CHAPTER 3

PRIME RADICALS IN TERNARY SEMIGROUPS
CHAPTER -3

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INTRODUCTION:

KRULL [27] proved that the nil-radical of an ideal A in a commutative ring is equal to the intersection of all minimal prime ideals containing A. SATYANARAYANA [46] has developed some literature on prime ideals and prime radicals for commutative semigroups and obtained KRULL’s theorem [27] for commutative semigroups. GIRI and WAZALWAR [15] studied about prime radicals in general semigroups. In this thesis we study about the prime ideals, completely prime ideals, semiprime ideals and completely semiprime ideals, prime radicals and generalize all these results in general ternary semigroups.

This chapter is divided into 3 sections. In section 1, the term, ‘completely prime ideal’ of a ternary semigroup is introduced. It is proved that an ideal A of a ternary semigroup T is completely prime iff TP is either a ternary subsemigroup of T or empty. We introduced the notion of ‘prime ideal’ in a ternary semigroup. It is proved that in a ternary semigroup (i) A is a prime ideal of T, (ii) For a, b, c ∈ T; < a > < b > < c > ⊆ A implies a ∈ A or b ∈ A or c ∈ A, (iii) For a; b; c ∈ T; T^1T^1aT^1T^1b T^1T^1c T^1T^1 ⊆ A implies a ∈ A or b ∈ A or c ∈ A are equivalent. Further it is proved that an ideal A of a ternary semigroup T is a prime ideal of T if and only if TA is an m-system of T or empty. It is also proved that every completely prime ideal of a ternary semigroup is prime. In a globally idempotent ternary semigroup, it is proved that every maximal ideal is prime. It is also proved that a globally idempotent ternary semigroup having a maximal ideal contains semisimple elements.

In section 2, the term, completely semiprime ideal of a ternary semigroup is introduced. It is proved that an ideal A of a ternary semigroup T is completely semiprime if and only if x ∈ T, x^3 ∈ A implies x ∈ A. It is proved that if A is a completely semiprime ideal of a ternary semigroup T, then x, y, z ∈ T, xyz ∈ A implies that xyTTz ⊆ A, xTTyz ⊆ A and xTyTz ⊆ A. The term; semiprime ideal in a ternary semigroup is introduced. It is also proved that every completely semiprime ideal of a ternary semigroup is semiprime. It is proved that an ideal A of a ternary semigroup T is completely semiprime if and only if
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$T \setminus A$ is a $d$-system of $T$ or empty. It is also proved that the nonempty intersection of a family of (1) completely prime ideals of a ternary semigroup is completely semiprime (2) prime ideals of a ternary semigroup is semiprime. And also proved that an ideal $Q$ of a semigroup $T$ is (1) semiprime iff $T \setminus Q$ is either an $n$-system or empty. It is proved that if $N$ is an $n$-system in a ternary semigroup $T$ and $a \in N$, then there exist an $m$-system $M$ in $T$ such that $a \in M$ and $M \subseteq N$.

In section 3, to each ideal $A$ of a ternary semigroup $T$, we associate four types of sets namely $A_1$, $A_2$, $A_3$, $A_4$ and we proved that $A \subseteq A_1 \subseteq A_2 \subseteq A_3 \subseteq A_4$. In a commutative ternary semigroup, it is proved that $A_1 = A_2 = A_3 = A_4$ and in general ternary semigroups, it is proved that $A_1 \neq A_2 \neq A_3 \neq A_4$ by means of examples. The terms ‘radical’ and ‘complete radical’ of an ideal in a ternary semigroup are also introduced and some of their properties are obtained. It is proved that in a ternary semigroup $T$ if $A$, $B$ and $C$ are ideals of $T$, then i) $A \subseteq B \Rightarrow \sqrt{A} \subseteq \sqrt{B}$, ii) if $A \cap B \cap C \neq \emptyset$ then $\sqrt{ABC} = \sqrt{A \cap B \cap C} = \sqrt{A} \cap \sqrt{B} \cap \sqrt{C}$ and iii) $\sqrt{\sqrt{A}} = \sqrt{A}$. In a ternary semigroup $T$ if $A$ is an ideal, then $\sqrt{A}$ is a semiprime ideal of $T$. It is proved that in a ternary semigroup $T$ with identity there is a unique maximal ideal $M$ such that $\sqrt{M^n} = M$ for all odd natural numbers $n$. It is proved that an ideal $Q$ of a ternary semigroup $T$ is semiprime iff $\sqrt{Q} = Q$. Further it is proved that if $A$ is a semiprime ideal of a ternary semigroup $T$ and $M$ is a maximal $m$-system of $T$ such that $A \cap M = \emptyset$, then $T \setminus M$ is a minimal prime ideal of $T$ containing $A$. It is also proved that every prime ideal $P$ minimal relative to containing a completely semiprime ideal $A$ of a ternary semigroup $T$ is completely prime.

The contents of chapter 3 are published in “International Organization of Scientific Research Journal of Mathematics” under the title ‘Prime Radicals in Ternary Semigroups’ [42].

3.1. COMPLETELY PRIME IDEALS AND PRIME IDEALS:

In this section, the term, ‘completely prime ideal’ is introduced. It is proved that an ideal $A$ of a ternary semigroup $T$ is completely prime iff $T \setminus A$ is either a ternary subsemigroup of $T$ or empty. We introduced the notion of ‘prime ideal’ in a ternary semigroup. It is proved that in a ternary semigroup (i) $A$ is a prime ideal of $T$, (ii) For $a, b, c \in T; \langle a, b \rangle \subseteq A$ implies $a \in A$ or $b \in A$, (iii) For $a, b, c \in T; \langle T^1 \rangle a \langle T^1 \rangle b \langle T^1 \rangle c \subseteq A$ implies $a \in A$ or $b \in A$ or $c \in A$ are equivalent. Further it is
proved that an ideal A of a ternary semigroup T is a prime ideal of T if and only if T\A is an $m$-system of T or empty. It is also proved that every completely prime ideal of a ternary semigroup is prime. In a globally idempotent ternary semigroup, it is proved that every maximal ideal is prime. It is also proved that a globally idempotent ternary semigroup having a maximal ideal contains semisimple elements.

We are introduce the notion of a completely prime ideal of a ternary semigroup.

**DEFINITION 3.1.1**: An ideal A of a ternary semigroup T is said to be a completely prime ideal of T provided $x, y, z \in T$ and $xyz \in A$ implies either $x \in A$ or $y \in A$ or $z \in A$.

**EXAMPLE 3.1.2**: In the commutative ternary semigroup $\mathbb{Z}^{-}$ of all negative integers, the ideal $P = \{ 3k : k \in \mathbb{Z}^{-} \}$ is a completely prime ideal. For $x, y, z \in \mathbb{Z}^{-}$, $xyz \in P \iff xyz$ is divisible by 3 $\iff x$ is divisible by 3 or $y$ is divisible by 3 or $z$ is divisible by 3 $\iff x = 3k_1$ or $y = 3k_2$ or $z = 3k_3$ for $k_1, k_2, k_3 \in \mathbb{Z}^{-}$.

