CHAPTER - I
INTRODUCTION

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1.1 Continuity And Measurability

Sets and functions 1.1.1: A well-defined collection of objects is called a set and a member of the set is called the element. A null set is denoted by $\phi$. A member $x$ of the set $X$ is expressed as $x \in X$ and if $x$ is not $X$, we write $x \not\in X$. A subset $A$ of $X$ is denoted $A \subset X$ and if $A \subset X$, then the complement $A'$ of $A$ in $X$ is denoted by $X - A$ or $X \setminus A$ and $A' = \{x \in X \mid x \not\in A\}$. Some important sets are given below which are frequently used in the subsequent part of the thesis without mentioning anything in detail.

\[
\begin{align*}
\mathbb{R} & = \text{The set of real numbers,} \\
\mathbb{N} & = \text{The set of natural numbers,} \\
\mathbb{C} & = \text{The set of complex numbers,} \\
\mathbb{Z} & = \text{The set of integers,} \\
\mathbb{Q} & = \text{The set of rational numbers.}
\end{align*}
\]

Let $X$ be a non-empty set and let $A, B \subset X$. Then three set-operations union ($\cup$), intersection ($\cap$) and difference ($\setminus$) are defined by

\[
\begin{align*}
A \cup B & = \{x \in X \mid x \in A \text{ or } x \in B\}, \\
A \cap B & = \{x \in X \mid x \in A \text{ and } x \in B\}, \quad \text{and} \\
A \setminus B & = \{x \in A \mid x \not\in B\}.
\end{align*}
\]
Two important laws (De-Morgan’s laws) for set-operations are

(i) \[ \left( \bigcup_{\alpha} A_{\alpha} \right)' = \bigcap_{\alpha} A'_{\alpha} \quad \text{and} \]

(ii) \[ \left( \bigcap_{\alpha} A_{\alpha} \right)' = \bigcup_{\alpha} A'_{\alpha}, \]

where \( \{A_{\alpha}\} \) is a family of sets in \( X \).

A set is called finite if it contains a finite number of elements and it is called countable if there is a one-to-one correspondence between the elements of \( A \) with the set of natural numbers.

Let \( X \) and \( Y \) be two sets. A subset \( R \) of the Cartesian product \( X \times Y = \{(x, y) \mid x \in X \text{ and } y \in Y\} \) is called a relation. If \( (a, b) \in R \), then we say \( a \) is related or associated to \( b \) under the relation \( R \). In this case we write \( aRb \). A relation \( f \) is called function if every element of \( X \) is associated to a unique element of \( Y \) under the relation \( f \). If \( (x, y) \in f \), then in this case we write \( y = f(x) \). Then set \( X \) is called domain and the set \( Y \) is called the codomain of the function \( f \). A function \( f \) form \( X \) to the set \( Y \) is expressed as \( f : X \rightarrow Y \). The set of all elements in \( Y \) which are associated with the elements of \( X \) under the function \( f \) is called the range of \( f \) and is the image set \( f(X) \). The function \( f \) is called one-one if \( f(x_1) = f(x_2) \Rightarrow x_1 = x_2 \) and is called
onto if \( f(X) = Y \). If \( f \) is one-one and onto, it is called bijective. If \( y = f(x) \). Then \( x \in f^{-1}(y) \) and \( f^{-1} \) is called the inverse of \( f \). If \( f \) is onto and one-to-one, then for given \( y \in Y \) there is precisely one element \( x \in X \) such that \( x = f^{-1}(y) \). If \( f : X \to Y \) and \( g : Y \to Z \) be two functions, then their composition \( h = gof \) is a map from \( X \) to \( Z \) defined by \((gof)(x) = g(f(x))\).

**Continuity of functions 1.1.2** : A non-empty set together with a collection \( \tau \) of subsets of \( X \) is called a topological space, denoted by a pair \((X, \tau)\), it \( \tau \) satisfies the following properties :

(i) \( \phi, X \in \tau \),

(ii) \( \tau \) is closed under arbitrary union of members of \( \tau \), and

(iii) \( \tau \) is closed under finite intersection of members of \( \tau \).

