Chapter 4

Some auxiliary function spaces

The Sobolev space $\tilde{W}^{1,p}(\mathbb{C}^n)$

Let $L_j$ and $M_j$ be the differential operators defined by

$$L_j = \left( \frac{\partial}{\partial x_j} + \frac{i y_j}{2} \right) \quad \text{and} \quad M_j = \left( \frac{\partial}{\partial y_j} - \frac{i x_j}{2} \right), \quad j = 1, 2, \ldots, n. \quad (4.0.1)$$

We consider the space

$$\tilde{W}^{1,p}(\mathbb{C}^n) = \{ f \in L^p(\mathbb{C}^n) : L_j f, M_j f \in L^p(\mathbb{C}^n), 1 \leq j \leq n \}$$

with norm $\| f \| = \| f \|_{L^p(\mathbb{C}^n)} + \sum_{j=1}^{n} (\| L_j f \|_{L^p(\mathbb{C}^n)} + \| M_j f \|_{L^p(\mathbb{C}^n)})$. If $\{ f_k \}$ is a Cauchy sequence in $\tilde{W}^{1,p}(\mathbb{C}^n)$ then there exists $f, g_j, h_j \in L^p(\mathbb{C}^n)$ such that $f_k \to f, L_j f_k \to g_j, M_j f_k \to h_j$ in $L^p(\mathbb{C}^n)$ as $k \to \infty$ for $1 \leq j \leq n$. Since $L_j, M_j$ are skew adjoint operators, it is easy to see that $L_j f = g_j, M_j f = h_j$ in $S'(\mathbb{C}^n)$ for $1 \leq j \leq n$. This shows that $f \in \tilde{W}^{1,p}(\mathbb{C}^n)$ and $f_k \to f$ in $\tilde{W}^{1,p}(\mathbb{C}^n)$. Hence $\tilde{W}^{1,p}(\mathbb{C}^n)$ is a Banach space.

An interesting relation between the Sobolev space $\tilde{W}^{1,p}(\mathbb{C}^n)$ and the ordinary Sobolev space $W^{1,p}(\mathbb{C}^n)$ is the following: If $u \in \tilde{W}^{1,p}(\mathbb{C}^n)$, then $|u| \in W^{1,p}(\mathbb{C}^n)$.

Lemma 4.0.9 [Sobolev Embedding Theorem] We have the continuous inclusion

$$\tilde{W}^{1,p_1}(\mathbb{C}^n) \hookrightarrow L^{p_2}(\mathbb{C}^n) \quad \text{for} \quad p_1 \leq p_2 \leq \frac{2np_1}{2n-p_1} \quad \text{if} \quad p_1 < 2n$$

$$\text{for} \quad p_1 \leq p_2 < \infty \quad \text{if} \quad p_1 = 2n$$

$$\text{for} \quad p_1 \leq p_2 \leq \infty \quad \text{if} \quad p_1 > 2n$$
where $1 < p_1 < \infty$.

**Proof.** Let $f \in \tilde{W}^{1,p_1}(\mathbb{C}^n)$ and $\epsilon > 0$. Consider $u_\epsilon = e^{-\epsilon \mathcal{L}} f$. Note that (4.0.6) is also valid for $f \in L^{p_1}(\mathbb{C}^n)$. Since $K_\epsilon$ given by (4.0.7) is in $\mathcal{S}(\mathbb{C}^n)$ and from Lemma 4.0.17 $u_\epsilon = f \times K_\epsilon \in \tilde{W}^{1,p_1}(\mathbb{C}^n) \cap C^\infty(\mathbb{C}^n)$ and we have

$$2|u_\epsilon| \frac{\partial}{\partial x_j}u_\epsilon = \frac{\partial}{\partial x_j}(|u_\epsilon|^2) = 2 \Re \left( \frac{\partial}{\partial x_j}u_\epsilon \cdot \frac{\partial}{\partial x_j} \overline{u_\epsilon} \right) = 2 \Re \left( \frac{\partial}{\partial x_j} + \frac{i y_j}{2} \right) u_\epsilon.$$ 