**EXAMPLE 3.1.3**: In example 3.1.2., P is a completely prime ideal. But the ideal $Q = \{ 30k : k \in \mathbb{Z}^{-} \}$ is not a prime ideal of $\mathbb{Z}^{-}$, since $(-2) \cdot (-3) \cdot (-5) = -30 \in Q$ but $(-2) \not\in Q$, $(-3) \not\in Q$ and $(-5) \not\in Q$.

**THEOREM 3.1.4**: An ideal A of a ternary semigroup T is completely prime if and only if $x_1, x_2, \ldots, x_n \in T, n$ is an odd natural number, $x_1 x_2 \ldots x_n \in A \Rightarrow x_i \in A$ for some $i = 1, 2, 3, \ldots, n$.

**Proof**: Suppose that A is a completely prime ideal of T.

Let $x_1, x_2, \ldots, x_n \in T$ where $n$ is an odd natural number and $x_1 x_2 \ldots x_n \in A$.

If $n = 1$ then clearly $x_1 \in A$. If $n = 3$ then $x_1 x_2 x_3 \in A \Rightarrow x_1 \in A$ or $x_2 \in A$ or $x_3 \in A$.

If $n = 5$ then $x_1 x_2 x_3 x_4 x_5 \in A \Rightarrow x_1 x_2 x_3 \in A$ or $x_4 \in A$ or $x_5 \in A$.

Therefore by induction of $n$ is an odd natural number, then $x_1 x_2 \ldots x_n \in A \Rightarrow x_i \in A$ for some $i = 1, 2, 3, \ldots, n$. The converse part is trivial.

We now prove a necessary and sufficient condition for an ideal to be a completely prime ideal in a ternary semigroup.

**THEOREM 3.1.5**: An ideal A of a ternary semigroup T is completely prime if and only if $T \setminus A$ is either subsemigroup of T or empty.
Therefore X

Suppose if possible X

Let X, Y, Z be the three ideals of T and XYZ \subseteq A.

(ii) Suppose that T is a completely prime ideal of T. Then (i) is obvious.

(iii) Suppose that T is a prime ideal of T. Let a, b, c \in T and abc \in A.

We now introduce the notion of a prime ideal of a ternary semigroup which is due to MUHAMMAD SHABIR and MEHAR BANO [35].

DEFINITION 3.1.6: An ideal A of a ternary semigroup T is said to be a prime ideal of T provided X, Y, Z are ideals of T and XYZ \subseteq A \Rightarrow X \subseteq A or Y \subseteq A or Z \subseteq A.

THEOREM 3.1.7: In a ternary semigroup T, the following conditions are equivalent:

(i) A is a prime ideal of T.

(ii) For a, b, c \in T; < a > < b > < c > \subseteq A implies a \in A or b \in A or c \in A.

(iii) For a; b; c \in T; T^{1}T^{1}aT^{1}bT^{1}T^{1}cT^{1}T^{1} \subseteq A implies a \in A or b \in A or c \in A.

Proof: (i) \Rightarrow (ii): Suppose that A is a prime ideal of T. Then (i) \Rightarrow (ii) is obvious.

(ii) \Rightarrow (iii): Let a, b, c \in T such that T^{1}T^{1}aT^{1}bT^{1}T^{1}cT^{1}T^{1} \subseteq A.

Now < a > < b > < c > = (T^{1}T^{1}aT^{1}bT^{1}T^{1}) (T^{1}T^{1}cT^{1}T^{1}) \subseteq T^{1}T^{1}aT^{1}T^{1}bT^{1}T^{1}cT^{1}T^{1} \subseteq A \Rightarrow a \in A or b \in A or c \in A.

(iii) \Rightarrow (i): Suppose that a, b, c \in T; T^{1}T^{1}aT^{1}bT^{1}T^{1}cT^{1}T^{1} \subseteq A \Rightarrow a \in A or b \in A or c \in A.

Let X, Y, Z be the three ideals of T and XYZ \subseteq A.

Suppose if possible X \nsubseteq A, Y \nsubseteq A, Z \nsubseteq A.

X \nsubseteq A, Y \nsubseteq A, Z \nsubseteq A, there exists a, b, c such that a \in X and a \nsubseteq A, b \in Y and b \nsubseteq A and c \in Z and c \nsubseteq A. a \in X, b \in Y, c \in Z \Rightarrow abc \in XYZ \subseteq A.

Now T^{1}T^{1}aT^{1}T^{1}bT^{1}T^{1}cT^{1}T^{1} \subseteq XYZ \subseteq A \Rightarrow a \in A or b \in A or c \in A. It is a contradiction. Therefore X \subseteq A or Y \subseteq A or Z \subseteq A and hence A is a prime ideal of T.
THEOREM 3.1.8 : An ideal A of a ternary semigroup T is prime if and only if \( X_1, X_2, \ldots, X_n \subseteq T \), \( n \) is an odd natural number, \( X_1 X_2 \ldots X_n \subseteq A \Rightarrow X_i \in A \) for some \( i = 1, 2, 3, \ldots n \).

**Proof** : Suppose that A is a prime ideal of T.
Let \( X_1, X_2, \ldots, X_n \subseteq T \), \( n \) is an odd natural number and \( X_1 X_2 \ldots X_n \subseteq A \)
If \( n = 1 \) then clearly \( X_1 \in A \).
If \( n = 3 \) then \( X_1 X_2 X_3 \subseteq A \Rightarrow X_1 \subseteq A \) or \( X_2 \subseteq A \) or \( X_3 \subseteq A \).
If \( n = 5 \) then \( X_1 X_2 X_3 X_4 X_5 \subseteq A \Rightarrow X_1 X_2 X_3 \in A \) or \( X_4 \in A \) or \( X_5 \in A \)
\( \Rightarrow X_1 \in A \) or \( X_2 \in A \) or \( X_3 \in A \) or \( X_4 \in A \) or \( X_5 \in A \).
Therefore by induction of \( n \) is an odd natural number, then \( X_1 X_2 \ldots X_n \subseteq A \Rightarrow X_i \subseteq A \) for some \( i = 1, 2, 3, \ldots n \).

The converse part is trivial.

THEOREM 3.1.9 : Every completely prime ideal of a ternary semigroup T is a prime ideal of T.

**Proof** : Suppose that A is a completely prime ideal of a ternary semigroup T.
Let \( a, b, c \in T \) and \( < a > < b > < c > \subseteq A \). Then \( abc \in A \). Since A is a completely prime, either \( a \in A \) or \( b \in A \) or \( c \in A \). Therefore A is a prime ideal of T.

The following theorem is duo to Kar.S and Maity.B.K. [25].

THEOREM 3.1.10 : Let T be a commutative ternary semigroup . An ideal P of T is a prime ideal if and only if P is a completely prime ideal.

We now introduce the notion of an \( m \)-system of a ternary semigroup.

