Chapter 4

Subclasses of Starlike Functions Associated with Caputo’s Fractional Operator

4.1 INTRODUCTION

Recently, Kiryakova (1994) discussed the Generalised Fractional Calculus (GFC) with various applications to the special functions and integral transforms to the hyper Bessel operators, ordinary differential equations dual and Volterra integral equations, univalent functions etc. All the classical Fractional Calculus (FC) operators (see Samko et al. 1993) and most of their generalizations by different authors fall in the Generalised Fractional Calculus (GFC). The classical definition of fractional calculus and their applications in the theory of analytic functions have been applied in defining a new subclass of analytic functions with negative coefficients and discuss some interesting properties of the generalised function classes.

Let $A$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$  \hspace{1cm} (4.1)

which are analytic and univalent in the open disc $U = \{z : z \in \mathbb{C}, |z| < 1\}$.

Also let $T$ be a subclass of $A$ consisting of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad a_n \geq 0, \quad z \in U,$$  \hspace{1cm} (4.2)
We recall the following definitions due to Owa (2008).

**Definition 4.1.1.** Let the function $f(z)$ be analytic in a simply-connected region of the $z-$plane containing the origin. The fractional integral of $f$ of order $\mu$ is defined by

$$D_{z}^{-\mu}f(z) = \frac{1}{\Gamma(\mu)} \int_{0}^{z} \frac{f(\xi)}{(z-\xi)^{1-\mu}} d\xi, \quad \mu > 0,$$

(4.3)

where the multiplicity of $(z-\xi)^{1-\mu}$ is removed by requiring $\log(z-\xi)$ to be real when $z-\xi > 0$.

**Definition 4.1.2.** The fractional derivatives of order $\mu$, is defined for a function $f(z)$, by

$$D_{z}^{\mu}f(z) = \frac{1}{\Gamma(1-\mu)} \frac{d}{dz} \int_{0}^{z} \frac{f(\xi)}{(z-\xi)^{\mu}} d\xi, \quad 0 \leq \mu < 1,$$

(4.4)

where the function $f(z)$ is constrained, and the multiplicity of the function $(z-\xi)^{-\mu}$ is removed as in Definition 4.1.1.

**Definition 4.1.3.** Under the hypothesis of Definition 4.1.2, the fractional derivative of order $n + \mu$ is defined by

$$D_{z}^{n+\mu}f(z) = \frac{d^{n}}{dz^{n}} D_{z}^{\mu}f(z), \quad (0 \leq \mu < 1 ; \ n \in \mathbb{N}_0).$$

(4.5)

With the aid of the above definitions and their known extensions involving fractional derivative and fractional integrals, the generalization of Salagean (1983) derivative operator and Libera integral operator (1969) was given by Owa (2008). Srivastava and owa (1987) introduced the operator

$$\Omega^{\delta} : \mathcal{A} \to \mathcal{A}$$
defined by,

\[ \Omega^\delta f(z) = \Gamma(2 - \delta)z^\delta D_z^\delta f(z) = z + \sum_{n=2}^{\infty} \Phi(n, \delta)a_n z^n, \quad (4.6) \]

where

\[ \Phi(n, \delta) = \frac{\Gamma(n+1)\Gamma(2 - \delta)}{\Gamma(n + 1 - \delta)} \text{ and } \delta \in \mathbb{R}, \delta \neq 2, 3, 4, \ldots \quad (4.7) \]

For \( f \in \mathcal{A} \) and various choices of \( \delta \), we get

\[ \Omega^0 f(z) := f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (4.8) \]

\[ \Omega^1 f(z) := zf'(z) = z + \sum_{n=2}^{\infty} na_n z^n, \quad (4.9) \]

\[ \Omega^j f(z) := \Omega(\Omega^{j-1} f(z)) = z + \sum_{n=2}^{\infty} n^j a_n z^n, (j = 1, 2, 3, \ldots), \quad (4.10) \]

which is known as Salagean operator (Salagean, 1983). Also note that

\[ \Omega^{-1} f(z) = \frac{2}{z} \int_0^z f(t) dt := z + \sum_{n=2}^{\infty} \left( \frac{2}{n + 1} \right) a_n z^n \]

and

\[ \Omega^{-j} f(z) := \Omega^{-1}(\Omega^{-j+1} f(z)) := z + \sum_{n=2}^{\infty} \left( \frac{2}{n + 1} \right)^j a_n z^n, (j = 1, 2, 3, \ldots) \quad (4.11) \]

called Libera integral operator (1969). We note that the Libera integral operator is generalized as Bernardi integral operator given by Bernardi (1969) is

\[ \frac{1 + \nu}{z^\nu} \int_0^z t^{\nu-1} f(t) dt := z + \sum_{n=2}^{\infty} \left( \frac{1 + \nu}{n + 1} \right) a_n z^n, (\nu = 1, 2, 3, \ldots). \]

Further we recall the definition of the fractional-order derivative due to Caputo’s (1967) given by

\[ D^\alpha f(t) = \frac{1}{\Gamma(n - \alpha)} \int_a^t \frac{f^{(n)}(\tau)}{(t - \tau)^{\alpha+1-n}} d\tau, \quad (4.12) \]

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where \( n - 1 < \mathcal{R}(\alpha) \leq n, \) \( n \in \mathbb{N} \) and the parameter \( \alpha \) is allowed to be real or even complex, \( a \) is the initial value of the function \( f \).

Making use of these results and the definition of Caputo’s fractional derivative (1967), recently Salah and Darus in (2010), introduced the following operator

\[
\mathbf{J}_\mu^\alpha f(z) = \frac{\Gamma(2 + \eta - \mu)}{\Gamma(\eta - \mu)} z^{\mu - \eta} \int_0^z \frac{\Omega^\eta f(t)}{(z - t)^{\mu + 1 - \eta}} dt, \tag{4.13}
\]

where \( \eta \) (real number) and \( (\eta - 1 < \mu < \eta < 2) \). That is

\[
\mathbf{J}_\mu^\alpha f(z) = z + \sum_{n=2}^{\infty} \frac{(\Gamma(n + 1))^2 \Gamma(2 + \eta - \mu) \Gamma(2 - \eta)}{\Gamma(n + \eta - \mu + 1) \Gamma(n - \eta + 1)} a_n z^n, \quad (z \in \mathbb{U}). \tag{4.14}
\]

We note that

\[
\mathbf{J}_0^0 f(z) = f(z) \quad \text{and} \quad \mathbf{J}_1^1 f(z) = zf'(z).
\]

Now, we introduce a new subclass of analytic functions with negative coefficients involving the operator \( \mathbf{J}_\mu^\alpha \) given by (4.14) and discuss some usual properties of the geometric function theory.

**Definition 4.1.4.** For fixed \(-1 \leq A \leq B \leq 1\) and \(0 < B \leq 1\), let \( \mathcal{S}J_\mu^\alpha(\alpha, \beta, \gamma, A, B) \) denote the subclass of \( \mathcal{A} \) consisting of functions \( f \) of the form (4.1) and satisfying the condition

\[
\left| \frac{z(\mathbf{J}_\mu^\alpha f(z))'}{\mathbf{J}_\mu^\alpha f(z)} - 1 \right| < \beta, \quad z \in \mathbb{U}, \tag{4.15}
\]

where \( \mathbf{J}_\mu^\alpha f(z) \) is given by (4.14) and

\[
0 \leq \alpha < 1, \quad 0 < \beta \leq 1, \quad \frac{B}{2(B - A)} < \gamma \leq \begin{cases} 
\frac{B}{2(B - A)\alpha} & \alpha \neq 0, \\
1 & \alpha = 0.
\end{cases}
\]

We also let \( \mathcal{T}J_\mu^\alpha(\alpha, \beta, \gamma, A, B) = \mathcal{S}J_\mu^\alpha(\alpha, \beta, \gamma, A, B) \cap \mathcal{T} \), where \( \mathcal{T} \) is given by (4.2)
Remark 4.1.1. By suitably specializing the values of $A, B, \alpha, \beta$ and $\gamma$ the class $TJ^n_{\mu}(\alpha, \beta, \gamma, A, B)$ leads to known subclasses studied in Aghalary and Kullkarni (2002), Khairanar and Meena (2008), Owa and Nishiwaki (2002) and various new subclasses.

For convenience in our study we consider $0 \leq \alpha < 1$, $0 < \beta \leq 1$,

$$\frac{B}{2(B - A)} < \gamma \leq \begin{cases} \frac{B}{2(B - A)\alpha} & \alpha \neq 0, \\ 1 & \alpha = 0, \end{cases}$$

for fixed $-1 \leq A \leq B \leq 1$ and $0 < B \leq 1$. Further

$$C_n(\eta, \mu) = \frac{(\Gamma(n + 1))^2\Gamma(2 + \eta - \mu)\Gamma(2 - \eta)}{\Gamma(n + \eta - \mu + 1)\Gamma(n - \eta + 1)} \quad (4.16)$$

and

$$C_2(\eta, \mu) = \frac{4\Gamma(2 + \eta - \mu)\Gamma(2 - \eta)}{\Gamma(3 + \eta - \mu)\Gamma(3 - \eta)} \quad (4.17)$$

In the following section we obtain the characterization property for functions $f \in TJ^n_{\mu}(\alpha, \beta, \gamma, A, B)$.

