B

Unstructured compatible Lagrangian hydrodynamics scheme

B.1 Introduction

The compatible unstructured Lagrangian hydrodynamics algorithm used in Chapter 4 is a staggered mesh, finite-difference scheme for solving the equations of fluid dynamics in Lagrangian form. The compatible algorithm is formulated [98] in such a way that it exactly conserves momentum and internal energy. The unstructured mesh offers more flexibility especially when the geometry is complex. We have also included the sub-zonal pressures and masses [99] to control artificial grid distortion. Further, to stabilize the grid a node based tensor viscosity [100] is used. In the following we have described the details of this compatible unstructured Lagrangian hydrodynamics scheme.
B.2 Staggered unstructured mesh

Spatially staggered unstructured mesh is constructed as follows. The construction of the grid begins with arbitrary placement of nodes which define the geometry of interest. The velocities and the accelerations are defined at these nodes. In order to define computational cells or zones uniquely, a connectivity among the nodes which define arbitrary shaped polygons are specified, see Fig. B.1. The density, pressure and internal energy are defined at the zone centers. The sub-cell or sub-zone of each node is formed by connecting the zone centroid and edge centers. The sub-zonal cell will be always a quadrilateral regardless of the shape of the unstructured mesh, see Fig. B.1. With node positions, their connectivity and the cells defined, an unstructured mesh is completely specified. Following conventions are used: The cells are denoted by $z$, and nodes are indexed by $p$. The set of nodes that defines a cell $z$ are $p_2 S(z)$, where the nodes are ordered counterclockwise. Similarly, the set of cells that shares a node $p$ is denoted by $z_2 S(p)$.

B.3 Discrete Lagrangian hydrodynamics

In the Lagrangian framework, the equations of conservation of mass, momentum and energy are written

$$\frac{d\rho}{dt} = -\rho \nabla \cdot \mathbf{v} \quad (B.1)$$

$$\frac{d\mathbf{v}}{dt} = -\frac{\nabla P}{\rho} \quad (B.2)$$

$$\frac{\rho de}{dt} = P \nabla \cdot \mathbf{v} \quad (B.3)$$

where $\rho$ is density, $P$ is pressure, $\mathbf{v}$ is the velocity vector and $e$ is the internal
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![Diagram of a typical unstructured staggered mesh, including zone $z$ and point $p$. The solid lines define the mesh and the dashed lines show the sub-cell or median mesh. The sub-cell of each node is formed by connecting the zone centroid and edge centers.](image)

Figure B.1: A typical unstructured staggered mesh, including zone $z$ and point $p$. The solid lines define the mesh and the dashed lines show the sub-cell or median mesh. The sub-cell of each node is formed by connecting the zone centroid and edge centers.

Energy. In Lagrangian scheme, the mass of the cell is assumed to be constant. The same assumption is used for the sub-cell, leading to the definition of sub-zonal corner volume, mass and pressure. The cell volumes, $V_z$, are related to the sub-cell volumes, $V_{zp}$, by

$$V_z = \sum_{p \in S(z)} V_{zp} \tag{B.4}$$

Let us denote the mass of this corner volume as $m_{zp}$, where the indices define the zone and point with which it is associated. Also, define $m_{zp} = m_p$, with a convention that the summation in any mathematical expression is always with respect to the lower index.

The corner mass is used to define both the zone $m_z$ and point mass $m_p$. 
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\[ m_z = \sum_{p \in S(z)} m^z_p \quad \text{(B.5)} \]

\[ m_p = \sum_{z \in S(p)} m^p_z \]

Since, the mass of a cell is constant the discrete forms for mass conservation equation is as below.

\[ \rho_z = \frac{m_z}{V_z} \quad \text{(B.6)} \]

\[ \rho^*_p = \frac{m^*_p}{V^*_p} \]

The algorithm is written in terms of corner forces. A corner force is the contribution of one computational zone to the total force at a point or node. The discrete momentum equation at a node \( p \) in terms of corner force, \( f^*_p \), is as given below.

\[ m_p \frac{dV_p}{dt} = F_p = - \sum_{z \in S(p)} f^*_p \quad \text{(B.7)} \]

The corner force, \( f^*_p \), is the force from zone \( z \) that acts on point \( p \). The summation of all the corner forces around a point \( p \) gives \( F_p \). The corner force due to hydrostatic pressure is evaluated as

\[ f^*_p = P_z (s^z_{p+1} - s^z_{p}) \quad \text{(B.8)} \]

where \( \mathbf{s} \) is the area vector defined in Fig. B.1. The change in mass specific internal energy can be calculated as below [98].
\[ m_z \frac{de_z}{dt} = - \sum_{p \in S(z)} f_p^z \cdot \mathbf{v}_p \] (B.9)

The derivation of the above equation and proof of energy conservation can be found in Ref. [98]. Since the conservation of total energy is used to derive the above equation, it ensures total energy is conserved to numerical round-off in calculations.

### B.4 Sub-zonal pressure method

The sub-zonal pressure method [99] is developed to control artificial grid distortion. The difference between the sub-zonal and the zonal density and hence in the pressure (assuming same specific internal energy for sub-zone and zone) is used to formulate a corner force which oppose the hourglass motion. The details of this formulation and physical backgrounds are described in Ref. [99]. For each sub-zone the pressure difference between the zone and sub-zone, \( \delta P_p^z \), is calculated using the following equation.

\[ \delta P_p^z = \left( \rho_p^z - \rho_z \right) c_z^2 \] (B.10)

Here, \( c_z \) is the zone sound speed. The corner force due to these pressure difference is then calculated by integrating around the boundary of the sub-zonal volume.

\[ f_{p,h}^z = C_h \left( \delta P_p^z \left( a_p^+ + a_p^- \right) + \frac{1}{2} \left( \left( \delta P_p^z - \delta P_{p+1}^z \right) s_{p+1}^z + \left( \delta P_{p-1}^z - \delta P_p^z \right) s_p^z \right) \right) \] (B.11)

Here, \( 0 < C_h \leq 1 \) is a dimensionless factor and the \( \mathbf{a} \) vectors are outward edge normals to the zone, see Fig. B.1. This sub-zonal corner force, \( f_{p,h}^z \), is added to
corner force, $f^z_p$.

B.5 Tensor artificial viscosity

To resolve shocks over just a few zones, the tensor artificial viscosity of Campbell and Shashkov [100] is included in the hydrodynamic equations. It assumes that the artificial viscosity tensor is a combination of a scalar coefficient and the gradient of the velocity tensor, $\nabla v$. The sub-zonal artificial viscous force, $f^z_{p,\nu}$, thus calculated is added to the corner force, $f^z_p$. The details of the algorithm can be found in Ref. [100].