**DEFINITION 3.1.11** : A nonempty subset A of a ternary semigroup T is said to be an \( m \)-system provided for any \( a, b, c \in A \) implies that \( T^1T^1T^1T^1T^1T^1T^1T^1T^1T^1T^1T^1T^1T^1T^1T^1T^1T^1 \cap A \neq \emptyset \).

We now prove a necessary and sufficient condition for an ideal to be a prime ideal in a ternary semigroup.

THEOREM 3.1.12 : An ideal A of a ternary semigroup T is a prime ideal of T if and only if \( T \setminus A \) is an \( m \)-system of T or empty.

**Proof** : Suppose that A is a prime ideal of a ternary semigroup T and \( T \setminus A \neq \emptyset \).
Let \( a, b, c \in T \setminus A \). Then \( a \not\in A \), \( b \not\in A \) and \( c \not\in A \).
Suppose if possible \( T^1T^1aT^1T^1bT^1T^1c \ T^1T^1 \cap T \setminus A = \emptyset \).
\( T^1T^1aT^1T^1bT^1T^1c \ T^1T^1 \cap T \setminus A = \emptyset \Rightarrow T^1T^1aT^1T^1bT^1T^1c \ T^1T^1 \subseteq A \).

Since \( A \) is prime, either \( a \in A \) or \( b \in A \) or \( c \in A \).

It is a contradiction. Therefore \( T^1T^1aT^1T^1bT^1T^1c \ T^1T^1 \cap T \setminus A \neq \emptyset \).

Hence \( T \setminus A \) is an \( m \)-system.

Conversely suppose that \( T \setminus A \) is either an \( m \)-system of \( T \) or \( T \setminus A = \emptyset \).

If \( T \setminus A = \emptyset \), then \( T = A \) and hence \( A \) is a prime ideal of \( T \).

Assume that \( T \setminus A \) is an \( m \)-system of \( T \). Let \( a, b, c \in T \) and \( <a> <b> <c> \subseteq A \).

Suppose if possible \( a \notin A, b \notin A \) and \( c \notin A \). Then \( a, b, c \in T \setminus A \). Since \( T \setminus A \) is an \( m \)-system, \( \Rightarrow T^1T^1aT^1T^1bT^1T^1c \ T^1T^1 \cap T \setminus A \neq \emptyset \Rightarrow T^1T^1aT^1T^1bT^1T^1c \ T^1T^1 \not\subseteq A \)
\( \Rightarrow <a><b><c> \not\subseteq A \). It is a contradiction.

Therefore \( a \in A \) or \( b \in A \) or \( c \in A \). Hence \( A \) is a prime ideal of \( T \).

THEOREM 3.1.13 : If \( T \) is a globally idempotent ternary semigroup then every maximal ideal of \( T \) is a prime ideal of \( T \).

**Proof** : Let \( M \) be a maximal ideal of \( T \). Let \( A, B, C \) be three ideals of \( T \) such that \( ABC \subseteq M \).

Suppose if possible \( A \not\subseteq M, B \not\subseteq M, C \not\subseteq M \).

Now \( A \not\subseteq M \Rightarrow M \cup A \) is an ideal of \( T \) and \( M \subseteq M \cup A \subseteq T \).

Since \( M \) is a maximal, \( M \cup A = T \).

Similarly \( B \not\subseteq M \Rightarrow M \cup B = T \), \( C \not\subseteq M \Rightarrow M \cup C = T \).

Now \( T = TTT = (M \cup A) (M \cup B) (M \cup C) \subseteq M \Rightarrow T \subseteq M \). Thus \( M = T \).

It is a contradiction. Therefore either \( A \subseteq M \) or \( B \subseteq M \) or \( C \subseteq M \).

Hence \( M \) is a prime.

THEOREM 3.1.14 : If \( T \) is a globally idempotent ternary semigroup having maximal ideals then \( T \) contains semisimple elements.

**Proof** : Suppose that \( T \) is a globally idempotent ternary semigroup having maximal ideals.

Let \( M \) be a maximal ideal of \( T \). Then by theorem 3.1.13., \( M \) is prime.

Now if \( a \in T \setminus M \) then \( <a> \not\subseteq M \) and \( <a>^3 \not\subseteq M \).

Then \( T = M \cup <a> = M \cup <a>^3 \).

Therefore \( a \in <a>^3 \) and hence \( <a> = <a>^3 \).

Thus \( a \) is a semisimple element.

Therefore \( T \) contains semisimple elements.
3.2. COMPLETELY SEMIPRIME IDEALS AND SEMIPRIME IDEALS:

In this section, the term, completely semiprime ideal of a ternary semigroup is introduced. It is proved that an ideal A of a ternary semigroup T is completely semiprime if and only if \( x \in T, x^3 \in A \) implies \( x \in A \). It is proved that if A is a completely semiprime ideal of a ternary semigroup T, then \( x, y, z \in T, xyz \in A \) implies that \( xyTz \subseteq A, xTTyz \subseteq A \) and \( xTyTz \subseteq A \). The term; semiprime ideal in a ternary semigroup is introduced. It is also proved that every completely semiprime ideal of a ternary semigroup is semiprime. It is proved that an ideal A of a ternary semigroup T is completely semiprime if and only if \( T \setminus A \) is a d-system of T or empty. It is also proved that the nonempty intersection of a family of (1) completely prime ideals of a ternary semigroup is completely semiprime (2) prime ideals of a ternary semigroup is semiprime. And also proved that an ideal Q of a semigroup T is (1) semiprime iff \( T \setminus Q \) is either an n-system or empty. It is proved that if N is an n-system in a ternary semigroup T and \( a \in N \), then there exist an m-system M in T such that \( a \in M \) and \( M \subseteq N \).

We now introduce the notion of a completely semiprime ideal of a ternary semigroup.

**DEFINITION 3.2.1**: An ideal A of a ternary semigroup T is said to be a **completely semiprime ideal** provided \( x \in T, x^n \in A \) for some odd natural number \( n > 1 \) implies \( x \in A \).

**EXAMPLE 3.2.2**: In commutative ternary semigroup \( \mathbb{Z}^- \) of all negative integers, the ideal \( Q = \{ 6k : k \in \mathbb{Z}^- \} \) is a semiprime ideal. For \( x \in \mathbb{Z}^-, x^3 \in Q \iff x^3 \) is divisible by 6 \( \iff x \) is divisible by 6 \( \iff x = 6k_1 \) for \( k_1 \in \mathbb{Z}^- \iff x \in Q \).

**THEOREM 3.2.3**: An ideal A of a ternary semigroup T is completely semiprime if and only if \( x \in T, x^3 \in A \) implies \( x \in A \).

**Proof**: Suppose that A is a completely semiprime ideal of T.

Then clearly \( x \in T, x^3 \in A \Rightarrow x \in A \).

Conversely suppose that \( x \in T, x^3 \in A \Rightarrow x \in A \).

