Chapter :- 3
TOTAL k-DOMINATION, K-TUPLE DOMINATION AND K-DEPENDENT K-DOMINATION
In this chapter we consider the notions of total k–domination, k-tuple domination and k-dependent k-domination for graphs. (k ≥ 2) It may be noted that if a graph has a vertex of degree less than k then there does not exist a totally k-dominating set in the graph. Similarly if a graph has a vertex of degree less than k - 1 then a k-tuple dominating set does not exist. In this chapter we consider and characterize those vertices whose removal increases or decreases total k-domination number of the graph. We prove similar result for k-tuple domination and k-dependent k-domination.

**TOTAL k-DOMINATION**

In this section we introduced a totally k-dominating sets. We prove theorems similar to those of domination.

**Definition-3.1: Totally k-dominating set.**

Let k be an integer k ≥ 1. Let G be a graph and S ⊂ V(G). The set S is said to be totally k-dominating set if for every vertex v ∈ V(G), v is adjacent to at least k vertices of S.

Note that every totally k–dominating set is a k-dominating set. However the converse is not true.

**Example-3.2:**

Consider the above graph G with vertices 1, 2, 3, 4. Let S = {1, 3} if k = 2 then S is a 2–dominating set but it is not a totally 2–dominating set.
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Remark-3.3:
Note that if a graph G contains a vertex v with degree less than k then no subset of V(G) can be totally k-dominating set. (Although it may be k-dominating set.)
k-dominating set: A set S is k-dominating set if for every vertex v ∈ V(G) - S, v is adjacent to at least k vertices of S. i.e. |N(v) ∩ S| ≥ k.

Definition -3.4: Minimal totally k-dominating set.
Let S be a totally k-dominating set then S is said to be minimal totally k-dominating set if for every vertex v in S, S-{v} is not a totally k-dominating set.

Definition -3.5: Minimum totally k-dominating set.
A totally k-dominating set with minimum cardinality is called a minimum totally k-dominating set. It is called a \(\gamma_{Tk}\) set.

Definition-3.6: Total k-Domination Number.
The cardinality of a minimum totally k-dominating set is called total k-domination number of the graph G and it is denoted as \(\gamma_{Tk}(G)\).

Note that any totally k-dominating set must contain at least k+1 vertices therefore total k-domination number of any graph, if it is define is greater than or equal to k+1.

Definition -3.7: Total k-private neighborhood.
Let G be a graph and S ⊂ V(G) and v ∈ S then total k-private neighborhood of v with respect to the set S.
\[P_{Tk}[v,S] = \{ w ∈ V(G) : w \text{ is adjacent to exactly } k \text{ vertices of } S \text{ including } v. \} \]

Example -3.8: Consider the cycle \(C_5\) with five vertices \(v_1, v_2, v_3, v_4, v_5\):
(See Figure-0.2)
\[S = \{ v_1, v_3, v_4 \}. \text{ We consider the cycle } C_5 \text{ with vertices } v_1, v_2, v_3, v_4, v_5. \text{ Let } v = v_1 \text{ then } P_{T2}[v_1, S] = \{ v_2, v_5 \}\]
Theorem 3.9: Let $G$ be a graph. $k \geq 1$ (k is a positive integer.) A totally $k$-dominating set $S$ is minimal if and only if for every vertex $v$ of $S$, $P_{Tk}[v,S] \neq \emptyset$.

Proof:

Suppose $S$ is a minimal totally $k$-dominating set. Let $v \in S$ then $S \setminus \{v\}$ is not a totally $k$-dominating set. Hence there is a vertex $w$ in $V(G)$ which is adjacent to at most $k-1$ vertices of $S \setminus \{v\}$.

If $w = v$ then we have a contradiction because $v$ is adjacent to at least $k$ vertices of $S$. So, $w \neq v$.

Now $w$ is adjacent to at least $k$ vertices of $S$ and is adjacent to at most $k-1$ vertices of $S \setminus \{v\}$. This means that $w$ is adjacent to exactly $k$ vertices of $S$ including $v$. Hence $w \in P_{Tk}[v,S]$.

Now we prove converse.

Suppose $v \in S$. Let $w \in P_{Tk}[v,S]$. Now $w$ is adjacent to exactly $k$ vertices of $S$ including $v$ therefore $w$ is adjacent to $k-1$ vertices of $S \setminus \{v\}$. i.e. $S \setminus \{v\}$ is not a totally $k$-dominating set. This implies that $S$ is a minimal totally $k$-dominating set.

Comments 3.10:

As we have noted earlier a graph having vertices with degree less than $k$ can not have totally $k$-dominating set. Also it may happen that when a vertex is removed the resulting graph may have vertices having degree less than $k$.

Let $G$ be a graph. Let $I_k$ denote the set of vertices whose degree is less than $k$. 