4.2 CHARACTERIZATION PROPERTIES

Theorem 4.2.1. Let the function $f$ be defined by (1.6) is in the class $TJ^n_{\mu}(\alpha, \beta, \gamma, A, B)$ if and only if

$$\sum_{n=2}^{\infty} [2\beta\gamma(B - A)(n - \alpha) + (1 - B\beta)(n - 1)]C_n(\eta, \mu)|a_n| \leq 2\beta\gamma(1 - \alpha)(B - A), \quad (4.18)$$

where $C_n(\eta, \mu)$ is given by (4.16).
Proof. Suppose that $f$ satisfies (4.18), then
\[
\left| \frac{z(J^\mu f(z))'}{J^\mu f(z)} - 1 \right| < \beta \left| 2\gamma(B - A)(\frac{z(J^\mu f(z))'}{J^\mu f(z)} - \alpha) - B(\frac{z(J^\mu f(z))'}{J^\mu f(z)} - 1) \right|,
\]
that is
\[
\left| z(J^\mu f(z))' - J^\mu f(z) \right| < \beta \left| 2\gamma(B - A)(z(J^\mu f(z))' - \alpha) - B(z(J^\mu f(z))' - J^\mu f(z)) \right|.
\]
Substituting for $z(J^\mu f(z))'$ and $J^\mu f(z)$ in the above condition, we have
\[
\left| z(J^\mu f(z))' - J^\mu f(z) \right| < \beta \left| 2\gamma(B - A)(1 - \alpha)z + \sum_{n=2}^{\infty} 2\gamma(B - A)(n - \alpha)C_n(\eta, \mu)a_n z^n \right|
\]
\[
+ \sum_{n=2}^{\infty} (n - 1)BC_n(\eta, \mu)a_n z^n \right|,
\]
then for $|z| = r$ and $r \to 1^{-1}$ we have
\[
\sum_{n=2}^{\infty} (n - 1)C_n(\eta, \mu)a_n \leq 2\beta\gamma(B - A)(n - \alpha)
\]
\[
- \sum_{n=2}^{\infty} 2\gamma\beta(B - A)(n - \alpha)C_n(\eta, \mu)a_n + \sum_{n=2}^{\infty} (n - 1)B\beta C_n(\eta, \mu)a_n. \quad (4.19)
\]
That is,
\[
\sum_{n=2}^{\infty} [2\beta\gamma(B - A)(n - \alpha) + (1 - B\beta)(n - 1)]C_n(\eta, \mu)|a_n| \leq 2\beta\gamma(1 - \alpha)(B - A). \quad (4.20)
\]
Hence by given condition of the Theorem 4.2.1, $f \in \mathcal{J}_\mu^n(\alpha, \beta, \eta, A, B)$.

To prove the converse part, assume that
\[
\left| \frac{z(J^\mu f(z))'}{J^\mu f(z)} - 1 \right| < \beta \left| 2\gamma(B - A)(\frac{z(J^\mu f(z))'}{J^\mu f(z)} - \alpha) - B(\frac{z(J^\mu f(z))'}{J^\mu f(z)} - 1) \right| < \beta.
\]
That is
\[
\left| \sum_{n=2}^{\infty} (n - 1)C_n(\eta, \mu)a_n z^{n-1} \right| \leq 1, \quad (4.21)
\]

equivalently
\[ \Re \left( \frac{\sum_{n=2}^{\infty} (n-1)C_n(\eta, \mu)a_n z^{n-1}}{2\beta \gamma (1-\alpha) + \sum_{n=2}^{\infty} \left(2\beta \gamma (B-A)(n-\alpha) - (n-1)B\beta\right) C_n(\eta, \mu)a_n z^{n-1}} \right) < 1. \]

(4.22)

Since \( \Re(z) \leq |z| \) for all \( z \) choose values of \( z \) on the real axis so that \( T J_{\mu}(\alpha, \beta, \gamma, A, B) \) is real. Upon clearing the denominator in (4.22) and letting \( |z| = r, \ 0 < r < 1 \) and letting \( r \to 1^{-1} \)

we have
\[ \left| \sum_{n=2}^{\infty} (n-1)C_n(\eta, \mu)a_n z^{n-1} \right| \leq 2\beta \gamma (1-\alpha) + \sum_{n=2}^{\infty} \left(2\beta \gamma (B-A)(n-\alpha) - (n-1)B\beta\right) C_n(\eta, \mu)a_n z^{n-1}, \]

that is
\[ \sum_{n=2}^{\infty} \left[2\beta \gamma (B-A)(n-\alpha) + (1 - B\beta)(n-1)\right] C_n(\eta, \mu)|a_n| \leq 2\beta \gamma (1-\alpha)(B-A), \]

obtain the desired assertion (4.18). \( \square \)

**Corollary 4.2.1.** Let the function \( f \) defined by (4.2) be in the class \( T J_{\mu}(\alpha, \beta, \gamma, A, B) \), then we have
\[ |a_n| \leq \frac{2\beta \gamma (1-\alpha)(B-A)}{2\beta \gamma (B-A)(n-\alpha) + (1 - B\beta)(n-1) C_n(\eta, \mu)}, \]

(4.23)

the equation (4.23) is attained for the function
\[ f(z) = z - \frac{2\beta \gamma (1-\alpha)(B-A)}{2\beta \gamma (B-A)(n-\alpha) + (1 - B\beta)(n-1) C_n(\eta, \mu)} z^n \quad (n \geq 2), \]

(4.24)

where \( C_n(\eta, \mu) \) is given by (4.16).

For the sake of brevity, we let
\[ \Phi_n(\alpha, \beta, \gamma, A, B) = 2\beta \gamma (B-A)(n-\alpha) + (1 - B\beta)(n-1) \]

(4.25)
and
\[
\Phi_2(\alpha, \beta, \gamma, A, B) = 1 + 2\beta\gamma(B - A)(2 - \alpha) - B\beta
\]  \hspace{1cm} (4.26)

unless otherwise stated.

In the following theorem we state the distorsion bounds results for functions \( f \) defined by (4.2) belong to \( TJ_\mu^\eta(\alpha, \beta, \gamma, A, B) \).

**Theorem 4.2.2.** Let the function \( f \) defined by (4.2) belong to \( TJ_\mu^\eta(\alpha, \beta, \gamma, A, B) \), then
\[
|f(z)| \geq |z| \left\{ 1 - \frac{2\beta\gamma(1 - \alpha)(B - A)}{\Phi_2(\alpha, \beta, \gamma, A, B)C_2(\eta, \mu)} |z| \right\} \hspace{1cm} (4.27)
\]
and
\[
|f(z)| \leq |z| \left\{ 1 + \frac{2\beta\gamma(1 - \alpha)(B - A)}{\Phi_2(\alpha, \beta, \gamma, A, B)C_2(\eta, \mu)} |z| \right\}, \hspace{1cm} (4.28)
\]
where \( C_2(\eta, \mu) \) given by (4.17).

**Proof.** In the view of (4.18) and the fact that \( C_n(\eta, \mu) \) is non-decreasing for \( n \geq 2, 0 \leq \alpha < 1 \) we have
\[
\left[ 2\beta\gamma(B - A)(2 - \alpha) + (1 - B\beta) \right] C_2(\eta, \mu) \sum_{n=2}^\infty a_n \\
\leq \sum_{n=2}^\infty \Phi_n(\alpha, \beta, \gamma, A, B)C_n(\eta, \mu)a_n \\
\leq 2\beta\gamma(1 - \alpha)(B - A),
\]
which readily yields,
\[
\sum_{n=2}^\infty a_n \leq \frac{2\beta\gamma(1 - \alpha)(B - A)}{\Phi_2(\alpha, \beta, \gamma, A, B)C_2(\eta, \mu)}. \hspace{1cm} (4.29)
\]

Theorem 4.2.2 follows readily from (4.2) and (4.29). \( \Box \)

**Theorem 4.2.3.** (Extreme Points) Let
\[
f_1(z) = z
\]
and
\[ f_n(z) = z - \frac{2\beta\gamma(1 - \alpha)(B - A)}{\Phi_n(\alpha, \beta, \gamma, A, B)C_n(\eta, \mu)} z^n, \quad (n \geq 2) \]

where \( C_n(\eta, \mu) \) is given by (4.16). Then \( f \in T^\eta_\mu(\alpha, \beta, \gamma, A, B) \) if and only if it can be expressed in the form
\[
f(z) = \sum_{n=1}^{\infty} \omega_n f_n(z), \quad (4.30)
\]
where \( \omega_n \geq 0 \) \((n \geq 1)\) and \( \sum_{n=1}^{\infty} \omega_n = 1 \).

**Proof.** Suppose \( f \) can be written as in equation (4.30), then
\[
f_n(z) = z - \frac{2\beta\gamma(1 - \alpha)(B - A)}{\Phi_n(\alpha, \beta, \gamma, A, B)C_n(\eta, \mu)} z^n,
\]

now
\[
\sum_{n=2}^{\infty} \Phi_n(\alpha, \beta, \gamma, A, B)C_n(\eta, \mu) \frac{2\beta\gamma(1 - \alpha)(B - A)}{2\beta\gamma(1 - \alpha)(B - A)\omega_n} \omega_n = \sum_{n=2}^{\infty} \omega_n = 1 - \omega_1 \leq 1.
\]

Thus \( f \in T^\eta_\mu(\alpha, \beta, \gamma, A, B) \).

Conversely, let \( f \in T^\eta_\mu(\alpha, \beta, \gamma, A, B) \) then by using (4.23), we set
\[
\omega_n = \frac{\Phi_n(\alpha, \beta, \gamma, A, B)C_n(\eta, \mu)}{2\beta\gamma(1 - \alpha)(B - A)} a_n, \quad n \geq 2
\]
and
\[
\omega_1 = 1 - \omega_n.
\]

Then we have
\[
f(z) = \sum_{n=2}^{\infty} \omega_n f_n(z),
\]
hence this completes the proof of Theorem 4.2.3.

Next we prove the following closure theorem for \( f \in \mathcal{T}_J^\mu(\alpha, \beta, \gamma, A, B) \).