Hence on the set $A_\epsilon = \{ z \in \mathbb{C}^n \mid u_\epsilon(z) \neq 0 \}$, we have

$$\left| \frac{\partial}{\partial x_j}u_\epsilon \right| \leq \left| \frac{\partial}{\partial x_j}u_\epsilon \right|.$$ 

Similarly $\left| \frac{\partial}{\partial x_j}u_\epsilon \right| \leq |M_ju_\epsilon|$ on $A_\epsilon$. Note that $\| u_\epsilon \|_{L^2_2(\mathbb{C}^n)} = \| u_\epsilon \mathcal{X}_A \|_{L^2_2(\mathbb{C}^n)}$. By the usual Sobolev embedding on $\mathbb{C}^n$ and above observations, we have inequality $\| u_\epsilon \|_{L^2_2(\mathbb{C}^n)} \leq C \| u_\epsilon \mathcal{X}_A \|_{W^{1,1}_1} \leq C \| u_\epsilon \|_{W^{1,1}_1}$. Since $S e^{-\epsilon \mathcal{L}} f = e^{-\epsilon \mathcal{L}} S f$ for $S = L_j, M_j (1 \leq j \leq n)$ (see Lemma 4.0.10), therefore by Lemma 4.0.17 $u_\epsilon = e^{-\epsilon \mathcal{L}} f \to f$ in $\tilde{W}^{1,1}_1(\mathbb{C}^n)$ and also in $L^2(\mathbb{C}^n)$ as $\epsilon \to 0$. Therefore we have $\| f \|_{L^2_2(\mathbb{C}^n)} \leq C \| f \|_{W^{1,1}_1(\mathbb{C}^n)}$. Hence Lemma is proved.

**Lemma 4.0.10** Let $f \in S'(\mathbb{C}^n)$. Then for every $t, t_0 \in \mathbb{R}$, we have the following equalities in $S'(\mathbb{C}^n)$

$$L_j e^{-i(t-t_0)\mathcal{L}} f = e^{-i(t-t_0)\mathcal{L}} L_j f$$

$$M_j e^{-i(t-t_0)\mathcal{L}} f = e^{-i(t-t_0)\mathcal{L}} M_j f.$$ 

**Proof.** Since $f \in S'(\mathbb{C}^n)$, $L_j e^{-i(t-t_0)\mathcal{L}} f$, $M_j e^{-i(t-t_0)\mathcal{L}} f \in S'(\mathbb{C}^n)$. In view of (1.3.17), (1.3.18), (1.3.21) and (1.3.22) in [33], we have

$$L_j \Phi_{\mu,\nu} = \frac{i}{2} \left( (2\mu_j)^2 \Phi_{\mu - e_j, \nu} + (2\mu_j + 2)^2 \Phi_{\mu + e_j, \nu} \right)$$

$$M_j \Phi_{\mu,\nu} = \frac{1}{2} \left( (2\mu_j)^2 \Phi_{\mu - e_j, \nu} - (2\mu_j + 2)^2 \Phi_{\mu + e_j, \nu} \right).$$

Since $L_j, M_j$ are skew adjoint operators and finite linear combination of special Hermite functions are dense in $S(\mathbb{C}^n)$ (Theorem 1.4.4 in [34]), Lemma follows.
from the following observations

\[
\langle L_j e^{-i(t-t_0)c} f, \Phi_{\mu,\nu} \rangle = \langle e^{-i(t-t_0)c} L_j f, \Phi_{\mu,\nu} \rangle
\]

\[
\langle M_j e^{-i(t-t_0)c} f, \Phi_{\mu,\nu} \rangle = \langle e^{-i(t-t_0)c} M_j f, \Phi_{\mu,\nu} \rangle
\]

for every \( \mu, \nu \in (\mathbb{Z}_{\geq 0})^n \).