We prove that \( x \in T, x^n \in A \), for some odd natural number \( n > 1 \Rightarrow x \in A \rightarrow (1) \), by induction on \( n \). Clearly (1) is true for \( n = 3 \). Assume that (1) is true for \( n = k \), i.e., \( x^k \in A \Rightarrow x \in A \) for some odd natural number \( k > 3 \).

Suppose that \( x^{k+2} \in A \). Then \( x^{k+2} \in A \Rightarrow x^{k+2}x^{k+2} \in A \Rightarrow x^{3k} \in A \Rightarrow (x^3)^k \in A \Rightarrow x^k \in A \Rightarrow x \in A \). Therefore \( x^k \in A \Rightarrow x \in A \).
By induction, \( x^n \in A \) for some natural number \( n, n > 1 \) implies \( x \in A \).

Therefore \( A \) is completely semiprime.

**THEOREM 3.2.4**: If \( A \) is a completely semiprime ideal of a ternary semigroup \( T \), then \( x, y, z \in T, xyz \in A \) implies that \( xTTyz \subseteq A, xTTyz \subseteq A \) and \( xTyTz \subseteq A \).

**Proof**: Let \( A \) be a completely semiprime ideal of a semigroup \( T \). Let \( x, y, z \in T, xyz \in A \).

Now \( xyz \in A \Rightarrow (zxy)^3 = (zxy)(zxy)(zxy) = z(xyz)(xyz) xy \in A \).

\((zxy)^3 \in A, A \) is completely semiprime implies \( zxy \in A \).

Let \( s, t \in T \). Consider \((xyz)^3 = (xyz)(xyz)(xyz) = xyst(xy)st(xy)stxy \in A \).

\((xyz)^3 \in A, A \) is completely semiprime implies \( xyz \in A \).

Therefore \( x, y, z \in T, xyz \in A \Rightarrow xyz \in A \) for all \( s, t \in T \Rightarrow xTTyz \subseteq A \).

Now \( xyz \in A \Rightarrow (yxz)^3 = (yxz)(yxz)(yxz) = yxz(yxz)(xyz)z \in A \).

\((yxz)^3 \in A, A \) is completely semiprime implies \( yxz \in A \).

Let \( s, t \in T \). Consider \((xtyz)^3 = (xtyz)(xtyz)(xtyz) = xyst(zyx)st(zyx)styz \in A \).

\((xtyz)^3 \in A, A \) is completely semiprime implies \( xtyz \in A \).

Therefore \( x, y, z \in T, xtyz \in A \) for all \( s, t \in T \Rightarrow xTTyz \subseteq A \).

If \( s, t \in T \), then \((xtyz)^3 = (xtyz)(xtyz)(xtyz) = xyst[zxy(ty)(zxs)y]t \in A \).

\((xtyz)^3 \in A, A \) is completely semiprime implies \( xtyz \in A \).

Therefore \( x, y, z \in T, xtyz \in A \) for all \( s, t \in T \Rightarrow xTYTz \subseteq A \).

**COROLLARY 3.2.5**: If an ideal \( A \) of a ternary semigroup \( T \) is completely semiprime then \( x, y, z \in T, xyz \in A \Rightarrow < x > < y > < z > \subseteq A \).

**THEOREM 3.2.6**: Every completely prime ideal of a ternary semigroup \( T \) is a completely semiprime ideal of \( T \).

**Proof**: Let \( A \) be a completely prime ideal of a ternary semigroup \( T \). Suppose that \( x \in T \) and \( x^3 \in A \). Since \( A \) is a completely prime ideal of \( T \), \( x \in A \).

Therefore \( A \) is a completely semiprime ideal.

**THEOREM 3.2.7**: Let \( A \) be a prime ideal of a ternary semigroup \( T \). If \( A \) is completely semiprime ideal of \( T \) then \( A \) is completely prime.

**Proof**: Let \( x, y, z \in T \) and \( xyz \in A \). Since \( A \) is completely semiprime, by corollary 3.2.5., \( xyz \in A \Rightarrow < x > < y > < z > \subseteq A \Rightarrow x \in A \) or \( y \in A \) or \( z \in A \) and hence \( A \) is completely prime.
THEOREM 3.2.8: The nonempty intersection of any family of a completely prime ideal of a ternary semigroup $T$ is a completely semiprime ideal of $T$.

**Proof**: Let $\{A_\alpha\}_{\alpha \in \Delta}$ be a family of a completely prime ideals of $T$ such that $\bigcap_{\alpha \in \Delta} A_\alpha \neq \emptyset$.

It is clear that $\bigcap_{\alpha \in \Delta} A_\alpha$ is an ideal. Let $a \in T$ and $a^3 \in \bigcap_{\alpha \in \Delta} A_\alpha$. Then $a^3 \in A_\alpha$ for all $\alpha \in \Delta$.

Since $A_\alpha$ is completely prime, $a \in A_\alpha$ for all $\alpha \in \Delta$ and hence $a \in \bigcap_{\alpha \in \Delta} A_\alpha$.

Therefore $\bigcap_{\alpha \in \Delta} A_\alpha$ is a completely semiprime ideal of $T$.

We now introduce the notion of a $d$-system of a ternary semigroup.

**DEFINITION 3.2.9**: Let $T$ be a ternary semigroup. A non-empty subset $A$ of $T$ is said to be a $d$-**system** of $T$ if $a \in A \Rightarrow a^n \in A$ for all odd natural number $n$.

We now prove a necessary and sufficient condition for an ideal to be a completely semiprime ideal in a ternary semigroup.

**THEOREM 3.2.10**: An ideal $A$ of a ternary semigroup $T$ is completely semiprime if and only if $T \backslash A$ is a $d$-system of $T$ or empty.

**Proof**: Suppose that $A$ is a completely semiprime ideal of $T$ and $T \backslash A \neq \emptyset$.

Let $a \in T \backslash A$. Then $a \notin A$. Suppose if possible $a^n \notin T \backslash A$ for some odd natural number $n$.

Then $a^n \in A$. Since $A$ is a completely semiprime ideal then $a \in A$.

It is a contradiction. Therefore $a^n \in T \backslash A$ and hence $T \backslash A$ is a $d$-system.

Conversely suppose that $T \backslash A$ is a $d$-system of $T$ or $T \backslash A$ is empty.

If $T \backslash A$ is empty then $T = A$ and hence $A$ is completely semiprime.

Assume that $T \backslash A$ is a $d$-system of $T$. Let $a \in T$ and $a^n \in A$.

Suppose if possible $a \notin A$. Then $a \in T \backslash A$.

Since $T \backslash A$ is a $d$-system, $a^n \in T \backslash A$. It is a contradiction. Hence $a \in A$.

Thus $A$ is a completely semiprime ideal of $T$.

We now introduce the notion of a semiprime ideal of a ternary semigroup.

**DEFINITION 3.2.11**: An ideal $A$ of a ternary semigroup $T$ is said to be **semiprime ideal** provided $X$ is an ideal of $T$ and $X^n \subseteq A$ for some odd natural number $n$ implies $X \subseteq A$. 
THEOREM 3.2.12: An ideal A of a ternary semigroup T is semiprime if and only if X is an ideal of T, \( X^3 \subseteq A \) implies \( X \subseteq A \).