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**Notations:** We define the following notations.

\[ V^l_{Tk} : \{ v \in V(G) : G-\{v\} \text{ has vertex of degree less than } k \text{ in } (G-\{v\}) \} \]

\[ V^+_{Tk} : \{ v \in V(G) : \gamma_{Tk}(G-v) > \gamma_{Tk}(G) \} \]

\[ V^-_{Tk} : \{ v \in V(G) : \gamma_{Tk}(G-v) < \gamma_{Tk}(G) \} \]

\[ V^0_{Tk} : \{ v \in V(G) : \gamma_{Tk}(G-v) = \gamma_{Tk}(G) \} \]

**Theorem -3.11:** Let \( v \in V(G) \) such that \( d(v) \geq k \) and \( v \notin V^l_{Tk} \). If \( v \in V^-_{Tk} \) then 
\[ \gamma_{Tk}(G)-k \leq \gamma_{Tk}(G-v) \leq \gamma_{Tk}(G)-1. \]

**Proof:**

Let \( S_1 \) be a minimum totally k-dominating set of \( G-\{v\} \). Since \( v \in V^-_{Tk} \), \[ |S_1| < \gamma_{Tk}(G) \] and \( v \) is adjacent to at most \( k-1 \) vertices of \( S_1 \). Suppose \( v \) is not adjacent to any vertex of \( S_1 \). Let \( z_1, z_2, \ldots, z_k \) be \( k \) neighbor of \( v \).

Let \( S = S_1 \cup \{ z_1, z_2, \ldots, z_k \} \), then \( S \) is a totally \( k \)-dominating set in \( G \). Therefore \[ \gamma_{Tk}(G) \leq |S| = |S_1| + k = \gamma_{Tk}(G-v) + k. \] Therefore \[ \gamma_{Tk}(G) - k \leq \gamma_{Tk}(G-v). \]

If \( v \) is adjacent to \( m \) vertices say \( z_1, z_2, \ldots, z_m \) (\( m < k \)). Let \( z_{m+1}, z_{m+2}, \ldots, z_k \) be the vertices adjacent to \( v \) and not in \( S_1 \).

Let \( S = S_1 \cup \{ z_{m+1}, z_{m+2}, \ldots, z_k \} \), then as above \( S \) is a totally \( k \)-dominating set in \( G \) and by similar argument \[ \gamma_{Tk}(G) - k \leq \gamma_{Tk}(G) - (k-m) \leq \gamma_{Tk}(G-v). \]

Thus in both the cases the inequality holds. ■
Theorem 3.12: Suppose $v \in V(G)$, $d(v) \geq k$ and $v \not\in V_k^i$ then $v \in V_k^+$ if and only if the following conditions hold.

(1) $v$ is contained in every $\gamma_k$ set of $G$.

(2) No subset $S$ of $V(G)$ which intersects $N[v]$ in at most $k-1$ vertices of $N[v]$ and with $|S| \leq \gamma_k(G)$ can be a totally $k$-dominating set of $G-\{v\}$.

Proof:

(1) Suppose $v \in V_k^+$. Suppose $S_0$ is a $\gamma_k$ of $G$ such that $v \not\in S_0$. Let $v_1$ be any vertex of $G-\{v\}$. Since $v \not\in V_k^i$, $d(v_1) \geq k$ in $G-\{v\}$ and hence $G$ also. Thus, $v_1$ is adjacent to at least $k$ vertices of $S_0$. Thus, $S_0$ is a totally $k$-dominating set of $G-\{v\}$. Thus, $\gamma_k(G-v) \leq |S_0| = \gamma_k(G)$. That is $v \not\in V_k^+$, a contradiction.

(2) Suppose there is a set $S_0$ which intersects $N[v]$ in at most $k-1$ vertices, and $|S_0| \leq \gamma_k(G)$ and $S_0$ is a totally $k$-dominating set of $G-\{v\}$. Then $\gamma_k(G-v) \leq |S_0| \leq \gamma_k(G)$. This is again a contradiction. Therefore condition (2) holds.

Now we prove converse.

Suppose $v \in V_k^i$. Let $S$ be a minimum totally $k$-dominating set of $G-\{v\}$. If $v$ is adjacent to at least $k$ vertices of $S$ then $S$ is a minimum totally $k$-dominating set of $G$ not containing $v$, which contradict (1).

Suppose $v$ is adjacent to $m$ vertices of $S$ where $0 \leq m < k$. Then $S$ is a set which intersects $N[v]$ in at most $k-1$ vertices, $|S| \leq \gamma_k(G)$ and $S$ is a totally $k$-dominating set of $G-\{v\}$ which contradicts (2).

Suppose $v \in V_k^-$. Then $\gamma_k(G)-k \leq \gamma_k(G-v) \leq \gamma_k(G)-1$.