**Lemma 4.0.11** Let \( t_0 \in \mathbb{R} \) and \( I \) an open interval containing \( t_0 \). Let \( g \in L^q_{\text{loc}}(I, \dot{W}^{1,p'}(\mathbb{C}^n)) \), where \((q,p)\) be an admissible pair. Then \( \int_{t_0}^t e^{-i(t-s)c} g(z,s)ds \in C(I, \dot{W}^{1,2}(\mathbb{C}^n)) \). Moreover for each \( t \in I \), we have the following equalities in \( L^2(\mathbb{C}^n) \)

\[
L_j \int_{t_0}^t e^{-i(t-s)c} g(z,s)ds = \int_{t_0}^t e^{-i(t-s)c} L_j g(z,s)ds
\]

\[
M_j \int_{t_0}^t e^{-i(t-s)c} g(z,s)ds = \int_{t_0}^t e^{-i(t-s)c} M_j g(z,s)ds.
\]

**Proof.** Since \( g \in L^q_{\text{loc}}(I, \dot{W}^{1,p'}(\mathbb{C}^n)) \), by Strichartz estimates (Theorem 3.0.7), \( \int_{t_0}^t e^{-i(t-s)c} S g(z,s)ds \in C(I, L^2(\mathbb{C}^n)) \), where \( S = L_j, M_j, 1 \leq j \leq n \). In view of Theorem 1.4.4 in [34], Lemma follows from the following observations

\[
\langle L_j \int_{t_0}^t e^{-i(t-s)c} g(z,s)ds, \Phi_{\mu,\nu} \rangle = \langle \int_{t_0}^t e^{-i(t-s)c} L_j g(z,s)ds, \Phi_{\mu,\nu} \rangle
\]

\[
\langle M_j \int_{t_0}^t e^{-i(t-s)c} g(z,s)ds, \Phi_{\mu,\nu} \rangle = \langle \int_{t_0}^t e^{-i(t-s)c} M_j g(z,s)ds, \Phi_{\mu,\nu} \rangle
\]

for every \( \mu, \nu \in (\mathbb{Z}_{\geq 0})^n \).

**The Sobolev space \( \dot{W}^{1,1}(\mathbb{C}^n) \)**

The local well posedness of the nonlinear Schrödinger equation for the twisted Laplacian is discussed in chapter [6] for initial values in \( \dot{W}^{1,2}(\mathbb{C}^n) \). However this approach does not give the energy conservation. We overcome this difficulty by introducing the Sobolev space \( \dot{W}^{1,1}(\mathbb{C}^n) \) defined using the operators \( Z_j \) and \( \overline{Z}_j \)

\[
Z_j = \frac{\partial}{\partial z_j} + \frac{1}{2} \overline{z}_j, \quad \overline{Z}_j = -\frac{\partial}{\partial \overline{z}_j} + \frac{1}{2} z_j,
\]
which is the natural one in this context. Here \( \frac{\partial}{\partial z_j}, \frac{\partial}{\partial \bar{z}_j} \) denote the complex derivatives \( \frac{\partial}{\partial z_j} \pm i \frac{\partial}{\partial \bar{z}_j} \) respectively.

**Definition 4.0.12** Let \( m \) be a nonnegative integer and \( 1 \leq p < \infty \). We define space \( \tilde{W}^{m,p}_L(\mathbb{C}^n) \) by the following

\[
\tilde{W}^{m,p}_L(\mathbb{C}^n) = \{ f \in L^p(\mathbb{C}^n) : S^\alpha f \in L^p(\mathbb{C}^n), |\alpha| \leq m \}
\]

where \( S^\alpha \) denotes monomial in \( Z_1, \ldots, Z_n, \bar{Z}_1, \ldots, \bar{Z}_n \) of degree \( |\alpha| = \alpha_1 + \cdots + \alpha_{2n} \). \( \tilde{W}^{m,p}_L(\mathbb{C}^n) \) is a Banach space with norm given by

\[
\| f \|_{\tilde{W}^{m,p}_L} = \sum_{|\alpha| \leq m} \| S^\alpha f \|_{L^p}.
\]