**Proof:** Suppose that A is a semiprime ideal. Then clearly \( X^3 \subseteq A \Rightarrow X \subseteq A \).

Conversely suppose that X is an ideal of T, \( X^3 \subseteq A \Rightarrow X \subseteq A \).

We prove that \( X^n \subseteq A \), for some odd natural number \( n \Rightarrow X \subseteq A \). Assume that \( X^k \subseteq A \) for some odd natural number, \( 1 \leq k < n \Rightarrow X \subseteq A \). Now \( X^k + 2 \subseteq A \Rightarrow X^k + 2 \cdot X^k - 4 \subseteq \Rightarrow \ldots \) by assumption. By induction \( X^n \subseteq A \) for some odd natural number \( n \Rightarrow X \subseteq A \).

Therefore A is semiprime.

THEOREM 3.2.13: Every prime ideal of a ternary semigroup T is semiprime.

**Proof:** Suppose that A is a prime ideal of a ternary semigroup T. Let X be an ideal of T such that \( X^3 \subseteq A \). Since A is prime, \( X \subseteq A \). Hence A is semiprime.

THEOREM 3.2.14: If A is an ideal of a ternary semigroup T then the following are equivalent.

1. A is a semiprime ideal.
2. For \( a \in T; < a >^3 \subseteq A \) implies \( a \in A \).
3. For \( a \in T; T^i T^j a T^k T^l a T^m T^n a T^o \subseteq A \) implies \( a \in A \).

**Proof:** (i) \( \Rightarrow \) (ii): Suppose that A is a semiprime ideal of T. Then (i) \( \Rightarrow \) (ii) is obvious.

(ii) \( \Rightarrow \) (iii): Let \( a \in T \) such that \( T^i T^j a T^k T^l a T^m T^n \subseteq A \).

Now \( < a >^3 = (T^i T^j a T^k T^l)( T^i T^j a T^k T^l)( T^i T^j a T^k T^l) \subseteq T^i T^j a T^k T^l a T^m T^n \subseteq A \) \( \Rightarrow a \in A \).

(iii) \( \Rightarrow \) (i): Suppose that \( a \in T; T^i T^j a T^k T^l a T^m T^n a T^o \subseteq A \Rightarrow a \in A \).

Let X be the an ideals of T and \( X^2 \subseteq A \).

Suppose if possible \( X \nsubseteq A \).

Let \( a \in X \) and \( a \not\in A \). \( a \in X \Rightarrow a^3 \in X^3 \subseteq A \).

Now \( T^i T^j a T^k T^l a T^m T^n a T^o \subseteq X^3 \subseteq A \Rightarrow a \in A \). It is a contradiction.

Therefore \( X \subseteq A \) and hence A is a semiprime ideal of T.

THEOREM 3.2.15: Every completely semiprime ideal of a ternary semigroup T is a semiprime ideal of T.
**Proof**: Suppose that $A$ is a completely semiprime ideal of a ternary semigroup $T$. Let $a \in T$ and $\langle a^n \rangle \subseteq A$ for some odd natural number $n$.

Now $aaa\ldots a(n$ odd terms$) \in \langle a^n \rangle \subseteq A \Rightarrow a^n \in A \Rightarrow a \in A \Rightarrow \langle a \rangle \subseteq A$.

Therefore $A$ is a semiprime ideal of $T$.

**THEOREM 3.2.16**: Let $T$ be a commutative ternary semigroup. An ideal $A$ of $T$ is completely semiprime if and only if it is semiprime.

**Proof**: Suppose that $A$ is a completely semiprime ideal of $T$. By theorem 3.2.15, $A$ is a semiprime ideal of $T$.

Conversely suppose that $A$ is a semiprime ideal of $T$. Let $x \in T$ and $x^n \in A$ for some odd natural number $n$.

Now $x^n \in A \Rightarrow \langle x^n \rangle \subseteq A \Rightarrow \langle x \rangle \subseteq A \Rightarrow x \in A$. Since $A$ is semiprime.

Therefore $A$ is a completely semiprime ideal of $T$.

**THEOREM 3.2.17**: The nonempty intersection of any family of prime ideals of a ternary semigroup $T$ is a semiprime ideal of $T$.

**Proof**: Let $\{A_{\alpha}\}_{\alpha \in \Delta}$ be a family of prime ideals of $T$ such that $\bigcap_{\alpha \in \Delta} A_{\alpha} \neq \emptyset$. It is clear that $\bigcap_{\alpha \in \Delta} A_{\alpha}$ is an ideal. Let $a \in T$, $\langle a \rangle^3 \subseteq \bigcap_{\alpha \in \Delta} A_{\alpha}$ then $\langle a \rangle^3 \subseteq A_{\alpha}$ for all $\alpha \in \Delta$.

Since $A_{\alpha}$ is a prime, $\langle a \rangle \subseteq A_{\alpha}$ for all $\alpha \in \Delta$. So $\langle a \rangle \subseteq \bigcap_{\alpha \in \Delta} A_{\alpha}$.

Therefore $\bigcap_{\alpha \in \Delta} A_{\alpha}$ is a semiprime ideal of $T$.

We now introduce the notion of an $n$-system of a ternary semigroup.

**DEFINITION 3.2.18**: A non-empty subset $A$ of a ternary semigroup $T$ is said to be an $n$-system provided for any $a \in A$ implies that $T^3T^1aT^1T^1aT^1T^1 \cap A \neq \emptyset$.

**THEOREM 3.2.19**: Every $m$-system in a ternary semigroup $T$ is an $n$-system.

**Proof**: Let $A$ be an $m$-system of a ternary semigroup $T$. Let $a \in A$. Since $A$ is an $m$-system, $a \in A$, $T^3T^1aT^1T^1aT^1T^1 \cap A \neq \emptyset$. Therefore $A$ is an $n$-system of $T$. 
We now prove a necessary and sufficient condition for an ideal to be a semiprime ideal in a ternary semigroup.

**THEOREM 3.2.20** : An ideal Q of a ternary semigroup T is a semiprime ideal if and only if T\(Q\) is an n-system of T (or) empty.

**Proof** : Suppose that A is a semiprime ideal of a ternary semigroup T and T\(\setminus\)A \(\neq\) \(\emptyset\).

Let \(a \in T\setminus A\). Then \(a \notin A\).

Suppose if possible \(T^{1}T^{1}aT^{1}T^{1}aT^{1}T^{1}T^{1} \cap T\setminus A = \emptyset\).

\(T^{1}T^{1}aT^{1}T^{1}aT^{1}T^{1}T^{1} \cap T\setminus A = \emptyset \Rightarrow T^{1}T^{1}aT^{1}T^{1}aT^{1}T^{1}T^{1} \subseteq A\).

Since A is semiprime, either \(a \in A\).

It is a contradiction. Therefore \(T^{1}T^{1}aT^{1}T^{1}aT^{1}T^{1}T^{1} \cap T\setminus A \neq \emptyset\).