The collection \( \tau \) is called a topology on \( X \) and the members of \( \tau \) are called open sets of \( X \).

**Example 1.1.3** : \((\Theta, Y)\) is a topological space, where \( Y \) is a standard topology on \( \Theta \) generated by the open intervals of \( \Theta \).

**Example 1.1.4** : A metric space \((X, d)\) is a topological space and the collection \( \{B_r(x) : x \in X\} \) of open ball generates a topology on \( X \), where open ball \( B_r(x) \) is defined by \( B_r(x) = \{y \in X \mid d(x, y) < r\} \) for
some real number $r > 0$. Metric spaces are very much useful for studying the uniform continuity of the functions.

**Definition 1.1.5** : Let $(X, \tau_1)$ and $(Y, \tau_2)$ be two topological spaces. A mapping $f : X \rightarrow Y$ is said to be continuous at $x \in X$ if for every open set $V$ in $Y$ containing $f(x)$, there is an open set $U$ such that $x \in U \subset F^{-1}(V)$. In general, $f$ is called continuous if inverse image $f^{-1}(V)$ of every open set in $Y$ is open in $X$.

**Example 1.1.6** : Let $(X, d_1)$ and $(Y, d_2)$ be two metric spaces. A function $f : X \rightarrow Y$ is said to be continuous at a point $a \in X$ if for $\varepsilon > 0$ there is a $\delta > 0$ such that $d_2(f(x), f(a)) < \varepsilon$ whenever $d_1(x, a) < \delta$ and which is further equivalent to $f(B_\delta(a)) \subset B_\varepsilon(f(a))$.

The following result is very much useful in determining the continuity of mapping in abstract spaces.

**Proposition 1.1.7** : Let $(X, d_1)$ and $(Y, d_2)$ be two metric spaces. A mapping $f : X \rightarrow X$ is continuous at a point $x \in X$ iff the sequence \{\(f_{x_n}\)\} converges to $f(x)$ in $Y$, whenever \{\(x_n\)\} converges to $x$ in $X$.

A topological space $(X, \tau)$ is called compact if every open cover for $X$ has a finite subcover. A subset $A$ of $X$ is called totally bounded or relatively compact if $\overline{A}$ is compact in the relative...
topology of $A$ in $X$. The following result is sometimes useful in the subsequent part of the thesis.

**Proposition 1.1.8**: Let $(X, d)$ be a compact metric space. If a function $f : X \to X$ is continuous then it is uniformly continuous on $X$ and attains its supremum or infimum on $X$.

Note that every uniformly continuous function is continuous but the converse may not be true. However, both the notions coincide on compact subsets of topological spaces.

**Measurability of functions 1.1.9**: Let $\Omega$ be a non-empty set. A collection of subsets $A$ of $\Omega$ is called algebra if

(i) $\emptyset, X \in A$

(ii) $\bigcup A_\alpha \in A$, whenever $A_\alpha \in A$, and

(iii) if $A \in A$, then $A' \in A$,

A algebra $A$ is called $\sigma$-algebra if the condition (ii) is replaced by

(ii) $\bigcup_{i=1}^{\infty} A_i \in A$ for any countable sets $A_i$. in $A$

A non-empty set $\Omega$ together with a $\sigma$ is called a measurable space, and it is denoted by $(\Omega, A)$. Notice that a $\sigma$-algebra of subsets of $\Omega$ is closed under set-operations, viz, $\bigcup$, $\bigcap$ and $\sim$. 

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A function \( \mu : A \to \mathbb{R}^+ \cup \{\infty\} \) is called a measure if

(i) \( \mu(\emptyset) = 0 \)

(ii) \( A \subseteq B \Rightarrow \mu(A) \leq \mu(B) \), and

(iii) if \( \{A_i\} \) are disjoint countable sets,

Then \( \mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i) \).