**Lemma 4.0.13** Let \( f \in \tilde{W}^{1,2}_L(\mathbb{C}^n) \). Then we have

\[
\| f \|_{\tilde{W}^{1,2}_L} \approx \sum_{1 \leq j \leq n} (\| Z_j f \|_{L^2} + \| \bar{Z}_j f \|_{L^2}).
\]

**Proof.** Clearly

\[
\| f \|_{\tilde{W}^{1,2}_L} \geq \sum_{1 \leq j \leq n} (\| Z_j f \|_{L^2} + \| \bar{Z}_j f \|_{L^2}).
\]

Now we show that

\[
\| f \|_{\tilde{W}^{1,2}_L} \leq 2 \sum_{1 \leq j \leq n} (\| Z_j f \|_{L^2} + \| \bar{Z}_j f \|_{L^2}).
\]

Enough to show that \( \| Z_j f \|_2 \geq \| f \|_2, \ 1 \leq j \leq n \). This follows from the Plancherel theorem for the special Hermite expansion

\[
\| f \|_2^2 = \sum_{\mu, \nu} |\langle f, \Phi_{\mu, \nu} \rangle|^2,
\]

for \( f \in L^2(\mathbb{C}^n) \). In view of (2.0.1) and \( Z_j, \bar{Z}_j \) are adjoint of each other, we have

\[
\bar{Z}_j f = \sum_{\mu, \nu} \langle \bar{Z}_j f, \Phi_{\mu, \nu} \rangle \Phi_{\mu, \nu} = - \sum_{\mu, \nu_j \geq 1} i(2\nu_j)^\frac{1}{2} \langle f, \Phi_{\mu, \nu_j - e_j} \rangle \Phi_{\mu, \nu}.
\] (4.0.2)

Thus in view of equation (4.0.2), we have
\[ \|\tilde{Z}_j f\|_2^2 = \sum_{\mu, \nu \geq 1} 2\nu_j |\langle f, \Phi_{\mu \nu - \epsilon_j}\rangle|^2 = \sum_{\mu, \nu} (2\nu_j + 2)|\langle f, \Phi_{\mu \nu}\rangle|^2 \geq \|f\|_2^2, \]

which completes the proof.

Though the operators \( Z_j \) and \( \tilde{Z}_j \) \((1 \leq j \leq n)\) do not commute with \( e^{-i\mathcal{L}t} \), they have a reasonable commutation relation, suitable for our purpose, see Lemma 4.0.15. The advantage of working with this Sobolev space is that we get energy conservation in this case. Using this we can show that there is no finite time blow up in the defocussing case and also in the focusing case with \( 0 \leq \alpha < \frac{2}{n} \), which yields the global existence in the Sobolev space \( \tilde{W}^{1,2}_L(\mathbb{C}^n) \).

We have the following embedding theorem for the Sobolev space \( \tilde{W}^{1,p}_L(\mathbb{C}^n) \).

**Lemma 4.0.14 (Sobolev Embedding Theorem)** We have the continuous inclusion

\[ \tilde{W}^{1,p}_L(\mathbb{C}^n) \hookrightarrow L^p(\mathbb{C}^n) \]

for \( p_1 \leq p_2 \leq \frac{2mp_1}{2m-p} \) if \( p_1 < 2n \)

for \( p_1 \leq p_2 < \infty \) if \( p_1 = 2n \)

for \( p_1 \leq p_2 \leq \infty \) if \( p_1 > 2n \)

where \( 1 < p_1 < \infty \).