Hence T\(\setminus\)A is an n-system.

Conversely suppose that T\(\setminus\)A is either an n-system or T\(\setminus\)A = \(\emptyset\).

If T\(\setminus\)A = \(\emptyset\) then T = A and hence A is a semiprime ideal.

Assume that T\(\setminus\)A is an n-system of T. Let \(a \in T\) and \(< a > \subseteq A\).

Let \(a \in T\setminus A\), T\(\setminus\)A is an n-system of T ⇒ \(T^{1}T^{1}aT^{1}T^{1}aT^{1}T^{1}T^{1} \cap T\setminus A \neq \emptyset\).

Suppose if possible \(a \notin A\). Then \(a \in T\setminus A\). Since T\(\setminus\)A is an m-system.

Then \(T^{1}T^{1}aT^{1}T^{1}aT^{1}T^{1}T^{1} \subseteq T\setminus A \Rightarrow T^{1}T^{1}aT^{1}T^{1}aT^{1}T^{1}T^{1} \notin A \Rightarrow < a > \notin A\).

It is a contradiction. Therefore \(a \in A\). Hence A is a semiprime ideal of T.

**THEOREM 3.2.21** : If N is an n-system in a ternary semigroup T and \(a \in N\), then there exist an m-system M in T such that \(a \in M\) and \(M \subseteq N\).

**Proof** : We construct a subset M of N as follows:

Define \(a_{i} = a\). Since \(a_{i} \in N\) and N is an n-system, \((T^{1}T^{1}a_{i}T^{1}T^{1}a_{i}T^{1}T^{1}a_{i}T^{1}T^{1}) \cap N \neq \emptyset\).

Let \(a_{j} \in (T^{1}T^{1}a_{i}T^{1}T^{1}a_{i}T^{1}T^{1}a_{i}T^{1}T^{1}) \cap N\). Since \(a_{j} \in N\) and N is an n-system, \((T^{1}T^{1}a_{j}T^{1}T^{1}a_{j}T^{1}T^{1}a_{j}T^{1}T^{1}) \cap N \neq \emptyset\) and so on.

In general, if \(a_{i}\) has been defined with \(a_{i} \in N\), choose \(a_{i+1}\) as an element of \((T^{1}T^{1}a_{i}T^{1}T^{1}a_{i}T^{1}T^{1}a_{i}T^{1}T^{1}) \cap N\). Let \(M = \{a_{i}, a_{i}, \ldots, a_{i}, a_{i}, \ldots\}\). Now \(a \in M\) and \(M \subseteq N\).

We now show that M is an m-system. Let \(a_{i}, a_{j}, a_{k} \in M\) (for \(i \leq j \leq k\)).

Then \(a_{k+1} \in T^{1}T^{1}a_{k}T^{1}T^{1}a_{k}T^{1}T^{1}a_{k}T^{1}T^{1} \subseteq T^{1}T^{1}a_{j}T^{1}T^{1}a_{j}T^{1}T^{1}a_{k}T^{1}T^{1}\)

\(\subseteq T^{1}T^{1}a_{j}T^{1}T^{1}a_{j}T^{1}T^{1}a_{k}T^{1}T^{1}\)

\(\Rightarrow a_{k+1} = T^{1}T^{1}a_{j}T^{1}T^{1}a_{k}T^{1}T^{1}a_{k}T^{1}T^{1}\). But \(a_{k+1} \in M\), so \(a_{k+1} \in T^{1}T^{1}a_{j}T^{1}T^{1}a_{j}T^{1}T^{1}a_{k}T^{1}T^{1} \cap M\), therefore M is an m-system.
3.3. PRIME RADICAL AND COMPLETELY PRIME RADICAL:

In this section, to each ideal \( A \) of a ternary semigroup \( T \), we associate four types of sets namely \( A_1, A_2, A_3, A_4 \) and we proved that \( A \subseteq A_1 \subseteq A_3 \subseteq A_2 \subseteq A_4 \). In a commutative ternary semigroup, it is proved that \( A_1 = A_2 = A_3 = A_4 \) and in general ternary semigroups, it is proved that \( A_i \neq A_j \neq A_k \neq A_l \) by means of examples. The terms ‘radical’ and ‘complete radical’ of an ideal in a ternary semigroup are also introduced and some of their properties are obtained. It is proved that in a ternary semigroup \( T \) if \( A, B \) and \( C \) are ideals of \( T \), then

i) \( A \subseteq B \Rightarrow \sqrt{A} \subseteq \sqrt{B} \),

ii) if \( A \cap B \cap C \neq \emptyset \) then

\[ \sqrt{ABC} = \sqrt{A \cap B \cap C} = \sqrt{A} \cap \sqrt{B} \cap \sqrt{C} \] and

iii) \( \sqrt{A} = \sqrt{A} \). In a ternary semigroup \( T \) if \( A \) is an ideal, then \( \sqrt{A} \) is a semiprime ideal of \( T \). It is proved that in a ternary semigroup \( T \) with identity there is a unique maximal ideal \( M \) such that \( \sqrt{M^n} = M \) for all odd natural numbers \( n \). It is proved that an ideal \( Q \) of a ternary semigroup \( T \) is semiprime iff \( \sqrt{Q} = Q \). Further it is proved that if \( A \) is a semiprime ideal of a ternary semigroup \( T \) and \( M \) is a maximal \( m \)-system of \( T \) such that \( A \cap M = \emptyset \), then \( \emptyset \) is a minimal prime ideal of \( T \) containing \( A \). It is also proved that every prime ideal \( P \) minimal relative to containing a completely semiprime ideal \( A \) of a ternary semigroup \( T \) is completely prime.

We use the following notation.

**NOTATION 3.3.1**: If \( A \) is an ideal of a ternary semigroup \( T \), then we associate the following four types of sets.

- \( A_1 = \) The intersection of all completely prime ideals of \( T \) containing \( A \).
- \( A_2 = \) \( \{ x \in T : x^n \in A \text{ for some odd natural numbers } n \} \)
- \( A_3 = \) The intersection of all prime ideals of \( T \) containing \( A \).
- \( A_4 = \) \( \{ x \in T : < x >^n \subseteq A \text{ for some odd natural number } n \} \)

**THEOREM 3.3.2**: If \( A \) is an ideal of a ternary semigroup \( T \), then

\( A \subseteq A_1 \subseteq A_3 \subseteq A_2 \subseteq A_4 \).

**Proof**: i) \( A \subseteq A_1 \) : Let \( x \in A \). Then \( < x > \subseteq A \) and hence \( x \in A_1 \)

Therefore \( A \subseteq A_1 \).

ii) \( A_4 \subseteq A_3 \) : Let \( x \in A_4 \). Then \( < x >^n \subseteq A \text{ for some odd natural number } n \).
Let P be any prime ideal of T containing A.
Then \( <x>^n \subseteq A \) for some odd natural number \( n \Rightarrow <x>^n \subseteq P \).