Let the functions \( f_j(z) (j \in \mathbb{N}) \) be defined by

\[
f_j(z) = z - \sum_{n=2}^{\infty} a_{n,j} z^n \quad \text{for} \quad a_{n,j} \geq 0, \quad z \in \mathbb{U}.
\] (4.32)

**Theorem 4.2.4.** (Closure Theorem) Let the functions \( f_j(j = 1, 2, \ldots m) \) defined by (4.32) be in the classes \( \mathcal{T}_J^\mu(\alpha_j, \beta, \gamma, A, B) \) \((j = 1, 2, \ldots m)\) respectively. Then the function \( h(z) \) defined by

\[
h(z) = z - \frac{1}{m} \sum_{n=2}^{\infty} \left( \sum_{j=1}^{m} a_{n,j} \right) z^n
\] (4.33)

is in the class \( \mathcal{T}_J^\mu(\alpha, \beta, \gamma, A, B) \), where \( \alpha = \min_{1 \leq j \leq m} \{ \alpha_j \}, \ 0 \leq \alpha_j \leq 1 \).

**Proof.** Let the function \( h \) be defined by (4.33). Since \( f_j \in \mathcal{T}_J^\mu(\alpha_j, \beta, \gamma, A, B) \), \((j = 1, 2, \ldots m)\) by applying Theorem 4.2.1, we observe that

\[
\sum_{n=2}^{\infty} \Phi_n(\alpha, \beta, \gamma, A, B) C_n(\eta, \mu) \left( \frac{1}{m} \sum_{j=1}^{m} a_{n,j} \right) \\
= \frac{1}{m} \sum_{j=1}^{m} \left( \sum_{n=2}^{\infty} \Phi_n(\alpha, \beta, \gamma, A, B) C_n(\eta, \mu) a_{n,j} \right) \\
\leq \frac{1}{m} \sum_{j=1}^{m} 2\beta\gamma(1 - \alpha_j)(B - A) \leq 2\beta\gamma(1 - \alpha)(B - A).
\]

By Theorem 4.2.1, we have \( h \in \mathcal{T}_J^\mu(\alpha, \beta, \gamma, A, B) \).

Next we obtain the radii of close-to-convexity, starlikeness and convexity for the class \( \mathcal{T}_J^\mu(\alpha, \beta, \gamma, A, B) \).
Theorem 4.2.5. Let the function $f$ defined by (4.2) belong to the class $\mathcal{T}_J^\mu(\alpha, \beta, \gamma, A, B)$. Then $f$ is close-to-convex of order $\sigma$ ($0 \leq \sigma < 1$) in the disc $|z| < r_1$, where

$$r_1 := \inf \left[ \frac{(1 - \sigma)\Phi_n(\alpha, \beta, \gamma, A, B)C_n(\eta, \mu)}{2n\beta\gamma(B - A)(1 - \alpha)} \right]^{\frac{1}{1-\sigma}} (n \geq 2), \tag{4.34}$$

where $C_n(\eta, \mu)$ is given by (4.16). The result is sharp, with extremal function $f$ given by (4.24).

Proof. Given $f \in \mathcal{T}$ and $f$ is close-to-convex of order $\sigma$, we have

$$|f'(z) - 1| < 1 - \sigma.$$

That is

$$|f'(z) - 1| \leq \sum_{n=2}^{\infty} na_n|z|^{n-1}.$$

The last expression is less than $1 - \sigma$ if

$$\sum_{n=2}^{\infty} \frac{n}{1 - \sigma} |a_n||z|^{n-1} < 1.$$

Since $f \in \mathcal{T}_J^\mu(\alpha, \beta, \gamma, A, B)$. By Theorem 4.2.1, we have

$$\sum_{n=2}^{\infty} \frac{\Phi_n(\alpha, \beta, \gamma, A, B)C_n(\eta, \mu)}{2\beta\gamma(B - A)(1 - \alpha)} |a_n| \leq 1.$$

Hence by solving for $|z|$, we get

$$\frac{n}{1 - \sigma} |z|^{n-1} \leq \frac{\Phi_n(\alpha, \beta, \gamma, A, B)C_n(\eta, \mu)}{2\beta\gamma(B - A)(1 - \alpha)}.$$

The last inequality lead us immediately to the disk $|z| < r_1$ where $r_1$ given by (4.34), which completes the proof.

Theorem 4.2.6. Let $f \in \mathcal{T}_J^\mu(\alpha, \beta, \gamma, A, B)$. Then
(i) \( f \) is starlike of order \( \sigma (0 \leq \sigma < 1) \) in the disc \( |z| < r_2 \); that is, \( \Re \left( \frac{zf'(z)}{f(z)} \right) > \sigma \), where

\[
r_2 = \inf \left[ \frac{1 - \sigma}{n - \sigma} \frac{\Phi_n(\alpha, \beta, \gamma, A, B)C_n(\eta, \mu)}{2\beta\gamma(B - A)(1 - \alpha)} \right]^{\frac{1}{n-1}} (n \geq 2)
\]

(4.35)

and

(ii) \( f \) is convex of order \( \sigma (0 \leq \sigma < 1) \) in the disc \( |z| < r_3 \), that is \( \Re \left( 1 + \frac{zf''(z)}{f'(z)} \right) > \sigma \), where

\[
r_3 = \inf \left[ \frac{1 - \sigma}{n(n - \sigma)} \frac{\Phi_n(\alpha, \beta, \gamma, A, B)C_n(\eta, \mu)}{2\beta\gamma(B - A)(1 - \alpha)} \right]^{\frac{1}{n-1}} (n \geq 2),
\]

(4.36)

where \( C_n(\eta, \mu) \) is given by (4.16). Each of these results are sharp for the extremal function \( f \) given by (4.24).

**Proof.** Using the techniques of Srivatava and Aouf (1992, 1995), we can easily prove (i).

Given \( f \in \mathcal{T} \) and \( f \) is starlike of order \( \sigma \), we have

\[
|zf'(z) - 1| \leq 1 - \sigma.
\]

(4.37)

For the left hand side of (4.37), we have

\[
|zf'(z) - 1| \leq \frac{\sum_{n=2}^{\infty} (n - 1)a_n|z|^{n-1}}{1 - \sum_{n=2}^{\infty} a_n|z|^{n-1}}.
\]

(4.38)

The last expression is less than \( 1 - \sigma \), if

\[
\sum_{n=2}^{\infty} \frac{n - \sigma}{1 - \sigma} |a_n||z|^{n-1} < 1.
\]

Using the fact that \( f \in \mathcal{T}J^\mu(\delta, \beta, \gamma, A, B) \), if and only if

\[
\sum_{n=2}^{\infty} \frac{\Phi_n(\alpha, \beta, \gamma, A, B)C_n(\eta, \mu)}{2\beta\gamma(1 - \alpha)(B - A)} a_n \leq 1,
\]

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we can say (4.37) is true if
\[
\frac{n - \sigma}{1 - \sigma}|z|^{n-1} < \frac{\Phi_n(\alpha, \beta, \gamma, A, B)C_n(\eta, \mu)}{2\beta\gamma(1 - \alpha)(B - A)}
\]
(or) equivalently
\[
|z|^{n-1} < \frac{\Phi_n(\alpha, \beta, \gamma, A, B)C_n(\eta, \mu)(1 - \sigma)}{2\beta\gamma(1 - \alpha)(B - A)(n - \sigma)},
\]
which yields the starlikeness of the family.

(ii) Using the fact that \( f \) is convex, if and only if \( zf' \) is starlike, we have
\[
\sum_{n=2}^{\infty} \frac{n(n - \sigma)}{1 - \sigma}|a_n||z|^{n-1} < 1,
\]
we can prove (ii) on lines similar the proof of (i). \(\square\)

4.3 MODIFIED HADAMARD PRODUCTS

Let the functions \( f_j (j = 1, 2) \) be defined by (4.32). The modified Hadamard product of \( f_1 \) and \( f_2 \) is defined by
\[
(f_1 \ast f_2)(z) = z - \sum_{n=2}^{\infty} a_{n,1}a_{n,2} z^n.
\]
Using the techniques of Schild and Silverman (1975), we prove the following results.

**Theorem 4.3.1.** For functions \( f_j (j = 1, 2) \) defined by (4.32), let \( f_1 \in T_J^\mu(\alpha, \beta, \gamma, A, B) \) and \( f_2 \in T_J^\mu(\delta, \beta, \gamma, A, B) \). Then \((f_1 \ast f_2) \in T_J^\mu(\xi, \beta, \gamma, A, B)\), where
\[
\xi = 1 - \frac{2\beta\gamma(B - A)(1 - \alpha)(1 - \delta)(1 + 2\beta\gamma(B - A) - B\beta)}{\Phi_2(\alpha, \beta, \gamma, A, B)\Phi_2(\delta, \beta, \gamma, A, B)C_2(\eta, \mu) - 4\beta^2\gamma^2(B - A)^2(1 - \alpha)(1 - \delta)},
\]
(4.39)
where $\Phi_2(\alpha, \beta, \gamma, A, B)$ is given by (4.26), $C_2(\eta, \mu)$ is given by (4.17) and $\Phi_2(\delta, \beta, \gamma, A, B) = \left[2\beta\gamma(B - A)(2 - \delta) + (1 - B\beta)\right]$.

**Proof.** In view of Theorem 4.2.1, it suffice to prove that
\[
\sum_{n=2}^{\infty} \frac{2\beta\gamma(B - A)(n - \xi) + (1 - B\beta)(n - 1)C_n(\eta, \mu)}{2\beta\gamma(1 - \xi)(B - A)} a_{n1}a_{n2} \leq 1, \quad (0 \leq \xi < 1),
\]
where $\xi$ is defined by (4.39). On the other hand, under the hypothesis, it follows from (4.18) and the Cauchy’s-Schwarz inequality that
\[
\sum_{n=2}^{\infty} \frac{[\Phi_n(\alpha, \beta, \gamma, A, B)]^{1/2}[\Phi_n(\delta, \beta, \gamma, A, B)]^{1/2}}{\sqrt{(1 - \alpha)(1 - \delta)(C_n(\eta, \mu))^{-1}} \sqrt{a_{n1}a_{n2}}} \leq 1, \quad (4.40)
\]
where $\Phi_n(\alpha, \beta, \gamma, A, B)$ is given by (4.25) and $\Phi_n(\delta, \beta, \gamma, A, B) = \left[2\beta\gamma(B - A)(n - \delta) + (1 - B\beta)(n - 1)\right]$.