Let $S$ be a minimum totally $k$-dominating set of $G-\{v\}$. If $v$ is adjacent to at least $k$ vertices of $S$ then $S$ is a totally $k$-dominating set of $G$ with $|S| < \gamma_k(G)$. That is
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$\gamma_{Tk}(G) < \gamma_{Tk}(G)$ - a contradiction. So $v$ is adjacent to at most $k-1$ vertices of $S$ then also $S$ is a set which intersects $N[v]$ in at most $k-1$ vertices and $|S| \leq \gamma_{Tk}(G)$ is a totally $k$-dominating set of $G-\{v\}$, which contradicts (2).

Thus, $v$ can not be in $V_{Tk}$ or $V_{Tk}^0$. Hence $v \in V_{Tk}^+$

Next we prove the following theorem.

**Theorem -3.13:** Suppose $d(v) \geq k$ and $v \in V_{Tk}^+$. Then for any $\gamma_{Tk}$ set $S$, $v \in S$ and $P_{Tk}[v,S]$ contains at least two vertices.

**Proof:**

Let $S$ be any $\gamma_{Tk}$ set of $G$. Since $v \in V_{Tk}^+$, $v \in S$. Since $S$ is a minimum set, $P_{Tk}[v,S]$ contains at least one vertex.

Suppose $P_{Tk}[v,S]$ contains only one vertex say $w$.

**Claim:** $w \notin S$.

**Proof of the Claim:** Suppose $w \in S$. If $w$ is not adjacent to any vertex outside $S$ then $d(w) < k$ in $G-\{v\}$ which contradicts that $v \in V_{Tk}^+$. Thus, there is a vertex $w_1$ outside $S$ which is adjacent to $w$.

Now let $S_1 = S - \{v\} \cup \{w_1\}$. Then $S_1$ is a minimum totally $k$-dominating set of $G$ not containing $v$, which contradicts that $v \in V_{Tk}^+$.

This proves that $w \notin S$. Since $d(w) \geq k$, in $G-\{v\}$, $w$ is adjacent to some vertex $w_1$ which is outside $S$.

Now let $S_1 = S - \{v\} \cup \{w_1\}$. Then $S_1$ is a minimum totally $k$-dominating set of $G$ not containing $v$, which is a contradiction.

Thus, in any case we get a contradiction. Hence $P_{Tk}[v,S]$ contains at least two vertices.
Theorem 3.14: Let \( v \) be a vertex of \( G \) such that \( d(v) \geq k \) and \( v \notin V_{Tk} \). Then
\( v \in V_{Tk} \) if and only if there is a minimum totally k-dominating set \( S \) and \( k \) vertices \( w_1, w_2, \ldots, w_k \) in \( S \) such that \( P_{Tk}[w_i, S] = \{v\} \) for every \( i \).

Proof:
Suppose \( v \in V_{Tk} \). Let \( S_1 \) be a minimum totally k-dominating set of \( G-\{v\} \).

If \( v \) is not adjacent to any vertex of \( S_1 \) then let \( w_1, w_2, \ldots, w_k \) be \( k \) vertices adjacent to \( v \). Let \( S = S_1 \cup \{ w_1, w_2, \ldots, w_k \} \) then \( S \) is a minimum totally k-dominating set of \( G \). For each \( i \) \( v \) is adjacent to exactly \( k \) vertices of \( S \) including \( w_i \) (other vertices to which \( v \) is adjacent are \( w_1, w_2, \ldots, w_{i-1}, w_{i+1}, \ldots, w_k \)). Thus, \( v \in P_{Tk}[w_i, S] \) for every \( i \).

Let \( v_1 \) be a vertex different from \( v \). Since \( S_1 \) is a totally k-dominating set of \( G-\{v\} \), \( v_1 \) is adjacent to at least \( k \)-vertices of \( S_1 \) and no \( w_j \) is member of \( S_1 \). Therefore \( v_1 \notin P_{Tk}[w_i, S] \).

Hence \( P_{Tk}[w_i, S] = \{v\} \) for each \( i \).

To prove converse suppose \( S \) is a minimum totally k-dominating set of \( G \) and \( w_1, w_2, \ldots, w_k \) are vertices of \( S \) such that \( P_{Tk}[w_i, S] = \{v\} \) for every \( i \).

Let \( S_1 = S-\{w_1\} \). We will prove that \( S_1 \) is a totally k-dominating set of \( G-\{v\} \). Let \( z \) be any vertex of \( G-\{v\} \). First suppose that \( z = w_1 \). Since \( S \) is a totally k-dominating set in \( G \), \( z = w_1 \) is adjacent to at least \( k \) vertices of \( S_1 \).

Suppose \( z \neq w_1 \). Since \( z \neq v \), \( z \notin P_{Tk}[w_i, S] \). Hence if \( z \) is adjacent to \( w_1 \) in \( G \) then \( z \) must be adjacent to at least \( k \) other vertices of \( S \). This means that \( z \) is adjacent to at least \( k \) vertices of \( S_1 \). If \( z \) is not adjacent to \( w_1 \) then since \( S \) is a totally k-dominating set of \( G \), \( z \) is adjacent to at least \( k \) vertices of \( S_1 \).