**Proof.** Let \( f \in \tilde{W}^{1,p_1}_L(\mathbb{C}^n) \) and \( \epsilon > 0 \). Consider \( u_\epsilon = e^{-i\mathcal{L}\epsilon} f \). Then \( u_\epsilon \in \tilde{W}^{1,p_1}_L(\mathbb{C}^n) \cap C^\infty(\mathbb{C}^n) \) and we have

\[ 2|u_\epsilon| \frac{\partial}{\partial x_j}|u_\epsilon| = \frac{\partial}{\partial x_j}(\overline{u_\epsilon}|u_\epsilon|) = 2\mathfrak{R}\left( \overline{u_\epsilon} \frac{\partial}{\partial x_j} u_\epsilon \right) = 2\mathfrak{R}\left( \overline{u_\epsilon} \left( \frac{\partial}{\partial x_j} - \frac{i\epsilon y_j}{2} \right) u_\epsilon \right). \]

Note that

\[ \frac{1}{2} (Z_j + \tilde{Z}_j) = -i \left( \frac{\partial}{\partial y_j} + \frac{i\epsilon y_j}{2} \right), \quad \frac{1}{2} (Z_j - \tilde{Z}_j) = \left( \frac{\partial}{\partial x_j} - \frac{i\epsilon y_j}{2} \right). \quad (4.0.3) \]

Hence on the set \( A_\epsilon = \{ z \in \mathbb{C}^n \mid u_\epsilon(z) \neq 0 \} \), we have

\[ \left| \frac{\partial}{\partial x_j}|u_\epsilon| \right| = \mathfrak{R}\left( \overline{|u_\epsilon|} \left( \frac{\partial}{\partial x_j} - \frac{i\epsilon y_j}{2} \right) u_\epsilon \right) \leq \frac{1}{2}(|Z_j u_\epsilon| + |\tilde{Z}_j u_\epsilon|). \]

Similarly

\[ \frac{\partial}{\partial y_j}|u_\epsilon| \leq \frac{1}{2}(|Z_j u_\epsilon| + |\tilde{Z}_j u_\epsilon|) \]

on \( A_\epsilon \). Note that \( \|u_\epsilon\|_{L^p(\mathbb{C}^n)} = \|u_\epsilon \chi_{A_\epsilon}\|_{L^p(\mathbb{C}^n)} \). By the usual Sobolev embedding on \( \mathbb{C}^n \) and above observations, we have inequality \( \|u_\epsilon\|_{L^p(\mathbb{C}^n)} \leq C\|u_\epsilon \chi_{A_\epsilon}\|_{W^{1,p}_L} \leq C\|u_\epsilon\|_{W^{1,p}_L} \). By Lemma 4.0.1, \( u_\epsilon = \)
\( e^{-i\mathcal{L}} f \rightarrow f \) in \( \tilde{W}^{1,p_1}_L(\mathbb{C}^n) \) and also in \( L^p(\mathbb{C}^n) \) as \( \epsilon \rightarrow 0 \). Therefore we have \( \|f\|_{L^p(\mathbb{C}^n)} \leq C\|f\|_{\tilde{W}^{1,p_1}_L(\mathbb{C}^n)} \), where constant \( C \) is a generic constant independent of \( f \). Hence Lemma is proved.

**Lemma 4.0.15** (Quasi commutativity) Let \( f \in S'(\mathbb{C}^n) \). Then for every \( t, t_0 \in \mathbb{R} \), we have the following equalities in \( S'(\mathbb{C}^n) \)

\[
\begin{align*}
Z_j e^{-i(t-t_0)\mathcal{L}} f &= e^{-2i(t-t_0)} e^{-i(t-t_0)\mathcal{L}} Z_j f \\
\overline{Z}_j e^{-i(t-t_0)\mathcal{L}} f &= e^{2i(t-t_0)} e^{-i(t-t_0)\mathcal{L}} \overline{Z}_j f.
\end{align*}
\]