Since P is prime, \( <x> \subseteq P \) and hence \( x \in P \).

Since this is true for all prime ideals of P containing A, \( x \in A_3 \). Therefore \( A_4 \subseteq A_3 \).

iii) \( A_3 \subseteq A_2 \) : Let \( x \in A_3 \). Suppose if possible \( x \notin A_2 \).
Then \( x^n \notin A \) for all odd natural number \( n \).
Consider \( Q = \bigcup x^n \) for all odd natural number \( n \), and \( x \in T \).
Let \( a, b, c \in Q \).
Then \( a = (x)^r, b = (x)^s, c = (x)^t \) for some odd natural numbers \( r, s, t \).
Therefore \( abc = (x)^r (x)^s (x)^t = x^{r+s+t} \in Q \) and hence Q is a subsemigroup of T.

By theorem 3.1.5, \( P = T \setminus Q \) is a completely prime ideal of T and \( x \notin P \).
By theorem 3.1.9, P is a prime ideal of T and \( x \notin P \). Therefore \( x \notin A_3 \).
It is a contradiction. Therefore \( x \in A_2 \) and hence \( A_3 \subseteq A_2 \).

iv) \( A_2 \subseteq A_1 \) : Let \( x \in A_2 \). Now \( x \in A_2 \Rightarrow x^n \in A \) for some odd natural number \( n \).
Let P be any completely prime ideal of T containing A.
Then \( x^n \in A \subseteq P \Rightarrow x^n \in P \Rightarrow x \in P \). Therefore \( x \in A_1 \).
Hence \( A \subseteq A_4 \subseteq A_3 \subseteq A_2 \subseteq A_1 \).

THEOREM 3.3.3 : If A is an ideal of a commutative ternary semigroup T, then
\( A_1 = A_2 = A_3 = A_4 \)

Proof : By theorem 3.3.2, \( A \subseteq A_4 \subseteq A_3 \subseteq A_2 \subseteq A_1 \). By theorem 3.1.10, in a commutative ternary semigroup T, an ideal A is a prime ideal if A is completely prime ideal.
So \( A_4 = A_3 \). By theorem 3.2.16, in a commutative ternary semigroup T an ideal A is semiprime if and only if A is completely semiprime ideal.
So \( A_4 = A_2 \) and hence \( A_4 = A_2 = A_3 = A_1 \).

NOTE 3.3.4 : In an arbitrary ternary semigroup \( A \neq A_2 \neq A_3 \neq A_4 \).

EXAMPLE 3.3.5 : Let T be thess free ternary semigroup generated by \( a, b, c \).
It is clear that \( A = T a^3 T \) is an ideal of T. Since \( a^5 \in T a^3 T \), we have \( a \in A_2 \).
Evidently \( (abc)^n \notin T a^3 T \) for all odd natural numbers \( n \) and thus \( abc \notin A_2 \).
Thus \( A_2 \) is not an ideal of T. Therefore \( A_4 \neq A_2 \) and \( A_2 \neq A_1 \).
We now introduce prime radical and complete prime radical of an ideal in a ternary semigroup.

**DEFINITION 3.3.6**: If \( A \) is an ideal of a ternary semigroup \( T \), then the intersection of all prime ideals of \( T \) containing \( A \) is called **prime radical** or simply **radical** of \( A \) and it is denoted by \( \sqrt{A} \) or \( \text{rad } A \).

**DEFINITION 3.3.7**: If \( A \) is an ideal of a ternary semigroup \( T \), then the intersection of all completely prime ideals of \( T \) containing \( A \) is called **completely prime radical** or simply **complete radical** of \( A \) and it is denoted by \( \text{c.rad } A \).

**NOTE 3.3.8**: If \( A \) is an ideal of a ternary semigroup \( T \), then \( \text{rad } A = A \), \( \text{c.rad } A = A \) and \( \text{rad } A \subseteq \text{c.rad } A \).

**COROLLARY 3.3.9**: If \( a \in \sqrt{A} \), then there exist a positive integer \( n \) such that \( a^n \in A \) for some odd natural number \( n \in \mathbb{N} \).

**Proof**: By theorem 3.3.2, \( A_3 \subseteq A_2 \) and hence \( a \in \sqrt{A} = A_3 \subseteq A_2 \).

Therefore \( a^n \in A \) for some odd natural number \( n \in \mathbb{N} \).

**COROLLARY 3.3.10**: If \( A \) is an ideal of a commutative ternary semigroup \( T \), then \( \text{rad } A = \text{c.rad } A \).

**Proof**: By theorem 3.3.3, \( \text{rad } A = \text{c.rad } A \).

**COROLLARY 3.3.11**: If \( A \) is an ideal of a ternary semigroup \( T \) then \( \text{c.rad } A \) is a completely semiprime ideal of \( T \).

**Proof**: By theorem 3.2.6, \( \text{c.rad } A \) is a completely semiprime ideal of \( T \).

**THEOREM 3.3.12**: If \( A, B \) and \( C \) are any three ideals of a ternary semigroup \( T \), then

i) \( A \subseteq B \Rightarrow \sqrt{A} \subseteq \sqrt{B} \)

ii) if \( A \cap B \cap C \neq \emptyset \) then \( \sqrt{ABC} = \sqrt{A \cap B \cap C} = \sqrt{A} \cap \sqrt{B} \cap \sqrt{C} \)

iii) \( \sqrt{A} = \sqrt{A} \).

**Proof**: i) Suppose that \( A \subseteq B \). If \( P \) is a prime ideal containing \( B \) then \( P \) is a prime ideal containing \( A \). Therefore \( \sqrt{A} \subseteq \sqrt{B} \).
ii) Let $P$ be a prime ideal containing $ABC$. Then $ABC \subseteq P \Rightarrow A \subseteq P$ or $B \subseteq P$ or $C \subseteq P \Rightarrow A \cap B \cap C \subseteq P$. Therefore $P$ is a prime ideal containing $A \cap B \cap C$.

Therefore $\text{rad}(A \cap B \cap C) \subseteq \text{rad}(ABC)$.

Now let $P$ be a prime ideal containing $A \cap B \cap C$.

Then $A \cap B \cap C \subseteq P \Rightarrow ABC \subseteq A \cap B \cap C \subseteq P \Rightarrow ABC \subseteq P$.

Hence $P$ is a prime ideal containing $ABC$. Therefore $\text{rad}(ABC) \subseteq \text{rad}(A \cap B \cap C)$.

Therefore $\text{rad}(ABC) = \text{rad}(A \cap B \cap C)$.

Since $A \cap B \cap C \neq \emptyset$, it is clear that $A \cap B \cap C$ is an ideal in $T$. Let $x \in \sqrt{A \cap B \cap C}$.

Then there exists an odd natural number $n \in \mathbb{N}$ such that $x^n \in A \cap B \cap C$.

Therefore $x^n \in A$, $x^n \in B$ and $x^n \in C$. It follows that $x \in \sqrt{A}$, $x \in \sqrt{B}$ and $x \in \sqrt{C}$.