A measure \( \mu \) is called finite if it assume finite values on \( \Omega \), i.e. \( \mu(A) < \infty \) for each \( A \subset A \). \( \mu \) is called complete, if \( \mu(A_0) = 0 \) for some \( A_0 \in A \) then \( \mu(A) = 0 \) for every \( A \subset A_0 \). A triplet \((\Omega, A, \mu)\) is called a measure space.

**Borel sets 1.1.10**: Let \( X \) be Banach space. Then the smallest \( \sigma \)-algebra of all open subsets of \( X \) is called the Borel algebra of subsets of \( X \). A member of \( \sigma \)-algebra \( \beta_X \) of \( X \) is called a Borel set in \( X \). In other words, \( \sigma \)-algebra of closed sets in \( X \) is a Borel algebra is called a Borel set in \( X \). In particular, \( G_\delta \) and \( F_\sigma \) are the examples of the Borel sets in the topological space \( X \), where \( G_\delta \) is the intersection of a countable intersection of open subsets of \( X \) and \( F_\sigma \) is the union of countable closed sets in \( X \).
1. Derivatives and Integration

**Derivatives 1.2.1**: Let \( f : \mathbb{R} \to \mathbb{R} \) be a continuous function. In order to define the derivatives of the function \( f \), we need to define the following four quantities called the Dini derivatives of \( f \) on \( \mathbb{R} \) For any \( x \in \mathbb{R} \), denote

\[
D^+ f(x) = \lim_{h \to 0^+} \frac{f(x + h) - f(x)}{h} \\
D^- f(x) = \lim_{h \to 0^-} \frac{f(x) - f(x - h)}{h} \\
D_+ f(x) = \lim_{h \to 0^+} \frac{f(x + h) - f(x)}{h} \\
D_- f(x) = \lim_{h \to 0^-} \frac{f(x) - f(x - h)}{h}
\]

Clearly, we have \( D^+ f(x) \geq D_+ f(x) \) and \( D^- f(x) \geq D_- f(x) \) for all \( x \in \mathbb{R} \). If \( D^+ f(x) = D_+ f(x) = D^- f(x) = D_- f(x) \neq \pm \infty \), we say that \( f \) is differentiable at \( x \) and define \( f'(x) \) to be the common value of the Dini-derivatives at \( x \) if \( D^+ f(x) = D_+ f(x) \), we say that \( f \) has right hand derivative at \( x \), and define \( f'(x^+) \) to be their common value. Similarly, the left hand derivative \( f'(x^-) \) is defined.
A real-valued function $f$ defined on the closed and bounded interval $[a, b]$ of $\mathbb{R}$ is said to be absolutely continuous on $[a, b]$ if for given $\varepsilon > 0$ there is a $\delta > 0$ such that
\[
\sum_{i=1}^{n} |f(x_i) - f(x_{i+1})| < \varepsilon
\]
for every finite collection $\{(x_i, x_{i+1})\}$ of intervals of $[a, b]$ with
\[
\sum_{i=1}^{n} |x_i - x_{i+1}| < \delta.
\]

An absolutely continuous function is continuous and the following result has been frequently used in the rest part of the present work.

**Proposition 1.2.2** : Let the function $f : [a, b] \to \mathbb{R}$ is absolutely continuous, then $f$ has a derivative almost everywhere on $[a, b]$.

**Integration 1.2.3** : Let $(\Omega, \mathcal{A}, \mu)$ be a complete $\sigma$-finite measure space and Let $X$ be a Banach space. Let $\mathcal{B}_X$ be the Borel algebra of $X$. A function $f : \Omega \to X$ is called measurable if
\[
f^{-1}(B) = \{ \omega \in \Omega \mid f(\omega) \in B \} \in \mathcal{A}
\]
for all open subsets $B$ of $X$. If $\Omega = X$. Then $f$ is measurable if the set $\{ x \mid f(x) > \alpha \}$ or $\{ x \mid \delta(x) < \alpha \}$ is open for every $\alpha \in \mathbb{x}$. 
The function $\chi_A$ defined by

