $p_i$ can be constructed as:

$$\mu_i(x) = \begin{cases} 
1 & \text{for } B_i x \leq d_i, \\
1 - \frac{B_i x - d_i}{p_i} & \text{for } d_i \leq B_i x \leq d_i + p_i, \\
0 & \text{for } B_i x > d_i + p_i 
\end{cases}$$

$$i = 1, 2, \ldots, m+1.$$ (3.6.4)

After some rearrangements [Zimmermann (1976)] and with some additional assumptions, the fuzzy set “decision” $\tilde{D}$ of model (3.6.5) can be defined by the membership function

$$\mu_{\tilde{D}}(x) = \max_{x \in \Omega} \min_i \left( 1 - \frac{B_i x - d_i}{p_i} \right)$$

Then, introducing one new variable $\lambda$, $\lambda \in [0, 1]$ which corresponds to (3.6.5), we can arrive at an equivalent problem:

$$\max \lambda$$

such that

$$\begin{cases} 
\lambda p_i + B_i x \leq d_i + p_i \\
x \geq 0, \quad i = 1, 2, \ldots, m+1
\end{cases}$$

$$(3.6.6)$$
If an optimal solution to (3.6.6) is the vector \((\lambda^*, x^*)\), then, \(x^*\) is the maximizing solution of model (3.6.2). In 1984, Werners investigated a fuzzy linear programming problem involving fuzzy as well as crisp constraints with a crisp objective function. The problem can be defined in analogy to (3.6.1) as

\[
\begin{align*}
\text{Maximize} & \quad f(x) = c^T x \\
\text{Subject to} & \quad Ax \leq b, \\
& \quad Dx \leq b', \\
& \quad x \geq 0, \quad b', D \in \mathbb{R}^{m+n}
\end{align*}
\]  

(3.6.7)

where the feasible solution \(x\) lies in the fuzzy feasible region \(\tilde{S}\).

The membership function of the fuzzy constraints can be defined in analogy to (3.6.4) as

\[
\mu_i(x) = \begin{cases} 
1 & \text{for } A_i x \leq b_i \\
\frac{b_i + p_i - A_i x}{p_i} & \text{for } b_i < A_i x \leq b_i + p_i \\
0 & \text{for } A_i x > b_i + p_i
\end{cases}
\]  

(3.6.8)

The membership function of the objective function is determined as

\[
\mu_f(x) = \begin{cases} 
1 & \text{if } f_s \leq c^T x \\
\frac{c^T x - f_i}{f_s - f_i} & \text{if } f_i < c^T x < f_s \\
0 & \text{if } c^T x \leq f_i
\end{cases}
\]  

(3.6.9)

where \(f_s\) and \(f_i\) are the optimal values of \(f(x)\) for the following LP problems respectively:

\[
\begin{align*}
\text{Maximize} & \quad f(x) = c^T x \\
\text{Subject to} & \quad Ax \leq b + p, \\
& \quad Dx \leq b', \quad x \geq 0
\end{align*}
\]  

(3.6.10)
and

\[
\begin{align*}
\text{Maximize} & \quad f(x) = c^T x \\
\text{Subject to} & \quad Ax \leq b \\
& \quad Dx \leq b', \ x \geq 0
\end{align*}
\]  \hspace{1cm} (3.6.11)

The underlying assumption is that \( f_\circ \) is the supremum of the support \( \tilde{R} \) and \( f_i \) is the supremum of \( 1 \)-cut of \( \tilde{R}, \tilde{R}_i \).

Then, the equivalent crisp model to (3.6.6) is given by

\[
\begin{align*}
\max \quad & \lambda \\
\text{such that} & \quad \lambda(f_\circ - f) - c^T x \leq -f_i \\
& \quad \lambda + Ax \leq b + p \\
& \quad Dx \leq b' \\
& \quad \lambda \leq 1 \\
& \quad \lambda, x \geq 0
\end{align*}
\]  \hspace{1cm} (3.6.12)

The most general type of fuzzy linear programming [see, e.g. Klir & Yuan (2002)] is formulated as follows:

\[
\begin{align*}
\max \quad & \sum_{j=1}^{n} \tilde{C}_j x_j \\
\text{such that} & \quad \sum_{j=1}^{n} \tilde{A}_j x_j \leq \tilde{B}_i, \quad i = 1, 2, \ldots, m \\
& \quad x_j \geq 0, \quad j = 1, 2, \ldots, n.
\end{align*}
\]  \hspace{1cm} (3.6.13)

where \( \tilde{A}_j, \tilde{B}_i, \tilde{C}_j \) are fuzzy numbers. The operations of addition and multiplication are operations of fuzzy arithmetic and \( \leq \) denotes the ordering of fuzzy numbers. In many situations, these fuzzy numbers are designated by triangular fuzzy numbers (see definitions). By defining triangular fuzzy numbers \( \tilde{A}_j = (s_j, l_j, r_j), \tilde{B}_i = (t_i, u_i, v_i) \) and \( \tilde{C}_j = (c_j, 0, 0) \) and substituting these values in (3.6.13), the problem can be rewritten as
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\[ \begin{align*}
\text{max } & \sum_{j=1}^{n} c_j x_j \\
\text{such that } & \sum_{j=1}^{n} s_{ij} x_j \leq t_i \\
& \sum_{j=1}^{n} (s_{ij} - l_{ij}) x_j \leq t_i - u_i \\
& \sum_{j=1}^{n} (s_{ij} + r_{ij}) x_j \leq t_i + v_i \\
x_j & \geq 0, \quad i = 1, 2, ..., m; \quad j = 1, 2, ..., n
\end{align*} \]  

(3.6.14)

Then, the problem (3.6.14) can be solved by using standard linear programming technique.