Thus in any case \( z \) is adjacent to at least \( k \) vertices of \( S_1 \). This proves that \( S_1 \) is a totally k-dominating set of \( G-\{v\} \) and hence \( \gamma_{Tk}(G-v) \leq |S_1| < |S| = \gamma_{Tk}(G) \). This means that \( v \in V_{Tk} \).
Corollary -3.15: Suppose $v$ is a vertex in $V^+_Tk$ and $w$ is a vertex in $V^-Tk$ then $v$ and $w$ are non adjacent.

Proof:
There is a minimum totally $k$-dominating set $S$ and $k$ vertices $w_1, w_2, \ldots, w_k$ in $S$ such that $P_{Tk}[w_i, S] = \{w\}$ for each $i$. Since $v \in V^+_Tk$, $v \in S$. (Theorem -3.3). Note that $v \neq w_i$ for any $i$, because $P_{Tk}[v, S]$ contains at least two vertices while $P_{Tk}[w_i, S]$ contains only $w$. Now if $v$ and $w$ are adjacent then $w$ is adjacent to $k+1$ vertices of $S$ including $w_1$, which contradicts the fact that $P_{Tk}[w_1, S] = \{w\}$. Thus, $v$ and $w$ can not be adjacent. ■

**K-TUPLE DOMINATION**

The concept of $k$-tuple domination can be found in [44]. Note that every totally $k$-dominating set is a $k$-tuple dominating set but converse is not true. We begin with the definition of a $k$-tuple dominating set.

**Definition -3.16: $k$-tuple dominating set.** [44]

Let $G$ be a graph and $k$ be an integer greater than or equal to two. A subset $S$ of $V(G)$ is said to be a $k$-tuple dominating set if following conditions satisfied.

1. If $v \in S$ then $v$ is adjacent to at least $k-1$ vertices of $S$.
2. If $v \notin S$ then $v$ is adjacent to at least $k$ vertices of $S$.

**Definition -3.17: Minimal $k$-tuple dominating set.**

A $k$-tuple dominating set $S$ of $G$ is said to be a minimal $k$-tuple dominating set if for each vertex $v$ of $S$, $S-\{v\}$ is not a $k$-tuple dominating set.

**Definition -3.18: Minimum $k$-tuple dominating set.**

A $k$-tuple dominating set with minimum cardinality is called minimum $k$-tuple dominating set which also called $\gamma_{ku}$ set of $G$. 
Definition -3.19: k-tuple domination number.

The cardinality of a minimum k-tuple dominating set is called k-tuple domination number of the graph G. It is denoted by $\gamma_{ku}(G)$.

Remark-3.20: Note that any minimum totally k-dominating set is a k-tuple dominating set, but converse is not true. This means that $\gamma_{ku}(G) \leq \gamma_{Tk}(G)$.

Example -3.21: Consider the cycle $C_5$ with vertices $v_1, v_2, v_3, v_4, v_5$. Let $k=2$ then 2-tuple domination number of $C_5$ is 4 and total 2-domination number is 5. (See Figure –0.2)

Now we define so called k-tuple private neighborhood of a vertex v with respect to a set containing it.

Definition -3.22: k-tuple private neighborhood.

Let S be a subset of V(G) and $v \in S$. Then the k-tuple private neighborhood of v with respect to S. i.e. $P_{ku}[v,S] = S_1 \cup S_2 \cup S_3$

Where $S_1 = \{ w \in S: w \neq v \text{ and } w \text{ is adjacent to exactly } k-1 \text{ vertices of } S \text{ including } v \}$,
$S_2 = \{ w \in S: w = v \text{ and } w \text{ is adjacent to exactly } k-1 \text{ vertices of } S \}$,
$S_3 = \{ w \notin S: w \text{ is adjacent to exactly } k \text{ vertices of } S \text{ including } v \}$

For example if we consider the cycle graph $C_5$, (See Figure –0.2)

$S = \{ v_1, v_2, v_3, v_4 \}$, $v = v_1$ then $P_{ku}[v_1, S] = \{ v_1, v_3 \}$.

Note that in the above definition any one of $S_1$, $S_2$, $S_3$ can be an empty set.

Also note that every minimum k-tuple dominating set is a minimal k-tuple dominating set.

We state the following theorem without proof as it is similar to that of Theorem -3.9
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Theorem-3.23: A subset $S$ of $V(G)$ is a minimal $k$-tuple dominating set if and only if for each vertex $v$ of $S$ $P_{ku}[v,S] \neq \emptyset$. □

Now we introduced the following symbols.

$$V^{+}_{ku} = \{ v \in V(G): \gamma_{ku}(G-v) > \gamma_{ku}(G) \}.$$  

$$V^{-}_{ku} = \{ v \in V(G): \gamma_{ku}(G-v) < \gamma_{ku}(G) \}.$$  

$$V^{0}_{ku} = \{ v \in V(G): \gamma_{ku}(G-v) = \gamma_{ku}(G) \}.$$  

Theorem-3.24: Let $v \in V(G)$ such that $d(v) \geq k$ and $v \notin V^{+}_{Tk}$. Then $v \in V^{-}_{ku}$ if and only if $\gamma_{ku}(G) - k \leq \gamma_{ku}(G-v) < \gamma_{ku}(G)$.