$$
\chi_A(x) = \begin{cases} 
1, & \text{if } x \in A \\
0, & \text{if } x \notin A 
\end{cases}
$$

is called the characteristic function of $A \in \mathcal{A}$. A linear combination

$$
\phi(x) = \sum_{i=1}^{n} a_i \chi_{A_i}
$$

is called simple function if $A_i \in \mathcal{A} \ \forall \ i, \ i = 1,2,...n$. In other words, $\phi$ is simple if it is measurable and assumes finite number of values on $\Omega$.

The integral of the simple function $\phi$ on $\Omega$ is defined by

$$
\int \phi(x)dx = \sum_{i=1}^{n} a_i \chi_{A_i} \mu(A_i).
$$

**Definition 1.2.4:** If $f$ is a bounded and measurable function defined on a measurable set $A$ with finite measure, then we define the integral of the function $f$ as

$$
\int_{A} f(x)d\mu = \inf_{A} \int_{A} \psi(x)d\mu
$$

for all simple function $\psi \geq f$.

The following results are frequently used in the subsequent part of the thesis.
Proposition 1.2.5 : If \( f : [a, b] \to \mathbb{R} \) is a bounded and Lebesgue integrable function such that \( F'(x) = f(t) \) a.e. \( t \in [a, b] \), then

\[
F(x) = F(a) + \int_a^x f(t) dt.
\]

Theorem 1.2.6 : (Dominated convergence theorem) Let \( \{f_n\} \) be a sequence of measurable functions on a measurable set \( A \). If for all \( t \in A, \; |f_n(t)| \leq f(t) \) for some measurable function \( f \) on \( A \) with \( \int_A f dm < \infty \) and \( f_n(t) \to f(t) \) for all \( t \in A \), then

\[
\int_A f_n dm \to \int_A f dm.
\]

Theorem 1.2.7 : (Monotone convergence theorem) Let \( \{f_n\} \) be an increasing sequence of nonnegative measurable functions and let \( f = \lim_{n \to \infty} f_n \) a.e. Then

\[
\lim_{n \to \infty} \int_A f_n(x) dx = \int_A f(x) dx.
\]

1.3 Function Spaces

Banach space 1.3.1 : A non-empty set \( V \) together with two binary operation \( + : V \times V \to V \) and \( \cdot : F \times V \to V \) is called a vector space if
(i) \((V, +)\) is an abelian group,

(ii) \(\alpha(x + y) = \alpha x + \alpha y\),

(iii) \((\alpha + \beta)x = \alpha x + \beta x\),

(iv) \(1 \cdot x = x\)

for all \(x, y \in V\) and \(\alpha, \beta \in F\), where the field \(F\) is \(F = \mathbb{R}\) or \(F = \mathbb{C}\). A function \(\| \cdot \| : V \rightarrow \mathbb{R}^+\) is called a norm if it satisfies

(i) \(\| x \| \geq 0 \forall x \in V\),

(ii) \(\|\lambda x\| = |\lambda| \| x \|\) for \(\lambda \in \mathbb{R}\), and

(iii) \(\| x + y \| \leq \| x \| + \| y \|\) for all \(x, y \in V\).

The vector space \(V\) together with the function \(\| \cdot \|\) is called a normed linear space, and we denote it by \((V, \| \cdot \|)\). If \(V\) is complete through the metric \(d : V \times V \rightarrow \mathbb{R}^+\) defined by \(d(x, y) = \| x - y \|, x, y \in V\),

Then \(V\) is called a complete normed linear space or simply Banach space.