**Proof.** Note that both \( Z_j e^{-i(t-t_0)\mathcal{L}} f \) and \( \overline{Z}_j e^{-i(t-t_0)\mathcal{L}} f \) are in \( S'(\mathbb{C}^n) \) for \( f \in S'(\mathbb{C}^n) \). Since every tempered distribution has a special Hermite expansion, enough to show the identities

\[
\langle Z_j e^{-i(t-t_0)\mathcal{L}} f, \Phi_{\mu,\nu} \rangle = e^{-2i(t-t_0)} \langle e^{-i(t-t_0)\mathcal{L}} Z_j f, \Phi_{\mu,\nu} \rangle \\
\langle \overline{Z}_j e^{-i(t-t_0)\mathcal{L}} f, \Phi_{\mu,\nu} \rangle = e^{2i(t-t_0)} \langle e^{-i(t-t_0)\mathcal{L}} \overline{Z}_j f, \Phi_{\mu,\nu} \rangle
\]

for every \( \mu, \nu \in (\mathbb{Z}_0)^n \).

Since \( Z_j \) and \( \overline{Z}_j \) are adjoint of each other, both identities in the Lemma can be easily verified using the relations

\[
\begin{align*}
e^{-i(t-t_0)\mathcal{L}} Z_j \Phi_{\mu,\nu} &= e^{2i(t-t_0)} Z_j e^{-i(t-t_0)\mathcal{L}} \Phi_{\mu,\nu} \quad (4.0.4) \\
e^{-i(t-t_0)\mathcal{L}} Z_j \Phi_{\mu,\nu} &= e^{-2i(t-t_0)} Z_j e^{-i(t-t_0)\mathcal{L}} \Phi_{\mu,\nu} \quad (4.0.5)
\end{align*}
\]

which follows from the relations (2.0.1) and the fact that \( e^{-i\mathcal{L}} \Phi_{\mu,\nu} = e^{-i(2k+n)} \Phi_{\mu,\nu} \).

**Lemma 4.0.16** (Quasi commutativity) Let \( t_0 \in \mathbb{R} \) and \( I \) an open interval such that \( t_0 \in I \). Let \( g \in L^q_{loc} \left( I, \tilde{W}^{1,p'}(\mathbb{C}^n) \right) \), where \((q,p)\) be an admissible pair. Then \( \int_{t_0}^t e^{-i(t-s)\mathcal{L}} g(z,s)ds \in C(I, \tilde{W}^{1,2}(\mathbb{C}^n)) \). Moreover for each \( t \in I \), we have the following equalities in \( L^2(\mathbb{C}^n) \)

\[
\begin{align*}
Z_j \int_{t_0}^t e^{-i(t-s)\mathcal{L}} g(z,s)ds &= e^{-2it} \int_{t_0}^t e^{-i(t-s)\mathcal{L}} e^{2is} Z_j g(z,s)ds \\
\overline{Z}_j \int_{t_0}^t e^{-i(t-s)\mathcal{L}} g(z,s)ds &= e^{2it} \int_{t_0}^t e^{-i(t-s)\mathcal{L}} e^{-2is} \overline{Z}_j g(z,s)ds.
\end{align*}
\]
Proof. Since $g \in L^q_{\text{loc}}(I, \tilde{W}^{1,p'}_e(C^n))$, by Strichartz estimates (Theorem 3.0.7) \[ \int_0^t e^{-i(t-s)\mathcal{L}} S g(z, s) ds \in C(I, L^2(C^n)) \], where $S = Z_j, \bar{Z}_j, 1 \leq j \leq n$. As discussed in Lemma 4.0.15, Lemma follows from the following observations

\[
\left\langle Z_j \int_0^t e^{-i(t-s)\mathcal{L}} g(z, s) ds, \Phi_{\mu, \nu} \right\rangle = e^{-2it} \left\langle \int_0^t e^{-i(t-s)\mathcal{L}} e^{2is} L_j g(z, s) ds, \Phi_{\mu, \nu} \right\rangle
\]

\[
\left\langle \bar{Z}_j \int_0^t e^{-i(t-s)\mathcal{L}} g(z, s) ds, \Phi_{\mu, \nu} \right\rangle = e^{2it} \left\langle \int_0^t e^{-i(t-s)\mathcal{L}} e^{-2is} M_j g(z, s) ds, \Phi_{\mu, \nu} \right\rangle
\]

for every $\mu, \nu \in (\mathbb{Z}_{\geq 0})^n$. These identities can be easily verified using the relations (4.0.4), (4.0.5).