Proof:

Suppose $v \in V^{-}_{ku}$. Let $S_1$ be a minimum $k$-tuple dominating set of $G-\{v\}$. Obviously $v$ is adjacent to at most $k-1$ vertices of $S_1$.

If $v$ is adjacent to exactly $k-1$ vertices of $S_1$ and in this case let $S = S_1 \cup \{v\}$. Then $S$ is a minimum $k$-tuple dominating set of $G$ and $|S| = |S_1| + 1$. This means that $\gamma_{ku}(G-v) = \gamma_{ku}(G) - 1$.

If $v$ is adjacent to no vertex of $S_1$ then let $w_1, w_2, \ldots, w_k$ be vertices adjacent to $v$. Let $S = S_1 \cup \{w_1, w_2, \ldots, w_k\}$, then $S$ is a $k$-tuple dominating set of $G$. Therefore $\gamma_{ku}(G) \leq |S| = |S_1| + k = \gamma_{ku}(G-v) + k$. This proves that $\gamma_{ku}(G) - k \leq \gamma_{ku}(G-v) < \gamma_{ku}(G)$.

Suppose $v$ is adjacent to $m$ vertices of $S_1$ say $w_1, w_2, \ldots, w_m$ ($1 \leq m < k$). Let $w_{m+1}, w_{m+2}, \ldots, w_k$ be vertices adjacent to $v$ and not in $S_1$. Let $S = S_1 \cup \{w_{m+1}, w_{m+2}, \ldots, w_k\}$. Then $S$ is a $k$-tuple dominating set of $G$ and $|S| = |S_1| + k$. Therefore $\gamma_{ku}(G) \leq \gamma_{ku}(G-v) + k - m < \gamma_{ku}(G-v) + k$.

Hence $\gamma_{ku}(G) - k \leq \gamma_{ku}(G-v) < \gamma_{ku}(G)$.

This proves the theorem. □
We state the following theorem without proof as it is similar to that of Theorem-3.12

**Theorem-3.25:** Let \( v \in V(G) \) such that \( d(v) \geq k \) and \( v \not\in V^i_{Tk} \). Then \( v \in V^+_{ku} \) if and only if each of the following two conditions is satisfied.

1. \( v \) is contained in every minimum \( k \)-tuple dominating set.
2. No subset \( S \) of \( V(G-v) \) which intersects \( N[v] \) in at most \( k-1 \) vertices and with
   \[
   |S| \leq \gamma_{ku}(G)
   \]
   can be a tuple dominating set of \( G-\{v\} \).

**Theorem-3.26:** Let \( v \in V(G) \) such that \( d(v) \geq k \) and \( v \not\in V^i_{Tk} \). If \( v \in V^+_{ku} \) and \( S \) is a minimum \( k \)-tuple dominating set then \( v \in S \) and \( P_{ku}[v,S] \) contains at least two vertices.

**Proof:**

By Theorem-3.23, \( v \in S \). Since \( S \) is a minimal \( k \)-tuple dominating set, \( P_{ku}[v,S] \) is non-empty.

First suppose that \( P_{ku}[v,S] \) consists only one vertex \( w \).

Let \( w \in P_{ku}[v,S] \). If \( w = v \) then \( S-\{v\} \) is a \( k \)-tuple dominating set of \( G-\{v\} \). This means that \( v \in V^+_{ku} \) and this is a contradiction. If \( w \neq v \) then there are two cases:

**Case-1:**

\( w \in S \). Then \( w \) is adjacent to exactly \( k-1 \) vertices including \( v \) of \( S \). Since \( d(w) \geq k \), there is a vertex \( w_1 \) outside \( S \) which is adjacent to \( w \). Let \( S_1 = S-\{v\}U\{w_1\} \), then \( S_1 \) is a minimum \( k \)-tuple dominating set of \( G \) not containing \( v \). This contradicts the assumption that \( v \in V^+_{ku} \).

**Case-2:**

\( w \not\in S \). Let \( S_1 = S-\{v\}U\{w\} \), then \( S_1 \) is a minimum \( k \)-tuple dominating set of \( G \) not containing \( v \), which is again a contradiction as \( v \in V^+_{ku} \).