\textbf{Example 1.3.2} : Let \(J = [a, b]\) be a closed and bounded interval in \(\mathbb{R}\) and let \(C(J, \mathbb{R})\) be the class of real-valued continuous functions on \(J\). Then \(C(J, \mathbb{R})\) is a vector space with respect to the addition “\(+\)” and the scalar multiplication “\(\cdot\)” defined by \((x + y)(t) = x(t) + y(t)\) and \((\lambda x)(t) = \lambda x(t)\) for all \(t \in J\). It is also known that the \(C(J, \mathbb{R})\) is a Banach space with respect to the norm \(\| \cdot \|\) defined by
\[ \| x \| = \sup_{t \in J} |x(t)|. \] The above norm is called supremum norm on \( C(J, \mathbb{R}) \).

**Example 1.3.3:** Let \( L^p(J, \mathbb{R}) \) \((p \geq 1)\) denote class of real-valued functions such that \( |x(t)|^p \) is Lebesgue integrable on \( J \). It is known that \( L^p(J, \mathbb{R}) \) is a Banach space with respect to the norm \( \| \cdot \|_{L^p} \) defined by

\[
\| x \|_{L^p} = \left( \int_a^b |x(t)|^p \right)^{\frac{1}{p}}.
\]

We frequently use the space \( L^1(J, \mathbb{R}) \) of Lebesgue integrable real-valued functions on \( J \) which is again a Banach space with respect to the norm \( \| \cdot \|_{L^1} \) given by

\[
\| X \|_{L^1} = \int_a^b |x(t)| \, dt.
\]

**Banach algebra 1.3.4:** Let \( X \) be a Banach space and let “ \( \cdot \) ” be the multiplication in \( V \), i.e. \( \cdot : V \times V \to V \) and satisfies

(i) \( x(yz) = (xy)z \) for all \( x, y, z \in V \) and

(ii) \( \| xy \| \leq \| x \| \| y \| \) for all \( x, y \in X \). Then \( X \) is called a Banach algebra.
Clearly the space $C(J, \mathbb{R})$ of continuous real-valued functions on $J$ is a Banach algebra with respect to the supremum norm $|| \cdot ||$ and the multiplication “$\cdot$” in $C(J, \mathbb{R})$ defined by

$$(x \cdot y)(t) = x(t) \cdot y(t), \quad t \in J.$$ 

**Compactness in function spaces 1.3.5:** Compactness of the abstract space is next to the finite set which is very much useful for dealing with the many problems of mathematical analysis. There is a lot of discussion on the noncompactness of a bounded set $A$ in the Banach space $X$. A Kuratowski measure of noncompactness of a bounded set $A$ in the Banach space $X$ is a nonnegative real number $\alpha(A)$ defined by

$$\alpha(A) = \inf \left\{ r > 0 : A = \bigcup_{i=1}^{n} A_i, \ diam(A_i) \leq r \right\}$$

Clearly $\alpha(A) = 0 \iff \overline{A}$ is compact and $\alpha(\overline{A}) = \alpha(\overline{co}(A)) = \alpha(A)$, where $\overline{A}$ and $\overline{co}(A)$ denote respectively the closure and closed convex hull of $A$ in $X$. The exhaustive discussion on measures of noncompactness is given in Granas et.al. (2008)

Let $E$ be the Banach space and Let $C(J,E)$ be the continuous $E$-valued functions on closed and bounded interval $J = [a,b]$ in $\mathbb{R}$. Then we have the following important results concerning the
noncompactness of bounded sets in $C(J, E)$ which are useful in the study of nonlinear differential and integral equations.

**Proposition 1.3.6 :** For any bounded and equi-continuous set $A$ in $C(J, E)$, one has $\alpha(A) = \sup_{t \in [0, T]} \alpha(A(t)) = \alpha(A([0, T]))$.

When $E = \mathbb{R}$, we have the following results for the compactness of a set is the space $C(J, \mathbb{R})$ and $L^p(J, \mathbb{R})$.

**Theorem 1.3.7 :** (Arzela-Ascoli theorem) If every uniformly bounded and equicontinuous sequence $\{f_n\}$ of functions in $C(J, \mathbb{R})$, then it has a convergent subsequence.