Thus we need to find the largest $\xi$ such that
\[
\sum_{n=2}^{\infty} \frac{[\Phi_n(\xi, \beta, \gamma, A, B)]C_n(\eta, \mu)}{2\beta\gamma(1 - \xi)(B - A)} a_{n1}a_{n2} \leq \frac{[\Phi_n(\alpha, \beta, \gamma, A, B)]^{1/2}[\Phi_n(\delta, \beta, \gamma, A, B)]^{1/2}}{\sqrt{(1 - \alpha)(1 - \delta)(C_n(\eta, \mu))^{-1}}} \sqrt{a_{n1}a_{n2}}
\]
or, equivalently that
\[
\sqrt{a_{n1}a_{n2}} \leq \frac{1 - \xi}{\sqrt{(1 - \alpha)(1 - \delta)}} \frac{[\Phi_n(\alpha, \beta, \gamma, A, B)]^{1/2}[\Phi_n(\delta, \beta, \gamma, A, B)]^{1/2}}{[\Phi_n(\xi, \beta, \gamma, A, B)]}, \quad (n \geq 2)
\]
where $\Phi_n(\xi, \beta, \gamma, A, B) = 2\beta\gamma(B - A)(n - \xi) + (1 - B\beta)(n - 1)$.

By view of (4.40) it is sufficient to find largest $\xi$ such that
\[
\frac{2\beta\gamma(B - A)\sqrt{(1 - \alpha)(1 - \delta)(C_n(\eta, \mu))^{-1}}}{[\Phi_n(\alpha, \beta, \gamma, A, B)]^{1/2}[\Phi_n(\delta, \beta, \gamma, A, B)]^{1/2}} \leq \frac{1 - \xi}{\sqrt{(1 - \alpha)(1 - \delta)}} \frac{[\Phi_n(\alpha, \beta, \gamma, A, B)]^{1/2}[\Phi_n(\delta, \beta, \gamma, A, B)]^{1/2}}{2\beta\gamma(B - A)(n - \xi) + (1 - B\beta)(n - 1)},
\]

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which yields

\[ \xi = \Psi(n) = 1 - \frac{2\beta\gamma(B - A)(1 - \alpha)(1 - \delta)(n - 1)(1 + 2\beta\gamma(B - A) - B\beta)}{[\Phi_n(\alpha, \beta, \gamma, A, B)\Phi_n(\delta, \beta, \gamma, A, B)] C_n(\eta, \mu) - 4\beta^2\gamma^2(B - A)^2(1 - \alpha)(1 - \delta)}, \]

(4.41)

for \( n \geq 2 \) is an increasing function of \( n \ (n \geq 2) \) and letting \( n = 2 \) in (4.41), we get the desired result.

By using arguments similar to those in proof of Theorem 4.3.1 and employing the techniques of Srivastava and Aouf (1992, 1995), we can easily prove the following results.

**Theorem 4.3.2.** Let the functions \( f_j(j = 1, 2) \) defined by (4.32), be in the class \( \mathcal{T}J_\mu^\alpha(\alpha, \beta, \gamma, A, B) \) then

\[ (f_1 * f_2) \in \mathcal{T}J_\mu^\alpha(\rho, \beta, \gamma, A, B), \]

where

\[ \rho = 1 - \frac{2\beta\gamma(B - A)(1 - \alpha)^2(1 + 2\beta\gamma(B - A) - B\beta)}{[\Phi_2(\alpha, \beta, \gamma, A, B)]^2C_2(\eta, \mu) - 4\beta^2\gamma^2(B - A)^2(1 - \alpha)^2} \]

and \( C_2(\eta, \mu) \) is given by (4.17).

**Proof.** By taking \( \delta = \alpha \), in the above theorem, the result follows. \( \square \)

**Theorem 4.3.3.** Let the function \( f \) defined by (4.2) be in the class \( \mathcal{T}J_\mu^\alpha(\alpha, \beta, \gamma, A, B) \). Also let \( g(z) = z - \sum_{n=2}^{\infty} b_n z^n \) for \( |b_n| \leq 1 \). Then

\[ (f * g) \in \mathcal{T}J_\mu^\alpha(\alpha, \beta, \gamma, A, B). \]
Proof.

\[
\sum_{n=2}^{\infty} \left[2\beta\gamma(B-A)(n-\alpha) + (1-B\beta)(n-1)\right] C_n(\eta, \mu) |a_n b_n| \\
\leq \sum_{n=2}^{\infty} \left[2\beta\gamma(B-A)(n-\alpha) + (1-B\beta)(n-1)\right] C_n(\eta, \mu) |a_n| \\
\leq 2\beta\gamma(1-\alpha)(B-A),
\]

it follows that \((f \ast g) \in T^\eta J^\mu(\alpha, \beta, \gamma, A, B)\), in view of Theorem 4.2.1.

**Theorem 4.3.4.** Let the functions \(f_j(j = 1, 2)\) defined by (4.32) be in the class \(T^\eta J^\mu(\alpha, \beta, \gamma, A, B)\). Then the function \(h(z)\) defined by

\[
h(z) = z - \sum_{n=2}^{\infty} (a_{n,1}^2 + a_{n,2}^2) z^n
\]

is in the class \(T^\eta J^\mu(\xi, \beta, \gamma, A, B)\), where

\[
\xi = 1 - \frac{4\beta\gamma(1-\alpha)^2(B-A)}{C_2(\eta, \mu) |\Phi_2(\alpha, \beta, \gamma, A, B)|^2 - 8\beta^2\gamma^2(B-A)^2(1-\alpha)^2}
\]

and \(C_2(\eta, \mu)\) is given by (4.17).

**Proof.** By virtue of Theorem 4.2.1, it is sufficient to prove that

\[
\sum_{n=2}^{\infty} \left[2\beta\gamma(B-A)(n-\xi) + (1-B\beta)(n-1)\right] C_n(\eta, \mu) |a_n^2 + a_{n,2}^2| \leq 1, \quad (4.42)
\]

where \(f_j \in T J^\mu(\alpha, \beta, \gamma, A, B)\). We find from (4.32) and Theorem 4.2.1, that,

\[
\sum_{n=2}^{\infty} \left[ \frac{[\Phi_n(\alpha, \beta, \gamma, A, B)] C_n(\eta, \mu)}{2\beta\gamma(1-\alpha)(B-A)} \right]^2 a_{n,j}^2 \leq 1, \quad (4.43)
\]

which yields

\[
\sum_{n=2}^{\infty} \frac{1}{2} \left[ \frac{[\Phi_n(\alpha, \beta, \gamma, A, B)] C_n(\eta, \mu)}{2\beta\gamma(1-\alpha)(B-A)} \right]^2 (a_{n,1}^2 + a_{n,2}^2) \leq 1. \quad (4.45)
\]
On comparing (4.43) and (4.45), it is easily seen that the inequality (4.42) will be satisfied if

\[
\frac{2\beta\gamma(B - A)(n - \xi) + (1 - B\beta)(n - 1)}{2\beta\gamma(1 - \xi)(B - A)} C_n(\eta, \mu) \leq \frac{1}{2} \left[ \frac{[\Phi_n(\alpha, \beta, \gamma, A, B)] C_n(\eta, \mu)}{2\beta\gamma(1 - \alpha)(B - A)} \right]^2, \text{ for } n \geq 2.
\]

That is if

\[
\xi = \Psi(n) = 1 - \frac{4\beta\gamma(1 - \alpha)^2(B - A)(n - 1)(1 + 2\beta\gamma(B - A) - B\beta)}{C_n(\eta, \mu)[\Phi_n(\alpha, \beta, \gamma, A, B)]^2 - 8\beta^2\gamma^2(B - A)^2(1 - \alpha)^2}. \tag{4.48}
\]

Since \( \Psi(n) \) is an increasing function of \( n \) \( (n \geq 2) \) and by taking \( n = 2 \) in (4.48), we get the desired result.

\[\square\]

### 4.4 HOLDER’S INEQUALITY

Followed by Nishiwaki et al. (2008) and Murugusundaramoorthy et al. (2013) in this section we study some results of Holder type inequalities for \( f \in \mathcal{T}J_\mu(\alpha, \beta, \gamma, A, B) \). Now we recall the generalisation of the convolution due to Cho et al. (1996) as given below

\[
\mathcal{H}_m(z) = z - \sum_{n=2}^{\infty} \left( \prod_{j=1}^{m} a_{n,j}^{p_j} \right) z^n, \quad (p_j > 0, \ j = 1, 2, \ldots m). \tag{4.49}
\]

Further for functions \( f_j \in \mathcal{T}J_\mu(\alpha, \beta, \gamma, A, B), (j = 1, 2, \ldots m) \) given by the familiar Holder inequality assumes the following form

\[
\sum_{n=2}^{\infty} \left( \prod_{j=1}^{m} a_{n,j} \right) \leq \prod_{j=1}^{m} \left( \sum_{n=2}^{\infty} a_{n,j}^{p_j} \right)^{\frac{1}{p_j}}, \tag{4.50}
\]