Thus, the assumption that the \( P_{ku}[v, S] \) contains only one vertex leads to a contradiction. Therefore it must contain at least two vertices.
Example-3.27: Consider the following graph to understand for the Theorem-3.26.

\[
\gamma_{2u} = \{2,3,4,5\} \quad \text{and} \quad k = 2, \quad (G) = 4.5 \in V^+_{k_d} \text{, Now for the graph } G - \{5\} \\
\gamma_{2u} = \{2,3,4,6,7\} = S, \quad \text{and} \quad k = 2, \quad \text{So, } P_{2u}[5,S] = \{6,7\}
\]

\[
\gamma_{2u} = \{2,3,4,6,7\} = S, \quad \text{and} \quad k = 2, \quad \text{So, } P_{2u}[5,S] = \{6,7\}
\]
**Theorem-3.28:** Let \( v \in V(G) \), \( d(v) \geq k \), and \( v \notin V^i_{Tk} \),

1. If \( v \in V_ku \) then there is a minimal \( k \)-tuple dominating set \( S \) containing \( v \) such that \( P_{ku}[v,S] = \{v\} \).

2. If there is a minimum \( k \)-tuple dominating set \( S \) containing \( v \) such that \( P_{ku}[v,S] = \{v\} \) then \( v \in V_ku \).

**Proof:** (1)

Suppose \( v \in V_ku \). Let \( S_1 \) is a minimum \( k \)-tuple dominating set of \( G-\{v\} \). Then \( v \) is adjacent to at most \( k-1 \) vertices of \( S_1 \).

**Case-1:** \( v \) is adjacent to no vertex of \( S_1 \).

Let \( w_1, w_2, \ldots, w_{k-1} \) be vertices not in \( S_1 \) such that \( w_i \) is adjacent to \( v \) for every \( i \). Let \( S = S_1 \cup \{ w_1, w_2, \ldots, w_{k-1}, v \} \). Then \( S \) is a minimal \( k \)-tuple dominating set of \( G \) containing \( v \).

Suppose \( v_1 \) is a vertex different from \( v \).

If \( v_1 \in S_1 \) then \( v_1 \) is adjacent to at least \( k-1 \) vertices of \( S_1 \). Thus, if \( v_1 \) is adjacent to \( v \) then \( v_1 \) is adjacent to at least \( k \) vertices of \( S \). Therefore \( v_1 \notin P_{ku}[v,S] \).

Suppose \( v_1 = w_i \) for some \( i \). Now \( w_i \notin S_1 \) and therefore \( w_i \) is adjacent to at least \( k \) vertices of \( S_1 \). Therefore if \( w_i \) is adjacent to \( v \) then \( w_i \) is adjacent to \( k+1 \) vertices of \( S \). Therefore \( w_i \notin P_{ku}[v,S] \).

Suppose \( v_1 \notin S \) then \( v_1 \) is adjacent to at least \( k \) vertices of \( S_1 \) therefore if \( v_1 \) is adjacent to \( v \) then \( v_1 \) is adjacent to \( k+1 \) vertices of \( S \). Therefore \( v_1 \notin P_{ku}[v,S] \).

**Case-2:** \( v \) is adjacent to \( m \) vertices \( w_1, w_2, \ldots, w_m \) of \( S_1 \) where \( 1 \leq m < k \)

Let \( w_{m+1}, w_{m+2}, \ldots, w_{k-1} \) be vertices not in \( S_1 \) and adjacent to \( v \). Let \( S = S_1 \cup \{ w_{m+1}, w_{m+2}, \ldots, w_{k-1}, v \} \) then \( S \) is a minimal \( k \)-tuple dominating set of \( G \) containing \( v \).
Let $v_1$ be a vertex different from $v$.

If $v_1 = w_i$ for some $i \in \{1, 2, 3, \ldots, m\}$ then if $w_i$ is adjacent to exactly $k-1$ vertices of $S_1$ then $w_i$ is adjacent to $k$ vertices of $S$ if $w_i$ is adjacent to $v$. Therefore $v_1 = w_i \not\in P_{ku}[v, S]$.

If $v_1 = w_i$ for some $i \in \{m+1, m+2, \ldots, k-1\}$ then since $w_i$ is adjacent to at least $k$ vertices of $S_1$, $w_i$ is adjacent to at least $k+1$ vertices of $S$, if $w_i$ is adjacent to $v$. Therefore $w_i \not\in P_{ku}[v, S]$.

Case-3: $v$ is adjacent to exactly $k-1$ vertices of $S_1$.

Let $S = S_1 \cup \{v\}$, then $S$ is a minimal $k$-tuple dominating set of $G$ containing $v$. Let $v_1$ be a vertex different from $v$.

If $v_1 = w_i$ for some $i$, then since $w_i$ is adjacent to at least $k-1$ vertices of $S_1$, $w_i$ is adjacent to at least $k$ vertices of $S$ including $v$, if $w_i$ is adjacent to $v$. Therefore $v_1 = w_i \not\in P_{ku}[v, S]$.

If $v_1 \in S_1$ then $v_1$ is adjacent to at least $k-1$ vertices of $S_1$. Therefore $v_1$ is adjacent to at least $k$ vertices of $S$ if $v_1$ is adjacent to $v$. Therefore $v_1 \not\in P_{ku}[v, S]$.