**Theorem 1.3.8 :** A metric space $X$ is compact iff every sequence in $X$ has a convergent subsequence.

**Theorem 1.3.9 :** (M Riesz theorem) For a family of functions $K = \{x_\lambda(t)\} \subset L_p[0,1]$ to be compact, it is necessary and sufficient that these functions are uniformly bounded in the norm and uniformly continuous in mean, namely that

(i) $\int_0^1 |x_\lambda(t)|^p dt \leq c^p$, and

(ii) $\int_0^1 |x_\lambda(t + h) - x_\lambda(t)|^p < \varepsilon^p$ for $0 < h < \delta(\varepsilon)$ simultaneously for all the functions $x_\lambda$ of the family $K$. 

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1.4 Operator Theory

Let $X$ be a Banach space. Then a mapping $T : X \rightarrow X$ is called a operator and functional equation $T(x) = x$ is called an operator equation. The problem of existence of the solutions to such operator equations in abstract spaces is of great interest since long time. A mathematical statement asserting the existence of solutions to the operator equations is called the fixed point theorem and the solutions themselves are called a fixed points of $T$ in $X$. Such fixed-point results are obtained by imposing certain condition either on $f$ or on $X$ or on both $f$ and $X$. These fixed point results are useful for proving the existence theorems for various nonlinear problems of the universe modeled on differential and integral equations, called the dynamical systems. Below we discuss some of the results in this direction.

**Fixed point theory 1.4.1:** As has been stated that the topic of fixed point theory provides powerful tools in the study of nonlinear equations, it is classified into three main categories, viz, algebraic, geometrical and topological fixed point theory. The above classification is not rigid, but based on the major hypotheses involved in the fixed point theorems in abstract spaces.

In the following, we state three basic theorems from the above categories useful for applications to allied areas of mathematics.
**Definition 1.4.2:** Let $X$ be a Banach space. A mapping $T : X \to X$ is called Lipschitz if there is a constant $\alpha > 0$ such that $\|Tx - Ty\| \leq \alpha \|x - y\|$ for all $x, y \in X$. If $\alpha < 1$, then $T$ is called a contraction on $X$ with the contraction constant $\alpha$.

**Theorem 1.4.3:** (Banach, 2002) Let $T$ be a contraction on a Banach space $X$ into itself. Then $T$ has a unique fixed point.

Before stating a basic topological fixed-point theorem, we need the following preliminaries.

Let $X$ be a Banach space and let $T : X \to X$, $T$ is called compact if $\overline{T(X)}$ is a compact subset of $X$. $T$ is called totally bounded if $\overline{T(S)}$ is a compact subset of $X$ for every bounded subset $S$ of $X$. $T$ is called completely continuous if it is continuous and totally bounded operator on $X$. It is clear that every compact operator is totally bounded but the converse need not be true.

**Theorem 1.4.4:** (Schauder) Let $S$ be a closed, convex and bounded subset of a Banach space $X$ and let $T : S \to X$ be a completely continuous operator. Then $T$ has a fixed point.

A non-empty closed subset $K$ of the Banach space $X$ is called cone if it satisfies the following properties.

(i) $K + K \subseteq K$,

(ii) $\lambda K \subseteq K$ for $\lambda \in \mathbb{R}^+$, and
(iii) \( K \cap \{-K\} = \{0\} \), where 0 is a zero element of \( X \).

We define an order relation \( \leq \) in \( X \) with the help of the cone \( K \) in \( X \) as follows. Let \( x, y \in X \). Then \( x \leq y \iff y - x \in K \). For any \( a, b \in X \), \( a \leq b \), the order interval \([a, b]\) to be a set in \( X \) defined by

\[
[a, b] = \{ x \in X \mid a \leq x \leq b \}.
\]

**Definition 1.4.5** : Let \( X \) be an ordered Banach space. A mapping \( Q : X \to X \) is called non-decreasing if \( Qx \leq Qy \) for all \( x, y \in X \) for which \( x \leq y \).