Therefore $x \in \sqrt{A} \cap \sqrt{B} \cap \sqrt{C}$.

Consequently, $x \in \sqrt{A} \cap \sqrt{B} \cap \sqrt{C}$ implies that there exists odd natural numbers $n, m, p \in \mathbb{N}$ such that $x^n \in A$, $x^m \in B$ and $x^p \in C$. Clearly, $x^{nmp} \in A \cap B \cap C$.

Thus $x \in \sqrt{A \cap B \cap C}$. Therefore if $A \cap B \cap C \neq \emptyset$ then $\sqrt{A \cap B \cap C} = \sqrt{A} \cap \sqrt{B} \cap \sqrt{C}$.

iii) $\sqrt{A} = \text{The intersection of all prime ideals of } T \text{ containing } A$.

Now $\sqrt{\sqrt{A}} = \text{The intersection of all prime ideals of } T \text{ containing } \sqrt{A}$.

= $\text{The intersection of all prime ideals of } T \text{ containing } A = \sqrt{A}$.

Therefore $\sqrt{\sqrt{A}} = \sqrt{A}$.

**THEOREM 3.3.13** : If $A$ is an ideal of a ternary semigroup $T$ then $\sqrt{A}$ is a semiprime ideal of $T$.

**proof** : By theorem 3.2.17, $\sqrt{A}$ is a semiprime ideal of $T$.

**THEOREM 3.3.14** : An ideal $Q$ of ternary semigroup $T$ is a semiprime ideal of $T$ if and only if $\sqrt{Q} = Q$.

**Proof** : Suppose that $Q$ is a semiprime ideal. Clearly $Q \subseteq \sqrt{Q}$.

Suppose if possible $\sqrt{Q} \not\subseteq Q$.

Let $a \in \sqrt{Q}$ and $a \notin Q$. Now $a \notin Q \Rightarrow a \in T \setminus Q$ and $Q$ is semiprime. By theorem 3.2.20, $T \setminus Q$ is an $n$-system. By theorem 3.2.21, there exists an $m$-system $M$ such that $a \in M \subseteq T \setminus Q$. 

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Q \subseteq T \setminus M and now \( T \setminus M \) is a prime ideal of \( T \), \( a \not\in T \setminus M \). It is a contradiction.

Therefore \( \sqrt{Q} \subseteq Q \). Hence \( \sqrt{Q} = Q \).

Conversely suppose that \( Q \) is an ideal of \( T \) such that \( \sqrt{Q} = Q \).

By corollary 3.3.13, \( \sqrt{Q} \) is a semiprime ideal of \( T \). Therefore \( Q \) is semiprime.

**COROLLARY 3.3.15:** An ideal \( Q \) of a ternary semigroup \( T \) is a semiprime ideal if and only if \( Q \) is the intersection of all prime ideal of \( T \) contains \( Q \).

**Proof:** By theorem 3.3.14., \( Q \) is semiprime iff \( Q \) is the intersection of all prime ideals of \( T \) contains \( Q \).

**COROLLARY 3.3.16:** If \( A \) is an ideal of a ternary semigroup \( T \), then \( \sqrt{A} \) is the smallest semiprime ideal of \( T \) containing \( A \).

**Proof:** We have that \( \sqrt{A} \) is the intersection of all prime ideals containing \( A \) in \( T \).

Since intersection of prime ideals is semiprime, we have \( \sqrt{A} \) is semiprime.

Further, let \( Q \) be any semiprime ideal containing \( A \), i.e. \( A \subseteq Q \). So \( \sqrt{A} \subseteq \sqrt{Q} \).

Since \( Q \) is semiprime, By theorem 3.3.14, \( \sqrt{Q} = Q \). Therefore \( \sqrt{A} \subseteq Q \).

Hence \( \sqrt{A} \) is the smallest semiprime ideal of \( T \) containing \( A \).

**THEOREM 3.3.16:** If \( P \) is a prime ideal of a ternary semigroup \( T \), then \( \sqrt{(P)^n} = P \) for all odd natural numbers \( n \in \mathbb{N} \).

**Proof:** We use induction on \( n \) to prove \( \sqrt{P^n} = P \).

First we prove that \( \sqrt{P} = P \). Since \( P \) is a prime ideal, \( P \subseteq \sqrt{P} \subseteq P \Rightarrow \sqrt{P} = P \).

Assume that \( \sqrt{P^k} = P \) for odd natural number \( k \) such that \( 1 \leq k < n \).

Now \( \sqrt{P^{k+2}} = \sqrt{P^k \cdot P \cdot P} = \sqrt{P^k} \cap \sqrt{P} \cap \sqrt{P} = \sqrt{P} \cap \sqrt{P} \cap \sqrt{P} = \sqrt{P} = P \).

Therefore \( \sqrt{P^{k+2}} = P \). By induction \( \sqrt{P^n} = P \) for all odd natural number \( n \in \mathbb{N} \).

**THEOREM 3.3.17:** In a ternary semigroup \( T \) with identity there is a unique maximal ideal \( M \) such that \( \sqrt{(M)^n} = M \) for all odd natural numbers \( n \in \mathbb{N} \).

**Proof:** Since \( T \) contains identity, \( T \) is a globally idempotent ternary semigroup.

Since \( M \) is a maximal ideal of \( T \), by theorem 3.1.13 \( M \) is prime.

By theorem 3.3.16, \( \sqrt{(M)^n} = M \) for all odd natural numbers \( n \).
Theorem 3.3.18: If A is an ideal of a ternary semigroup T then \( \sqrt{A} = \{ x \in T : \text{every m-system of T containing } x \text{ meets } A \} \) i.e., \( \sqrt{A} = \{ x \in T : M(x) \cap A \neq \emptyset \} \).

**Proof:** Suppose that \( x \in \sqrt{A} \). Let M be an m-system containing x.
Then \( T \setminus M \) is a prime ideal of T and \( x \notin T \setminus M \). If \( M \cap A = \emptyset \) then \( A \subseteq T \setminus M \).
Since \( T \setminus M \) is a prime ideal containing \( A \), \( \sqrt{A} \subseteq T \setminus M \) and hence \( x \in T \setminus M \).
It is a contradiction. Therefore \( M(x) \cap A \neq \emptyset \). Hence \( x \in \{ x \in T : M(x) \cap A \neq \emptyset \} \).
Conversely suppose that \( x \in \{ x \in T : M(x) \cap A \neq \emptyset \} \).
Suppose if possible \( x \notin \sqrt{A} \). Then there exists a prime ideal P containing \( A \) such that \( x \notin P \). Now \( T \setminus P \) is an m-system and \( x \in T \setminus P \).
\( A \subseteq P \Rightarrow T \setminus P \cap A = \emptyset \Rightarrow x \notin \{ x \in T : M(x) \cap A \neq \emptyset \} \).
It is a contradiction. Therefore \( x \in \sqrt{A} \). Thus \( \sqrt{A} = \{ x \in T : M(x) \cap A \neq \emptyset \} \).