\( (p_j > 1, \ j = 1, 2, \ldots m, \sum_{j=1}^{m} \frac{1}{p_j} \geq 1) \).
Theorem 4.4.1. If \( f_j \in T_{\mu}^J(\xi_j, \beta, \gamma, A, B) \), \(-1 \leq B < A \leq 1\), \(0 \leq \alpha < 1\), \(0 < \beta \leq 1\), \((j = 1, 2, \ldots m)\) then \( \mathcal{H}_m(z) \in T_{\mu}^J(\xi, \beta, \gamma, A, B) \) with

\[
\xi \leq 1 - \frac{(2\beta \gamma (B - A))^s \prod_{j=1}^{s} (1 - \xi_j)^{p_j} \left(1 - (1 - B \beta)2\beta \gamma (B - A)\right)}{\prod_{j=1}^{m} \left[2\beta \gamma (B - A)(2 - \xi_j) + (1 - B \beta)\right]^{p_j} [c_2(\eta, \mu)]^{p_j-1} - [2\beta \gamma (B - A)]^s \prod_{j=1}^{m} (1 - \xi_j)^{p_j}},
\]

where

\[
s = \sum_{j=1}^{m} p_j > 1;\quad p_j \geq \frac{1}{q_j} (j = 1, 2, 3 \ldots m),\quad q_j > 1 (j = 1, 2 \ldots m);\quad \sum_{j=1}^{m} q_j \geq 1.
\]

Proof. Let \( f_j \in T_J^p(\xi_j, \beta, \gamma, A, B) \), \((j = 1, 2 \ldots m)\) then we have

\[
\sum_{n=2}^{\infty} \frac{\left[2\beta \gamma (B - A)(n - \xi_j) + (1 - B \beta)(n - 1)\right] c_n(\eta, \mu)}{2\beta \gamma (1 - \xi_j)(B - A)} a_{n,j} \leq 1,
\]

which in turn implies that

\[
\left( \sum_{n=2}^{\infty} \frac{\left[2\beta \gamma (B - A)(n - \xi_j) + (1 - B \beta)(n - 1)\right] c_n(\eta, \mu)}{2\beta \gamma (1 - \xi_j)(B - A)} a_{n,j} \right)^{\frac{1}{q_j}} \leq 1,
\]

\[
\left(q_j > 1, (j = 1, 2, 3 \ldots m), \sum_{j=1}^{m} \frac{1}{q_j} = 1\right).
\]

Applying the inequality (4.50) we arrive at the following inequality

\[
\sum_{n=2}^{\infty} \left( \sum_{j=1}^{m} \frac{\left[2\beta \gamma (B - A)(n - \xi_j) + (1 - B \beta)(n - 1)\right] c_n(\eta, \mu)}{2\beta \gamma (1 - \xi_j)(B - A)} a_{n,j} \right)^{\frac{1}{q_j}} \frac{1}{a_{n,j}} \leq 1.
\]

Thus we determine the largest \( \xi \) such that
\[
\left( \sum_{n=2}^{\infty} \frac{2 \beta \gamma (B - A)(n - \xi) + (1 - B \beta)(n - 1)}{2 \beta \gamma (1 - \xi)(B - A)} c_n(\eta, \mu) \right) c_n(\eta, \mu) \prod_{j=1}^{m} a_{n,j}^{p_j} \leq 1.
\]
That is
\[
\left( \sum_{n=2}^{\infty} \frac{2 \beta \gamma (B - A)(n - \xi) + (1 - B \beta)(n - 1)}{2 \beta \gamma (1 - \xi)(B - A)} c_n(\eta, \mu) \right) \prod_{j=1}^{m} d_{n,j}^{p_j} \leq
\sum_{n=2}^{\infty} \left[ \left( \sum_{j=1}^{m} \frac{2 \beta \gamma (B - A)(n - \xi_j) + (1 - B \beta)(n - 1)}{2 \beta \gamma (1 - \xi_j)(B - A)} c_n(\eta, \mu) a_{n,j} \right)^{\frac{1}{q_j}} \right]^{\frac{1}{q_j}} a_{n,j}.
\]
Since
\[
\prod_{j=1}^{m} \left( \frac{2 \beta \gamma (B - A)(n - \xi_j) + (1 - B \beta)(n - 1)}{2 \beta \gamma (1 - \xi_j)(B - A)} c_n(\eta, \mu) \right)^{\frac{1}{q_j}} a_{n,j}^{p_j - \frac{1}{q_j}} \leq 1,
\]
\[
(p_j - \frac{1}{q_j}) \geq 0, \quad j = 1, 2, 3...m.
\]
We see that
\[
\prod_{j=1}^{m} a_{n,j}^{p_j - \frac{1}{q_j}} \leq \prod_{j=1}^{m} \left( \frac{2 \beta \gamma (B - A)(n - \xi_j) + (1 - B \beta)(n - 1)}{2 \beta \gamma (1 - \xi_j)(B - A)} c_n(\eta, \mu) \right)^{p_j - \frac{1}{q_j}}.
\] (4.51)
This last inequality (4.51) implies that
\[ 2\beta\gamma(B - A) \prod_{j=1}^{m}(2\beta\gamma(B - A))^{p_j-1}(1 - \xi)^{p_j-1} \]
\[ - \sum_{j=1}^{m} \left[ 2\beta\gamma(n - \xi_j)(B - A) + (1 - B\beta)(n - 1) \right]^{p_j} (c_n(\eta, \mu))^{p_j-1}(1 - \xi) \]
\[ \leq \left( -(n - 1)(1 - B\beta) \prod_{j=1}^{m}(2\beta\gamma(B - A))^{p_j-1}(1 - \xi_j)^{p_j} \right) \]
\[ + \left( (n - 1)2\beta\gamma(B - A) \prod_{j=1}^{m}(2\beta\gamma(B - A))^{p_j-1}(1 - \xi_j)^{p_j} \right), \]
where
\[ \Upsilon_j = \prod_{j=1}^{m}(2\beta\gamma(B - A))^{p_j}(1 - \xi_j)^{p_j}. \]

Which implies
\[ \Upsilon_j - \sum_{j=1}^{m} \left[ 2\beta\gamma(n - \xi_j)(B - A) + (1 - B\beta)(n - 1) \right]^{p_j} (c_n(\eta, \mu))^{p_j-1}(1 - \xi) \leq \]
\[- \left[ (n - 1)\Upsilon_j + (1 - B\beta)(n - 1) \prod_{j=1}^{m}(2\beta\gamma(B - A))^{p_j-1}(1 - \xi_j)^{p_j} \right]. \]

That is
\[ \xi \leq 1 - \frac{(n - 1)\Upsilon_j + (1 - B\beta)(n - 1) \prod_{j=1}^{m}(2\beta\gamma(B - A))^{p_j-1}(1 - \xi_j)^{p_j}}{\sum_{j=1}^{m} \left[ 2\beta\gamma(n - \xi_j)(B - A) + (1 - B\beta)(n - 1) \right]^{p_j} - \Upsilon_j} \]

Let
\[ \Phi(n) \leq 1 - \frac{(n - 1)\Upsilon_j + (1 - B\beta)(n - 1) \prod_{j=1}^{m}(2\beta\gamma(B - A))^{p_j-1}(1 - \xi_j)^{p_j}}{\sum_{j=1}^{m} \left[ 2\beta\gamma(n - \xi_j)(B - A) + (1 - B\beta)(n - 1) \right]^{p_j} - \Upsilon_j} . \]
which is an increasing function in \( n \) hence we have
\[
\xi \leq \Phi(2)
\]
\[
= 1 - \frac{(2\beta \gamma (B - A))^s \prod_{j=1}^{s} (1 - \xi_j)^{p_j} \left(1 - (1 - B\beta)2\beta \gamma (B - A)\right)}{\prod_{j=1}^{m} \left[2\beta \gamma (B - A)(2 - \xi_j) + (1 - B\beta)\right]^{p_j} [c_2(\eta, \mu)]^{p_j - 1} - [2\beta \gamma (B - A)]^s \prod_{j=1}^{m} (1 - \xi_j)^{p_j}}.
\]
Hence the proof.

In the following section we obtain some Inclusion relations of the class \( \mathcal{T}_{J}^{\mu}(\alpha, \beta, \gamma, A, B) \). Following Goodman (1957) and Ruscheweyh (1975), we define the \( \delta \)-neighbourhood of function \( f \in \mathcal{T} \) by
\[
N_{\delta}(f) := \left\{ h \in \mathcal{T} : h(z) = z - \sum_{n=2}^{\infty} d_n z^n \text{ and } \sum_{n=2}^{\infty} n|a_n - d_n| \leq \delta \right\}.
\]
Particularly for the identity function \( e(z) = z \), we have
\[
N_{\delta}(e) := \left\{ h \in \mathcal{T} : h(z) = z - \sum_{n=2}^{\infty} d_n z^n \text{ and } \sum_{n=2}^{\infty} n|d_n| \leq \delta \right\}.
\]

### 4.5 INCLUSION RELATIONS INVOLVING \( N_{\delta}(e) \)

**Theorem 4.5.1.** If
\[
\delta = \frac{4\beta \gamma (1 - \alpha)(B - A)}{[\Phi_2(\alpha, \beta, \gamma, A, B)\] C_2(\eta, \mu)},
\]
where \( C_2(\eta, \mu) \) is given by (4.17). Then \( \mathcal{T}_{J}^{\mu}(\alpha, \beta, \gamma, A, B) \subset N_{\delta}(e) \).