Now we are in a position to state a final algebraic fixed-point theorem useful for applications to differential and integral equations.

**Theorem 1.4.6** : (Heikkila et.al., 1994) Let \([a, b]\) be an order interval in a subset \( Y \) of the ordered Banach space \( X \) and Let \( Q : [a, b] \to [a, b] \) be nondecreasing mapping. If every sequence \( \{Qx_n\} \leq Q([a, b]) \) converges in \( X \), whenever \( \{x_n\} \) is a nondecreasing sequence in \([a, b]\), then \( Q \) has the least and greatest fixed point in \([a, b]\).

**Hybrid fixed point theory 1.4.7** : A fixed point theorem involving the mixed hypotheses from algebra, geometry and topology is called a hybrid fixed point theorem in abstract spaces. The study of hybrid fixed point theory is initiated by a Russian mathematician.
Krasnoselskii in the year 1964 and proved a fixed point theorem which combines the Banach fixed point theorem together with the Schauder fixed point theorem on subsets of a Banach space.

**Theorem 1.4.8** (Krasnoselskii, 1964) Let S be a closed convex and bounded subset of the Banach space X. Let \( A : X \rightarrow X \) and \( B : S \rightarrow X \) be two operators such that

(a) A is a contraction,

(b) \( B \) is completely continuous and

(c) \( Ax + By \in S \) for all \( x, y \in S \).

Then the operator equation \( Ax + Bx = x \) has a solution in \( S \).

Other two hybrid fixed point theorems involving two and three operators in a Banach algebras are as follows.

**Theorem 1.4.9** (Dhage, 2005-e) Let S be a closed convex and bounded subset of a Banach algebra \( X \) and let \( A : X \rightarrow X \) and \( B : S \rightarrow X \) be two operators such that

(a) A is Lipschitz with the Lipschitz constant \( \alpha \),

(b) \( B \) is completely continuous, and

(c) \( AxBy \in S \) for all \( x, y \in S \).
Then the operator equation $Ax Bx = x$ has a solution in $S$, whenever $\alpha M < 1$, where $M = \|B(S)\| = \sup \{ \|Bx\| : x \in S \}$.

**Theorem 1.4.10**: (Dhage, 2004-c) Let $S$ be a closed convex and bounded subset of the Banach algebra $X$ and let $A, C : X \to X$ and $B : S \to X$ be three operators satisfying

(a) $A$ and $C$ are Lipschitz with the Lipschitz constants $\alpha$ and $\beta$ respectively,

(b) $B$ is a completely continuous, and

(c) $Ax By + Cx \in S$ for all $x, y \in S$.

Then the operator equation $Ax Bx + Cx = x$ has a solution in $S$, whenever $\alpha M + \beta < 1$, where $M = \|B(S)\| = \sup \{ \|Bx\| : x \in X \}$.

We use some variants of the above two fixed point theorems in Banach algebras for proving the existence results for differential or integral equations in Banach algebras.

### 1.5 Differential And Integral Equations

**In Banach Algebras.**

The study of nonlinear differential equations is initiated by Dhage and Regan in the year 2001 and developed further in a series of papers by first authors and others (see Dhage [ , ] and the references...
therein). Let \( J = [0, T] \) be a closed and bounded interval in \( \mathbb{R} \). Then the general form of the nonlinear ordinary differential equations in Banach algebra is

\[
L \left( \frac{x(t)}{f(t, x(t))} \right) = g(t, x(t)) \quad \text{a.e.} \quad t \in J \tag{1.5.1}
\]

subject to either some initial condition

\[
x \in J. \tag{1.5.2}
\]

or some boundary condition

\[
x \in B \tag{1.5.3}
\]

where, the function \( f : J \times \mathbb{R} \rightarrow \mathbb{R} \setminus \{0\} \) is continuous,