The model of fuzzy linear programming was extended to fuzzy integer and 0-1 programming problems. Almost all the progress in fuzzy integer programming is due to Febian & Stoica (1984). A solution technique for solving fuzzy integer programming models with multiple criteria appeared first in Ignizio & Daniels (1983). The pioneering works on fuzzy 0-1 programming are those of Zimmermann & Pollatschek (1984). These extended Zimmermann’s fuzzy linear programming model[of model (3.6.2)] by adding the requirements \( x_i \in \{0, 1\} \) can be defined as

\[ \begin{align*}
\text{Maximize } & \mathbf{c}^T \mathbf{x} \geq z \\
\text{subject to } & \mathbf{A} \mathbf{x} \leq \mathbf{b} \\
x_i & \in \{0, 1\}, \quad i = 1, 2, ..., n.
\end{align*} \]  

(3.6.15)

Then, following in principle the line of procedure (3.6.3) to (3.6.6), a conventional crisp equivalent LP problem of (3.6.15) can be derived. A branch and bound method can be employed to obtain the solution of (3.6.15).

3.7. Fuzzy Dynamic Programming

Fuzzy dynamic programming as an extension of crisp dynamic programming was suggested for the first time by Zadeh & Bellman (1970). Their considerations are based on the symmetrical model of an optimization problem as
defined in definitions 3.3.2 and 3.3.3. The fuzzification of classical (crisp) dynamic can be seen in the following framework:

Let \( X = \{ x \} = \{ s_1, s_2, \ldots, s_s \} \) be a state space, \( U = \{ u \} = \{ c_1, c_2, \ldots, c_m \} \) be a control space, the temporal evolution of a dynamic system under control be described by its state equation

\[
x_{i+1} = f(x_i, u_i)
\]  

(3.7.1)

where \( x_i, x_{i+1} \in X \) are states at time \( t \) and \( t+1 \) respectively and \( u_i \in U \) is a control at time \( t \). Let \( x_0 \in X \) be an initial state and \( N \) be a fixed and specified termination time.

We assume that for each \( t = 0, 1, \ldots, N-1 \), a fuzzy constraint \( \mu_C(t(u_i)) \) and only for \( t = N \), a fuzzy goal \( \mu_G N(x_N) \) are defined. Then, the problem is to find an optimal sequence of controls \( u_1^*, u_2^*, \ldots, u_{N-1}^* \) such that the fuzzy decision \( \tilde{D} \) is maximized [Kacpryzk & Orlovski (1987)] i.e.

\[
\mu_{\tilde{D}}(u_1^*, u_2^*, \ldots, u_{N-1}^* \mid x_i) = \max_{u_1, u_2, \ldots, u_{N-1}} \mu_{\tilde{D}}(u_1, u_2, \ldots, u_{N-1} \mid x_i)
\]

(3.7.2)

\[
= \max_{u_1, u_2, \ldots, u_{N-1}} \left( \mu_C(u_i) \wedge \ldots \wedge \mu_C(u_{N-1}) \wedge \mu_G(x_N) \right)
\]

where \( x_{i+1} = f(x_i, u_i) \); \( t = 0, 1, \ldots, N-1 \).

Then, this problem can be solved by dynamic programming through the following set of recurrence equations:

\[
\mu_{G^{\tilde{D}}} (u_{N-,i}) = \max_{u_{N-,i}} \left( \mu_{G^{\tilde{D}}} (u_{N-,i}) \wedge \mu_{G^{\tilde{D}}} (x_{N-,i+1}) \right)
\]

(3.7.3)

\[
x_{N-,i+1} = f(x_{N-,i}, u_{N-,i}) \quad \text{for } i = 1, 2, \ldots, N
\]
Among the numerous applications of fuzzy dynamic programming, we should mention those for clustering, regional development [see e.g. Esogbue & Bellman (1984), Kacprzyk & Straszak (1984) etc.). Here, we will not consider the details of the problem.

### 3.8. Fuzzy Multi-criteria Optimization

It is known that the fuzzy multi-criteria optimization (decision making theory) is developed well enough. Many interesting results are proved and methods are studied in it. Among numerous analyses’, solution procedures and interactive approaches of multi-criteria fuzzy optimization, we should mention those works of Zhukovin (1983), Feng (1983,1987), Sakawa & Yuno (1987), Zimmermann (1978), Leung (1983,1987), Febian & Stoica (1984). Moreover fuzzy optimization in network [see e.g. Chanas (1987), Chanas et.al. (1984)]. Location models [see e.g. Darzentas (1987), Zimmermann (1996)], transportation problems[see e.g. Delgado et.al.(1987), Eigearlaigh (1982)] and optimal allocation of resources [see e.g. Mamik & Rimanek (1987)], clustering and pattern recognition [ see e.g. Oliverira and Pedrycz (2007), Bezekst et.al. (2005)] are also well developed enough. Here, we should not discuss the details of those models and solution procedures.

Recently, some newer approaches to fuzzy optimization have appeared. Basically in most of them, they try to further fuzzify(soften) the models presented in works of previous authors by representing some commonsense perceptions e.g. linguistic variables, linguistic hedges etc. The fuzzy optimization problems may be seen as attempts to develop what might be called knowledge based optimization and mathematical programming models as opposed to the data-based conventional ones. This should eventually lead to an expert-system based decision support for optimization which should greatly enhance implementability of optimization tools and techniques in real world problems.

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