Now we discuss the heat operator associated with the twisted Laplacian. For $\epsilon > 0$, consider the heat operator for the twisted Laplacian given by

\[ e^{-\epsilon \mathcal{L}} f = \sum_{k=0}^{\infty} e^{-\epsilon (2k+n)} P_k f \]

for $f \in L^2(\mathbb{C}^n)$. By orthogonality of the special Hermite functions $\Phi_{\mu, \nu}$, $P_k f$ are orthogonal projections. Hence it is clear that $e^{-\epsilon \mathcal{L}}$ is contraction on $L^2(\mathbb{C}^n)$.

\[ \|e^{-\epsilon \mathcal{L}} f\|_2^2 = \sum_{k=1}^{\infty} e^{-2\epsilon (2k+n)} \|P_k f\|_2^2. \]

The heat operator $e^{-\epsilon \mathcal{L}}$ has the following integral representation as a twisted convolution operator

\[ e^{-\epsilon \mathcal{L}} f = f \times K_\epsilon, \quad (4.0.6) \]

where

\[ K_\epsilon(z) = (2\pi)^{-n} e^{-n\epsilon} (1 - e^{-2\epsilon})^{-n} e^{-\frac{(1+e^{-2\epsilon})|z|^2}{2}} \quad (4.0.7) \]

see (2.10), (2.11) in [22]. Note that $\|K_\epsilon\|_{L^1(\mathbb{C}^n)} = 2^n e^{-n\epsilon} (1 + e^{-2\epsilon})^{-n} < 1$ and \[ \lim_{\epsilon \to 0} \|K_\epsilon\|_{L^1(\mathbb{C}^n)} = 1. \]

Lemma 4.0.17 For $\epsilon > 0$, $e^{-\epsilon \mathcal{L}} : L^p(\mathbb{C}^n) \to \tilde{W}^{m,p}_e(\mathbb{C}^n)$ defines a bounded operator for each nonnegative integer $m$ and $1 \leq p \leq \infty$. In particular we have the
following inequalities:

\[ \|e^{-\epsilon L} f\|_{L^p(\mathbb{C}^n)} \leq C \|f\|_{L^p(\mathbb{C}^n)} \]  
\[ \|e^{-\epsilon L} f\|_{\dot{W}^{1,p}_\epsilon(\mathbb{C}^n)} \leq C \|f\|_{\dot{W}^{1,p}_\epsilon(\mathbb{C}^n)} \]  
\[ \|e^{-\epsilon L} f\|_{W^{m,p}_\epsilon(\mathbb{C}^n)} \leq C \|f\|_{L^p(\mathbb{C}^n)} \]

with constant \( C \) in (4.0.8) and (4.0.9) is independent of \( \epsilon \in (0,1) \). Moreover, for \( f \in \dot{W}^{m,p}_\epsilon(\mathbb{C}^n) \), \( e^{-\epsilon L} f \rightarrow f \) in \( \dot{W}^{m,p}_\epsilon(\mathbb{C}^n) \), \( 1 < p < \infty \).

**Proof.** In view of (4.0.6) and the fact that \( |f \times g| \leq |f| \ast |g| \), we see that

\[ |e^{-\epsilon L} f| \leq |f| \ast K_\epsilon, \]  

where \( K_\epsilon \) is given by (4.0.7). Since

\[ \|K_\epsilon\|_{L^1(\mathbb{C}^n)} = 2^n e^{-n}(1 + e^{-2\epsilon})^{-n} \leq 1, \]

estimate (4.0.8) follows from Young’s inequality, see Folland [11] with \( C = 1 \).