If $v_1 \not\in S_1$ then $v_1$ is adjacent to at least $k$ vertices of $S_1$ and therefore adjacent to at least $k+1$ vertices of $S$ if $v_1$ is adjacent to $v$. Therefore $v_1 \not\in P_{ku}[v, S]$.

Note that $v \in P_{ku}[v, S]$. Hence $P_{ku}[v, S] = \{v\}$.

(2)

Suppose there is a minimum $k$-tuple dominating set $S$ of $G$ containing $v$ such that $P_{ku}[v, S] = \{v\}$.

Let $S_1 = S-\{v\}$. We will prove that $S_1$ is a $k$-tuple dominating set of $G-\{v\}$.
Let $v_1$ be any vertex of $G - \{v\}$.

**Case-1: $v_1 \in S_1$.**

Since $S$ is a $k$-tuple dominating set of $G$, $v_1$ is adjacent to at least $k-1$ vertices of $S$. Suppose $v_1$ is adjacent to $v$ in $G$ and $v_1$ is adjacent to exactly $k-1$ vertices of $S$ then $v_1$ is not adjacent to $v$ and $v_1 \in P_{ku}[v,S]$ which is not true. Since $P_{ku}[v,S] = \{v\}$. Therefore if $v_1$ is adjacent to $v$. Then $v_1$ is adjacent to at least $k-1$ other vertices of $S$. Thus, $v_1$ is adjacent to at least $k-1$ vertices of $S_1 = S - \{v\}$. If $v_1$ is not adjacent to $v$ then $v_1$ is adjacent to at least $k-1$ vertices of $S$ different from $v$. Therefore $v_1$ is adjacent to at least $k-1$ vertices of $S_1$.

Suppose $v_1 \notin S_1$. Now $v_1 \neq v$. Therefore $v_1 \notin S$. Now since $S$ is a $k$-tuple dominating set of $G$, $v_1$ is adjacent to at least $k$ vertices of $S$ different from $v$. Therefore $v_1$ is adjacent to at least $k$ vertices of $S_1$. Thus, $S_1$ is a $k$-tuple dominating set of $G - \{v\}$. Therefore,

$$\gamma_{ku}(G-v) \leq |S_1| < |S| \leq \gamma_{ku}(G)$$

Therefore,

$$\gamma_{ku}(G-v) < \gamma_{ku}(G)$$

Therefore,

$$v \in V_{ku}$$

The following definition of $k$-dependent set is due to J. F. Fink and M.S. Jacobson [21].


K-DEPENDENT K-DOMINATION

Definition -3.29: k-dependent set.[21]
Suppose \( k \geq 1 \). A set \( S \) subset of \( V(G) \) is said to be k-dependent set if for every vertex \( v \) in \( S \), \( v \) is adjacent to at most \( k-1 \) vertices of \( S \).
Note that if \( k=1 \) then 1-dependent set is just an independent set.

Definition -3.30: Maximal k-dependent set.
Let \( k \geq 1 \) and \( S \) be a subset of \( V(G) \). Then \( S \) is said to be a maximal k-dependent set if
(1) \( S \) is a k-dependent set.
(2) For every vertex \( v \) not in \( S \), \( S \cup \{v\} \) is not a k-dependent set.

Note that every maximum k-dependent set is a maximal k-dependent set.

If \( S \) is a maximal k-dependent set then obviously for every vertex \( v \) not in \( S \) \( v \) is adjacent to at least \( k \) vertices of \( S \). Thus, \( S \) is a k-dominating set. Hence every maximal k-dependent set is a k-dominating set.

Also if \( S \) is a k-dependent set and \( v \in S \) then \( v \) is adjacent to at most \( k-1 \) vertices of \( S \). Therefore \( v \) belongs to private k-neighborhood of \( v \) with respect to \( S \), which is denoted as \( P_k[v,S] \). That is \( P_k[v,S] \) is non empty.

Therefore \( S \) is a minimal k-dominating set of \( G \). [2] Thus, every maximal k-dependent set is a minimal k-dominating set.
Definition -3.31: k-dependent k-dominating set.

Let $k \geq 1$ and $S$ is subset of $V(G)$. Then $S$ is said to be k-dependent k-dominating set if

1. $S$ is a k-dependent set.
2. $S$ is a k-dominating set.

Definition -3.32: Minimal k-dependent k-dominating set.

Let $S$ be a k-dependent k-dominating set then $S$ is said to be minimal k-dependent k-dominating set if for each vertex $v \in S$, $S \setminus \{v\}$ is not a (k-dependent) k-dominating set.

Definition -3.33: Minimum k-dependent k-dominating set.

A k-dependent k-dominating set $S$ with minimum cardinality is called a minimum k-dependent k-dominating set. It is denoted by $i_k$ set.

Definition -3.34: k-dependent k-domination number.

The cardinality of a minimum k-dependent k-dominating set is called k-dependent k-domination number of the graph $G$. It is denoted as $i_k(G)$.