**Proof.** For \( f \in \mathcal{T}_{J}^{\mu}(\alpha, \beta, \gamma, A, B) \), Theorem 4.2.1 immediately yields
\[
\Phi_2(\alpha, \beta, \gamma, A, B)\] C_2(\eta, \mu) \sum_{n=2}^{\infty} a_n \leq 2\beta \gamma (1 - \alpha)(B - A),
\]
so that
\[ \sum_{n=2}^{\infty} a_n \leq \frac{2\beta\gamma(1 - \alpha)(B - A)}{[\Phi_2(\alpha, \beta, \gamma, A, B)]C_2(\eta, \mu)}. \] (4.55)

On the other hand, from (4.18) and (4.55) that
\[
[2\beta\gamma(B - A) + (1 - B\beta)]C_2(\eta, \mu) \sum_{n=2}^{\infty} na_n \leq 2\beta\gamma(1 - \alpha)(B - A) + (1 - B\beta)]C_2(\eta, \mu)
\]
\[
\times \left[ \Phi_2(\alpha, \beta, \gamma, A, B)C_2(\eta, \mu) \right]
\]
\[
= \frac{2[2\beta\gamma(1 - \alpha)(B - A)][2\beta\gamma(B - A) + (1 - B\beta)]}{\Phi_2(\alpha, \beta, \gamma, A, B)}.
\]

That is
\[ \sum_{n=2}^{\infty} na_n \leq \frac{4\beta\gamma(1 - \alpha)(B - A)}{\Phi_2(\alpha, \beta, \gamma, A, B)C_2(\eta, \mu)} := \delta, \] (4.56)

which in view of (4.53), which complete the proof of Theorem 4.5.1. \qed

Next we determine the neighborhood for the class \( T_{J_J}(\rho, \alpha, \beta, \gamma, A, B) \) which we define as follows. A function \( f \in T \) is said to be in the class \( T_{J_J}(\rho, \alpha, \beta, \gamma, A, B) \) if there exists a function \( h \in T_{J_J}(\rho, \alpha, \beta, \gamma, A, B) \) such that
\[ \left| \frac{f(z)}{h(z)} - 1 \right| < 1 - \rho, \quad (z \in U, \quad 0 \leq \rho < 1). \]

**Theorem 4.5.2.** If \( h \in T_{J_J}(\rho, \alpha, \beta, \gamma, A, B) \) and
\[ \rho = 1 - \frac{\Phi_2(\alpha, \beta, \gamma, A, B)\delta C_2(\eta, \mu)}{2 + 4\beta\gamma(B - A)(2 - \alpha) - B\beta} C_2(\eta, \mu) - 4\beta\gamma(1 - \alpha)(B - A), \] (4.57)

then \( N_\delta(h) \subset T_{J_J}(\rho, \alpha, \beta, \gamma, A, B) \).

**Proof.** Suppose that \( f \in N_\delta(h) \), we then find from (4.52) that
\[ \sum_{n=2}^{\infty} n|a_n - b_n| \leq \delta, \]
which implies that the coefficient inequality \( \sum_{n=2}^{\infty} |a_n - b_n| \leq \frac{\delta}{2} \).

Since \( h \in T^\eta_\mu(\alpha, \beta, \gamma, A, B) \), we have

\[
\sum_{n=2}^{\infty} b_n \leq \frac{2\beta\gamma(1 - \alpha)(B - A)}{\Phi_2(\alpha, \beta, \gamma, A, B)C_2(\eta, \mu)},
\]

so that

\[
\left| \frac{f(z)}{h(z)} - 1 \right| < \frac{\sum_{n=2}^{\infty} |a_n - b_n|}{1 - \sum_{n=2}^{\infty} b_n} \leq \frac{\frac{\delta}{2}}{1 - \frac{2\beta\gamma(1 - \alpha)(B - A)}{\Phi_2(\alpha, \beta, \gamma, A, B)C_2(\eta, \mu)}} = \frac{\Phi_2(\alpha, \beta, \gamma, A, B)\delta C_2(\eta, \mu)}{\Phi_2(\alpha, \beta, \gamma, A, B)\delta C_2(\eta, \mu)} = \frac{2 + 4\beta\gamma(B - A)(2 - \alpha) - B\beta}{2 + 4\beta\gamma(1 - \alpha)(B - A)} - 4\beta\gamma(1 - \alpha)(B - A)
\]

provided that \( \rho \) is given precisely by (4.57). Thus by definition,

\[
f \in T^\eta_\mu(\rho, \alpha, \beta, \gamma, A, B)
\]

where \( \rho \) given by (4.57), which completes the proof.

\[\square\]

### 4.6 INTEGRAL MEANS INEQUALITIES

In this section, we obtain integral means inequalities for the functions in the family \( T^\eta_\mu(\alpha, \beta, \gamma, A, B) \) due to Silverman (1997).

**Lemma 4.6.1.** (Littlewood, 1925) If the functions \( f \) and \( g \) are analytic in \( \mathbb{U} \) with \( g < f \), then for \( \eta > 0 \), and \( 0 < r < 1 \),

\[
\int_{0}^{2\pi} |g(re^{i\theta})|^\eta d\theta \leq \int_{0}^{2\pi} |f(re^{i\theta})|^\eta d\theta.
\]

(4.58)
In 1975, Silverman (1975), found that the function $f_2(z) = z - z^2$ is often extremal over the family $\mathcal{T}$ and applied this function to resolve his integral means inequality, conjectured in Silverman (1991) and settled in Silverman (1997), that

$$\int_0^{2\pi} |f(re^{i\theta})|^\eta \, d\theta \leq \int_0^{2\pi} |f_2(re^{i\theta})|^\eta \, d\theta,$$

for all $f \in \mathcal{T}$, $\eta > 0$ and $0 < \gamma < 1$. In Silverman (1997), also proved his conjecture for the subclasses $\mathcal{T}^*(\gamma)$ and $\mathcal{C}(\gamma)$ of $\mathcal{T}$.

Applying Lemma 4.6.1, Theorem 4.2.1 and Theorem 4.2.3, we prove the following result.

**Theorem 4.6.1.** Suppose $f \in \mathcal{T} J^\eta_{\mu}(\alpha, \beta, \gamma, A, B)$, $\eta > 0$, $0 \leq \lambda < 1$, $0 \leq \gamma < 1$, $\beta \geq 0$ and $f_2(z)$ is defined by

$$f_2(z) = z - \frac{2\beta\gamma(1-\alpha)(B-A)}{\Phi_2(\alpha, \beta, \gamma, A, B)} z^2,$$

where $\Phi_2(\alpha, \beta, \gamma, A, B)$ is given by (4.26), and $C_2(b, \mu)$ is given by (4.17). Then for $z = re^{i\theta}$, $0 < \gamma < 1$, we have

$$\int_0^{2\pi} |f(z)|^\eta \, d\theta \leq \int_0^{2\pi} |f_2(z)|^\eta \, d\theta. \quad (4.59)$$

**Proof.** For given $f$ of the form (4.2), from (4.59) is equivalent to proving that

$$\int_0^{2\pi} \left| 1 - \sum_{n=2}^\infty |a_n| z^{n-1} \right| \eta \, d\theta \leq \int_0^{2\pi} \left| 1 - \frac{(1-\gamma)}{\Phi(\lambda, \gamma, \beta, A, B)} z \right| \eta \, d\theta.$$

By Lemma 4.6.1, it suffices to show that

$$1 - \sum_{n=2}^\infty |a_n| z^{n-1} \prec 1 - \frac{2\beta\gamma(1-\alpha)(B-A)}{\Phi_2(\alpha, \beta, \gamma, A, B)} z.$$

Setting

$$1 - \sum_{n=2}^\infty |a_n| z^{n-1} = 1 - \frac{2\beta\gamma(1-\alpha)(B-A)}{\Phi_2(\alpha, \beta, \gamma, A, B)} w(z) \quad (4.60)$$
and using (4.18), we obtain

\[
|w(z)| = \left| \sum_{n=2}^{\infty} \frac{\Phi_n(\alpha, \beta, \gamma, A, B)}{2\beta \gamma (1 - \alpha)(B - A)} |a_n| z^{n-1} \right|
\leq |z| \left| \sum_{n=2}^{\infty} \frac{\Phi_n(\alpha, \beta, \gamma, A, B)}{2\beta \gamma (1 - \alpha)(B - A)} |a_n| \right|
\leq |z|,
\]

where \( \Phi_n(\alpha, \beta, \gamma, A, B) \) is given by (4.25).

This completes the proof by Theorem 4.6.1.

4.7 SUBORDINATION RESULTS

In this section we obtain subordination results for the new class \( \mathcal{T}_{\mathcal{J}_\mu}^{\eta}(\alpha, \beta, \gamma, A, B) \) due to Wilf (1961).

**Definition 4.7.1.** (Subordinating Factor Sequence) A sequence \( \{b_n\}_{n=1}^{\infty} \) of complex numbers is said to be a subordinating sequence if, whenever \( f(z) = \sum_{n=1}^{\infty} a_n z^n, \ a_1 = 1 \) is regular, univalent and convex in \( U \), we have

\[
\sum_{n=1}^{\infty} b_n a_n z^n \prec f(z), \ z \in U.
\] (4.61)

**Lemma 4.7.1.** (Wilf, 1961) The sequence \( \{b_n\}_{n=1}^{\infty} \) is a subordinating factor sequence if and only if

\[
\Re \left\{ 1 + 2 \sum_{n=1}^{\infty} b_n z^n \right\} > 0, \ z \in U.
\] (4.62)

For the sake of brevity, we let

\[
\psi_n(\alpha, \beta, \gamma, A, B) = \Phi_n(\alpha, \beta, \gamma, A, B) C_n(\eta, \mu)
\] (4.63)
and
\[ \psi_2(\alpha, \beta, \gamma, A, B) = \Phi_2(\alpha, \beta, \gamma, A, B)C_2(\eta, \mu). \] (4.64)

**Theorem 4.7.1.** Let \( f \in \mathcal{T}_J^\mu(\alpha, \beta, \gamma, A, B) \) and \( g(z) \) be any function in the usual class of convex functions \( C\mathcal{V} \), then
\[ \frac{\psi_2(\alpha, \beta, \gamma, A, B)}{2[2\beta\gamma(1-\alpha)(B-A) + \psi_2(\alpha, \beta, \gamma, A, B)]}(f \ast g)(z) < g(z), \] (4.65)
where \( 0 \leq \gamma < 1; \beta \geq 0 \) and \( 0 \leq \lambda < 1 \), and
\[ \Re(f(z)) > -\frac{[2\beta\gamma(1-\alpha)(B-A) + \psi_2(\alpha, \beta, \gamma, A, B)]}{\psi_2(\alpha, \beta, \gamma, A, B)}, \quad z \in \mathbb{U}. \] (4.66)

The constant factor \( \frac{[2\beta\gamma(1-\alpha)(B-A) + \psi_2(\alpha, \beta, \gamma, A, B)]}{\psi_2(\alpha, \beta, \gamma, A, B)} \) in (4.65) cannot be replaced by a larger number.