As in Lemma 4.0.15 we see that \( e^{-\epsilon L} f, Z_j e^{-\epsilon L} f, \overline{Z}_j e^{-\epsilon L} f \in \mathcal{S}'(\mathbb{C}^n) \), for \( \epsilon > 0 \), for \( f \in \mathcal{S}'(\mathbb{C}^n) \), and the following equalities hold:

\[ Z_j e^{-\epsilon L} f = e^{2\epsilon} e^{-\epsilon L} Z_j f, \quad \overline{Z}_j e^{-\epsilon L} f = e^{-2\epsilon} e^{-\epsilon L} \overline{Z}_j f, \]  

hence the estimate (4.0.9) follows from the estimate (4.0.8). To prove (4.0.10), enough to prove

\[ \|\tilde{S} e^{-\epsilon L} f\|_{L^p(\mathbb{C}^n)} \leq C_\epsilon \|f\|_{L^p(\mathbb{C}^n)} \]

for any monomial \( \tilde{S} \) in \((Z_1, \ldots, Z_n, \overline{Z}_1, \ldots, \overline{Z}_n)\) of degree at most \( m \). In view of (4.0.6) and equation (1.3.10) in Thangavelu [33], we have

\[ \tilde{S} e^{-\epsilon L} f = \tilde{S}(f \times K_\epsilon) = f \times \tilde{S} K_\epsilon. \]

Since \( K_\epsilon \in \mathcal{S}(\mathbb{C}^n) \), \( \tilde{S} K_\epsilon \in L^1(\mathbb{C}^n) \), hence (4.0.10) follows by Young’s inequality.

To prove the convergence in \( \dot{W}^{m,p}_\epsilon(\mathbb{C}^n) \), we first observe that for \( f \in L^2 \), \( e^{-\epsilon L} f \rightarrow f \) in \( L^2(\mathbb{C}^n) \) as \( \epsilon \rightarrow 0 \). This follows from the identity

\[ \|e^{-\epsilon L} f - f\|_{L^2(\mathbb{C}^n)}^2 = \sum_{k=0}^{\infty} \|1 - e^{-\epsilon(2k+n)}\|^2 \|P_k f\|_{L^2(\mathbb{C}^n)}^2 \]
by an application of the dominated convergence theorem applied to the sum.

First we consider the simple case $m = 0$. In view of the uniform estimate \((4.0.8)\), enough to prove the convergence on a dense set in $L^p$. For $2 < p < \infty$, writing $\frac{1}{p} = \frac{\delta}{2} + \frac{1-\delta}{\infty} = \frac{\delta}{2}$ and an application of Hölder’s inequality using the estimate \((4.0.8)\), we see that

\[
\|e^{-\epsilon L} f - f\|_{L^p(C^n)} \leq \|e^{-\epsilon L} f - f\|^\beta_{L^2(C^n)} \|e^{-\epsilon L} f - f\|^{1-\beta}_{L^\infty(C^n)} \\
\leq \|e^{-\epsilon L} f - f\|^\beta_{L^2(C^n)} (2\|f\|_{L^\infty(C^n)})^{1-\beta}
\]

which goes to zero as $\epsilon \to 0$, for $f \in L^2 \cap L^\infty(C^n)$. Similarly we can prove convergence in $L^p(C^n)$ for $1 < p < 2$.

For $m \neq 0$, we need to show $\tilde{S}(e^{-\epsilon L} f - f) \to 0$ in $L^p(C^n)$ as $\epsilon \to 0$. But in view of \((4.0.12)\), we have

\[
\tilde{S}(Z, \bar{Z})(e^{-\epsilon L} f) = \tilde{S}(e^{2\epsilon}, e^{-2\epsilon}) e^{-\epsilon L} (\tilde{S}(Z, \bar{Z}) f).
\]

Hence applying the previous argument to the $L^p$ function $g = \tilde{S}(Z, \bar{Z}) f$, and observing that $\tilde{S}(e^{2\epsilon}, e^{-2\epsilon}) \to 1$, the result follows.