\( L : C^n(J, \mathbb{R}) \rightarrow L^1(J, \mathbb{R}) \) is a \( n^{th} \) order linear differential operator defined by

\[
L = a_0 \frac{d^n}{dt^n} + a_1 \frac{d^{n-1}}{dt^{n-1}} + \ldots + a_n
\]

for some continuous functions \( a_i : J \rightarrow \mathbb{R} \); and \( N : C(J, \mathbb{R}) \rightarrow L^1(J, \mathbb{R}) \) is a Nemetsky operator defined by

\[
Nx = g \left( t, \frac{dx}{dt}, \ldots, \frac{d^{n-1}x}{dt^{n-1}} \right)
\]
The problem (1.5.1) – (1.5.2) is called \( n^{th} \) order initial value problem of ordinary nonlinear differential equations and the problem (1.5.1) – (1.5.3) is called \( n^{th} \) order boundary value problem of ordinary nonlinear differential equations in Banach algebras.

The first order initial value problem of ordinary nonlinear differential equations,

\[
\begin{aligned}
\frac{d}{dt} \left[ \frac{x(t)}{f(t,x(t))} \right] &= g(t,x(t)) \quad \text{a.e.} \quad t \in J \\
x(0) &= x_0 \in \mathbb{R}
\end{aligned}
\]

(1.5.4)

has been studied in Dhage and Regan, (2000) for existence results under suitable conditions. Similarly, periodic boundary value problem of first order ordinary nonlinear differential equations,

\[
\begin{aligned}
\frac{d}{dt} \left[ \frac{x(t)}{f(t,x(t))} \right] &= g(t,x(t)) \quad \text{a.e.} \quad t \in J \\
x(0) &= x(T)
\end{aligned}
\]

(1.5.5)

is discussed in Dhage et. al. (2007-b) via Caratheodory and monotone theory for nonlinear differential equations. Further, the initial value problems of ordinary second order nonlinear differential equations
\[
\frac{d^2}{dt^2} \left[ \frac{x(t)}{f(t,x(t))} \right] = g(t,x(t)) \quad \text{a.e. } t \in J \\
x(0) = x_0, \quad x'(0) = x_1
\]

is also discussed in Dhage (2006-g) for existence as well as for existence of extremal solution under certain Caratheodory and monotonicity conditions.

In the present work we will discuss some nonlinear differential equations in Banach algebras involving initial as well as boundary conditions which extend and generalize earlier known existence results for the differential equations (1.5.4), (1.5.5) and (1.5.6).

**Nonlinear integral equations in Banach algebras 1.5.7:**
The origin of the nonlinear integral equations lies in the work of Chandrasekhar (1960) on radiative heat transfer in thermodynamics. Nowadays it has become clear that such quadratic integral equation occur in the several phenomena of the universe, viz, queuing theory and population dynamics etc. The previous methods for proving the existence results for such equations were much cumbersome, so this topic is not developed much during the initial stage of investigations. But, since the formulation of functional analytic methods, in particular, fixed point theory in Banach algebras, there is a considerable development of the nonlinear integral equations in recent
years (see Dahge (2001, 2004-c) and the references therein). The quadratic nonlinear integral equation of the form

\[ x(t) = \left[ f\left(t, x(t)\right) \right] \left( \int_0^t g\left(s, x(s)\right) ds \right), \quad t \in J \quad \text{(1.5.8)} \]

is discussed in Dhage (1994) under some hybrid conditions of geometry and topology for the existence of solutions. Similarly, nonlinear quadratic functional integral equation,

\[ x(t) = \left[ f\left(t, x(\theta(t))\right) \right] \left( q(t) + \int_0^{\sigma(t)} g\left(s, x(\eta(s))\right) ds \right) \quad \text{(1.5.9)} \]

is discussed in Dhage and Regan (2000) for the existence of the solutions under some mixed conditions of geometry and topology.

In the present work, we shall discuss a more general nonlinear integral equation in Banach algebras for existence as well as for existence of the extremal solutions which includes the existence results for the quadratic integral equations (1.5.8) and (1.5.9) as special cases.

▽ ▽ ▽ ▽