**Proof.** Let \( f \in \mathcal{T}_J^\mu(\alpha, \beta, \gamma, A, B) \) and suppose that \( g(z) = z + \sum_{n=1}^{\infty} b_n z^n \in \mathbb{C} \). Then
\[ \frac{\psi_2(\alpha, \beta, \gamma, A, B)}{2[2\beta\gamma(1-\alpha)(B-A) + \psi_2(\alpha, \beta, \gamma, A, B)]}(f \ast g)(z) = \frac{\psi_2(\alpha, \beta, \gamma, A, B)}{2[2\beta\gamma(1-\alpha)(B-A) + \psi_2(\alpha, \beta, \gamma, A, B)]} \left( z + \sum_{n=2}^{\infty} b_n a_n z^n \right). \] (4.67)

Thus, by Definition 4.7.1, the subordination result holds true if
\[ \left\{ \frac{\psi_2(\alpha, \beta, \gamma, A, B)}{2[2\beta\gamma(1-\alpha)(B-A) + \psi_2(\alpha, \beta, \gamma, A, B)]} \right\}_{n=1}^{\infty}, \]
is a subordinating factor sequence, with \( a_1 = 1 \). In view of Lemma 4.7.1, this is equivalent to the following inequality
\[ \Re\left\{ \frac{\psi_2(\alpha, \beta, \gamma, A, B)}{2[2\beta\gamma(1-\alpha)(B-A) + \psi_2(\alpha, \beta, \gamma, A, B)]} a_n z^n \right\} > 0, \quad z \in \mathbb{U}. \] (4.68)

By noting the fact that \( \psi_n(\alpha, \beta, \gamma, A, B) \) is increasing function for \( n \geq 2 \) and in particular
\[ \frac{\psi_2(\alpha, \beta, \gamma, A, B)}{2\beta\gamma(1-\alpha)(B-A)} \leq \frac{\psi_n(\alpha, \beta, \gamma, A, B)}{2\beta\gamma(1-\alpha)(B-A)}, \quad n \geq 2, \]
therefore, for \( |z| = r < 1 \), we have

\[
\Re \left\{ 1 + \frac{\psi_2(\alpha, \beta, \gamma, A, B)}{2[2\beta \gamma(1-\alpha)(B-A) + \psi_2(\alpha, \beta, \gamma, A, B)]} \sum_{n=1}^{\infty} a_n z^n \right\} = \Re \left\{ 1 + \frac{\psi_2(\alpha, \beta, \gamma, A, B)}{2[2\beta \gamma(1-\alpha)(B-A) + \psi_2(\alpha, \beta, \gamma, A, B)]} z \right\} + \psi_2(\alpha, \beta, \gamma, A, B) \sum_{n=2}^{\infty} a_n z^n \]

\[
\geq 1 - \frac{1}{2[2\beta \gamma(1-\alpha)(B-A) + \psi_2(\alpha, \beta, \gamma, A, B)]} \left[ \frac{1}{2\beta \gamma(1-\alpha)(B-A) \psi_2(\alpha, \beta, \gamma, A, B)} \right]^{r > 0}, \quad (|z| = r),
\]

where we have also made use of the assertion (4.18) of Theorem 4.2.1. This evidently proves the inequality (4.68) and hence also the subordination result (4.65) asserted by Theorem 4.7.1. The inequality (4.66) follows from (4.65) by taking

\[
g(z) = \frac{z}{1-z} = z + \sum_{n=2}^{\infty} z^n \in \mathbb{C}.
\]

Next we consider the function

\[
F(z) := z - \frac{2\beta \gamma(1-\alpha)(B-A)}{\psi_2(\alpha, \beta, \gamma, A, B)} z^2,
\]

where \( 0 \leq \gamma < 1, \beta \geq 0, 0 \leq \lambda < 1 \). Clearly \( F \in T^{\eta}_{\mu}(\rho, \alpha, \beta, \gamma, A, B) \). For this function (4.65) becomes

\[
\frac{\psi_2(\alpha, \beta, \gamma, A, B)}{2[2\beta \gamma(1-\alpha)(B-A) + \psi_2(\alpha, \beta, \gamma, A, B)]} F(z) \prec \frac{z}{1-z}.
\]

It is easily verified that

\[
\min \left\{ \Re \left( \frac{\psi_2(\alpha, \beta, \gamma, A, B)}{2[2\beta \gamma(1-\alpha)(B-A) + \psi_2(\alpha, \beta, \gamma, A, B)]} F(z) \right) \right\} = -\frac{1}{2}, \quad z \in \mathbb{U}.
\]
This shows that the constant
\[
\frac{\psi_2(\alpha, \beta, \gamma, A, B)}{2[2\beta \gamma (1 - \alpha)(B - A) + \psi_2(\alpha, \beta, \gamma, A, B)]}
\]
cannot be replaced by any larger one. 

\section*{4.8 PARTIAL SUM RESULTS}

Silvia (1985) studied the partial sums of convex functions of order $\alpha$. Later on, Abubaker and Darus (2010), Frasin (2005, 2008), Raina and Bansal (2008) and Rosy et al. (2003) determined the sharp lower bound on the real part of the quotients between the normalized starlike or convex functions, viz., $\Re \left( \frac{f(z)}{f_k(z)} \right)$, $\Re \left( \frac{f(z)}{f_k(z)} \right)$, $\Re \left( \frac{f(z)}{f_k(z)} \right)$ and $\Re \left( \frac{f(z)}{f_k(z)} \right)$ for their sequences of partial sums $f_k(z) = z + \sum_{n=2}^{k} a_n z^n$ of the analytic function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$.

In the following theorems we discuss the results on partial sums for $f \in T_J^p(\alpha, \beta, \gamma, A, B)$.

\textbf{Theorem 4.8.1.} If $f$ of the form (4.1) satisfies the condition (4.18), then
\[
\Re \left( \frac{f(z)}{f_k(z)} \right) \geq \frac{\rho_{k+1}(\alpha, \beta, \gamma, A, B) - 2\beta \gamma (1 - \alpha)(B - A)}{\rho_{k+1}(\alpha, \beta, \gamma, A, B)}, \quad (z \in \mathbb{U}),
\]
where
\[
\rho_n = \rho_{k+1}(\alpha, \beta, \gamma, A, B) \geq \begin{cases} 
2\beta \gamma (1 - \alpha)(B - A), & \text{if } n = 2, 3, \ldots, k \\
\rho_{k+1}, & \text{if } n = k + 1, k + 2, \ldots.
\end{cases}
\]

The result (4.69) is sharp with the function given by
\[
f(z) = z + \frac{2\beta \gamma (1 - \alpha)(B - A)}{\rho_{k+1}} z^{n+1}.
\]
Proof. Define the function \( w(z) \) by

\[
1 + \frac{w(z)}{1 - w(z)} = \frac{\rho_{k+1}}{2\beta\gamma(1 - \alpha)(B - A)} \left[ \frac{f(z)}{f_k(z)} - \frac{\rho_{k+1} - 2\beta\gamma(1 - \alpha)(B - A)}{\rho_{k+1}} \right]
\]

\[
= 1 + \sum_{n=2}^{k} a_n z^{n-1} + \left( \frac{\rho_{k+1}}{2\beta\gamma(1 - \alpha)(B - A)} \right) \sum_{n=k+1}^{\infty} a_n z^{n-1}
\]

\[
= \frac{1 + \sum_{n=2}^{k} a_n z^{n-1} + \left( \frac{\rho_{k+1}}{2\beta\gamma(1 - \alpha)(B - A)} \right) \sum_{n=k+1}^{\infty} a_n z^{n-1}}{1 + \sum_{n=2}^{k} a_n z^{n-1}}.
\]

(4.72)

It suffices to show that \(|w(z)| \leq 1\). Now, from (4.72) we can write

\[
w(z) = \frac{\left( \frac{\rho_{k+1}}{2\beta\gamma(1 - \alpha)(B - A)} \right) \sum_{n=k+1}^{\infty} a_n z^{n-1}}{2 + 2 \sum_{n=2}^{k} a_n z^{n-1} + \left( \frac{\rho_{k+1}}{2\beta\gamma(1 - \alpha)(B - A)} \right) \sum_{n=k+1}^{\infty} a_n z^{n-1}}.
\]

Hence we obtain

\[
|w(z)| \leq \frac{\left( \frac{\rho_{k+1}}{2\beta\gamma(1 - \alpha)(B - A)} \right) \sum_{n=k+1}^{\infty} |a_n|}{2 - 2 \sum_{n=2}^{k} |a_n| - \left( \frac{\rho_{k+1}}{2\beta\gamma(1 - \alpha)(B - A)} \right) \sum_{n=k+1}^{\infty} |a_n|}.
\]

Now \(|w(z)| \leq 1\), if and only if

\[
2 \left( \frac{\rho_{k+1}}{2\beta\gamma(1 - \alpha)(B - A)} \right) \sum_{n=k+1}^{\infty} |a_n| \leq 2 \sum_{n=2}^{k} |a_n|.
\]

Or, equivalently,

\[
\sum_{n=2}^{k} |a_n| + \sum_{n=k+1}^{\infty} \frac{\rho_{k+1}}{2\beta\gamma(1 - \alpha)(B - A)} |a_n| \leq \sum_{n=2}^{\infty} \frac{\rho_n}{2\beta\gamma(1 - \alpha)(B - A)} |a_n|.
\]

From the condition (4.18), it is sufficient to show that

\[
\sum_{n=2}^{k} |a_n| + \sum_{n=k+1}^{\infty} \frac{\rho_{k+1}}{2\beta\gamma(1 - \alpha)(B - A)} |a_n| \leq \sum_{n=2}^{\infty} \frac{\rho_n}{2\beta\gamma(1 - \alpha)(B - A)} |a_n|,
\]

which is equivalent to

\[
\sum_{n=2}^{k} \left( \frac{\rho_n - 2\beta\gamma(1 - \alpha)(B - A)}{2\beta\gamma(1 - \alpha)(B - A)} \right) |a_n| + \sum_{n=k+1}^{\infty} \left( \frac{\rho_n - \rho_{k+1}}{2\beta\gamma(1 - \alpha)(B - A)} \right) |a_n| \geq 0.
\]

(4.73)
To see that the function given by (4.71) gives the sharp result, we observe that for $z = re^{i\pi/k}$

$$\frac{f(z)}{f_k(z)} = 1 + \frac{2\beta\gamma(1 - \alpha)(B - A)}{\rho_{k+1}} z^k \to 1 - \frac{2\beta\gamma(1 - \alpha)(B - A)}{\rho_{k+1}}$$

$$= \frac{\rho_{k+1} - 2\beta\gamma(1 - \alpha)(B - A)}{\rho_{k+1}} \quad \text{when } r \to 1^-.$$

\[\square\]

We next determine bounds for $\Re\left(f_k(z)/f(z)\right)$.

**Theorem 4.8.2.** If $f$ of the form (4.1) satisfies the condition (4.18), then

$$\Re\left(\frac{f_k(z)}{f(z)}\right) \geq \frac{\rho_{k+1}}{\rho_{k+1} + 2\beta\gamma(1 - \alpha)(B - A)}, \quad (z \in \mathbb{U}),$$

where $\rho_{k+1} \geq 2\beta\gamma(1 - \alpha)(B - A)$ and

$$\rho_n(\lambda, \gamma, \eta) \geq \begin{cases} 
2\beta\gamma(1 - \alpha)(B - A), & \text{if } n = 2, 3, \ldots, k \\
\rho_{k+1}, & \text{if } n = k+1, k+2, \ldots.
\end{cases}$$

(4.75)

The result (4.74) is sharp with the function given by (4.71).

**Proof.** We write

$$\frac{1 + w(z)}{1 - w(z)} = \frac{\rho_{k+1} + 2\beta\gamma(1 - \alpha)(B - A)}{2\beta\gamma(1 - \alpha)(B - A)} \left[ \frac{f_k(z)}{f(z)} - \frac{\rho_{k+1}}{\rho_{k+1} + 2\beta\gamma(1 - \alpha)(B - A)} \right]$$

$$= 1 + \sum_{n=2}^{k} a_n z^{n-1} - \left( \frac{\rho_{k+1}}{2\beta\gamma(1 - \alpha)(B - A)} \right) \sum_{n=k+1}^{\infty} a_n z^{n-1}$$

$$= 
\frac{1 + \sum_{n=2}^{k} a_n z^{n-1} - \left( \frac{\rho_{k+1} + 2\beta\gamma(1 - \alpha)(B - A)}{2\beta\gamma(1 - \alpha)(B - A)} \right) \sum_{n=k+1}^{\infty} a_n z^{n-1}}{1 + \sum_{n=2}^{\infty} a_n z^{n-1}},$$

where

$$\left| w(z) \right| \leq \frac{\left( \frac{\rho_{k+1} + 2\beta\gamma(1 - \alpha)(B - A)}{2\beta\gamma(1 - \alpha)(B - A)} \right) \sum_{n=k+1}^{\infty} |a_n|}{2 - 2 \sum_{n=2}^{k} |a_n| - \left( \frac{\rho_{k+1} - 2\beta\gamma(1 - \alpha)(B - A)}{2\beta\gamma(1 - \alpha)(B - A)} \right) \sum_{n=k+1}^{\infty} |a_n|} \leq 1.$$
This last inequality is equivalent to
\[
\sum_{n=2}^{k} |a_n| + \sum_{n=k+1}^{\infty} \frac{\rho_{k+1}}{2\beta\gamma(1-\alpha)(B-A)} |a_n| \leq 1.
\]
Making use of (4.18) to get (4.73). Finally, equality holds in (4.74) for the extremal function \( f(z) \) given by (4.71).
\[
\square
\]

We next turn to ratios involving derivatives.

**Theorem 4.8.3.** If \( f \) of the form (4.1) satisfies the condition (4.18), then
\[
\Re \left( \frac{f'(z)}{f'(z)} \right) \geq \frac{\rho_{k+1} - 2\beta\gamma(k+1)(1-\alpha)(B-A)}{\rho_{k+1}}, \quad (z \in \mathbb{U}),
\]
and
\[
\Re \left( \frac{f_k'(z)}{f'_k(z)} \right) \geq \frac{\rho_{k+1}}{\rho_{k+1} + 2\beta\gamma(k+1)(1-\alpha)(B-A)}, \quad (z \in \mathbb{U}),
\]
where \( \rho_{k+1} \geq 2\beta\gamma(k+1)(1-\alpha)(B-A) \) and
\[
\rho_k \geq \begin{cases} 
2\beta\gamma k(1-\alpha)(B-A), & i.f. \quad n = 2, 3, \ldots, k \\
n \left( \frac{\rho_{k+1}}{k+1} \right), & i.f. \quad n = k+1, k+2, \ldots
\end{cases}
\]
(4.78)
The results are sharp with the function given by (4.71).

**Proof.** We write
\[
\frac{1 + w(z)}{1 - w(z)} = \frac{\rho_{k+1}}{2\beta\gamma(k+1)(1-\alpha)(B-A)} \left( \frac{f'(z)}{f'_k(z)} - \left( \frac{\rho_{k+1} - 2\beta\gamma(k+1)(1-\alpha)(B-A)}{\rho_{k+1}} \right) \right),
\]
where
\[
w(z) = \frac{\left( \frac{\rho_{k+1}}{2\beta\gamma(k+1)(1-\alpha)(B-A)} \right) \sum_{n=k+1}^{\infty} na_nz^{n-1}}{2 + 2 \sum_{n=2}^{k} na_nz^{n-1} + \left( \frac{\rho_{k+1}}{2\beta\gamma(k+1)(1-\alpha)(B-A)} \right) \sum_{n=k+1}^{\infty} na_nz^{n-1}}.
\]
Now \(|w(z)| \leq 1\) if and only if
\[
\sum_{n=2}^{k} n |a_n| + \frac{\rho_{k+1}}{2\beta\gamma(k+1)(1-\alpha)(B-A)} \sum_{n=k+1}^{\infty} n |a_n| \leq 1.
\]
From the condition (4.18), it is sufficient to show that
\[ \sum_{n=2}^{k} n |a_n| + \frac{\rho_{k+1}}{2\beta \gamma (k+1)(1-\alpha)(B-A)} \sum_{n=2}^{\infty} n |a_n| \leq \sum_{n=2}^{\infty} \frac{\rho_n}{2\beta \gamma (1-\alpha)(B-A)} |a_n|. \]

Which is equivalent to
\[ \sum_{n=2}^{k} \left( \frac{\rho_n - 2\beta \gamma (1-\alpha)(B-A)n}{2\beta \gamma (1-\alpha)(B-A)} \right) |a_n| + \sum_{n=k+1}^{\infty} \frac{(k+1) \rho_n - n\rho_{k+1}}{2\beta \gamma (k+1)(1-\alpha)(B-A)} |a_n| \geq 0. \]

To prove the result (4.77), define the function \( w(z) \) by
\[
\frac{1 + w(z)}{1 - w(z)} = \frac{(k+1)(2\beta \gamma (1-\alpha)(B-A)) + \rho_{k+1}}{2\beta \gamma (k+1)(1-\alpha)(B-A)} \left[ f_k'(z) - \frac{\rho_{k+1}}{2\beta \gamma (k+1)(1-\alpha)(B-A) + \rho_{k+1}} \right],
\]
where
\[ w(z) = \frac{- \left( 1 + \frac{\rho_{k+1}}{2\beta \gamma (k+1)(1-\alpha)(B-A)} \right) \sum_{n=k+1}^{\infty} n a_n z^{n-1}}{2 + 2 \sum_{n=2}^{k} n a_n z^{n-1} + \left( 1 - \frac{\rho_{k+1}}{2\beta \gamma (k+1)(1-\alpha)(B-A)} \right) \sum_{n=k+1}^{\infty} n a_n z^{n-1}}.
\]

Now \( |w(z)| \leq 1 \) if and only if
\[ \sum_{n=2}^{k} n |a_n| + \left( \frac{\rho_{k+1}}{2\beta \gamma (k+1)(1-\alpha)(B-A)} \right) \sum_{n=k+1}^{\infty} n |a_n| \leq 1. \] (4.79)

It suffices to show that the left hand side of (4.79) is bounded above by the condition
\[ \sum_{n=2}^{\infty} \frac{\rho_n}{2\beta \gamma (1-\alpha)(B-A)} |a_n|, \]
which is equivalent to
\[ \sum_{n=2}^{k} \left( \frac{\rho_n}{2\beta \gamma (1-\alpha)(B-A)} - n \right) |a_n| \]
\[ + \sum_{n=k+1}^{\infty} \left( \frac{\rho_n}{2\beta \gamma (1-\alpha)(B-A)} - \frac{\rho_{k+1}}{2\beta \gamma (k+1)(1-\alpha)(B-A)} \right) n |a_n| \geq 0. \]

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