Thus, by above remark every maximal k-dependent set is a minimal k-dependent k-dominating set.

Conserve is also true. That is every minimal k-dependent k-dominating set is also a maximal k-dependent set.

Thus, the minimum cardinality of a k-dependent k-dominating set = the minimum cardinality of a maximal k-dependent set. That is $i_k(G)$. 
We define the following symbols.

\[ V^+_{Ik} = \{ v \in V(G) : \gamma_{Ik}(G) < \gamma_{Ik}(G-v) \} \]

\[ V^-_{Ik} = \{ v \in V(G) : \gamma_{Ik}(G) > \gamma_{Ik}(G-v) \} \]

\[ V^0_{Ik} = \{ v \in V(G) : \gamma_{Ik}(G) = \gamma_{Ik}(G-v) \} \]

\[ V^i_{Tk} = \{ G-{v} has a vertex which degree is less than k. \} \]

Note that the above sets are mutually disjoint and their union is \( V(G) \).

We state the following theorem without proof.

**Theorem-3.35:** Let \( v \in V(G) \), \( d(v) \geq k \) and \( v \notin V^i_{Tk} \) then \( v \in V^+_{Ik} \) if and only if the following conditions holds.

1. \( v \) belongs to every minimum \( k \)-dependent \( k \)-dominating set of \( G \).
2. No subset \( S \) of \( G-{v} \) which intersects \( N[v] \) in at most \( k-1 \) vertices and \( |S| \leq i_k(G) \) can be a \( k \)-dependent \( k \)-dominating set of \( G-{v} \).

**Proof:** The proof of this theorem is similar to that of corresponding theorem for total \( k \)-domination.

**Example-3.36:**

1. **Consider the graph \( G = Petersen Graph \) (See Figure- 0.3)**
   For the Petersen Graph \( i_3 \) set is \( \{2, 4, 6, 7, 9, 10\} \) and \( i_3(G) = 6 \).
   and \( i_2 \) set is \( \{1, 3, 6, 9, 10\} \) and \( i_2(G) = 5 \).

2. **Consider the graph \( G = Hyper Qube \) (See Figure – 0.8)**
   For the Hyper Qube Graph \( i_3 \) set is \( \{2, 3, 4, 5, 6, 8\} \) and \( i_3(G) = 6 \).
   and \( i_2 \) set is \( \{2, 4, 6, 8\} \) and \( i_2(G) = 4 \).
**Definition -3.37: External private k-neighborhood.**

Let $S$ be a subset of $V(G)$ and $v \in S$, then the external private k-neighborhood of $v$ with respect to $S$, i.e $E_x[v,S]$

$$E_x[v,S] = \{w \in V(G)-S : \text{w is adjacent to exactly k vertices of S including v.} \}$$

Now we state and prove the equivalent conditions for vertex $v$ to be in $V_{i_k}$.

**Theorem-3.38:** Let $v \in V(G)$, $d(v) \geq k$, and $v \notin V_{i_k}$ then the following statements are equivalent.

1. $v \in V_{i_k}$.
2. There is a minimum k-dependent k-dominating set $S$ containing $v$ such that $E_{sk}[v,S]$ is empty.
3. There is a minimum k-dependent k-dominating set $S$ of $G$ containing $v$ such that $S-\{v\}$ is a k-dependent k-dominating set of $G-\{v\}$.

**Proof:**

(1) $\Rightarrow$ (2)

Let $S_1$ be a k-dependent k-dominating set of $G-\{v\}$. Then $|S_1| < i_k(G)$. If $v$ is adjacent to at least $k$ vertices of $S_1$ then $S_1$ is a k-dependent k-dominating set of $G$ and therefore $i_k(G) \leq |S_1| < i_k(G)$. This is a contradiction. Therefore $v$ is adjacent to at most $k-1$ vertices of $S_1$.

Let $S = S_1 \cup \{v\}$ then $S$ is a minimum k-dependent k-dominating set of $G$ containing $v$.

Suppose $w \in E_{sk}[v,S]$ then $w$ is adjacent to exactly $k$-vertices of $S$ including $v$ therefore $w$ is a vertex of $G-\{v\}$ such that $w \notin S_1$ and $w$ is adjacent to exactly $k-1$ vertices of $S_1$. This is a contradiction because $S_1$ is a maximal k-dependent set in $G-\{v\}$. Therefore $E_{sk}[v,S]$ is empty. Hence (1) $\Rightarrow$ (2) is proved.
Chapter-3 : Total k-Domination and k-tuple Domination and k-dependent k-Domination

Now (2) $\Rightarrow$ (3).

Let $S$ be a minimum k-dependent k-dominating set of $G$ containing $v$ such that $E_{sk}[v,S]$ is empty.

Consider the set $S_1 = S \setminus \{v\}$. We prove that $S_1$ is a k-dependent k-dominating set of $G \setminus \{v\}$.

Let $w$ be a vertex of $G \setminus \{v\}$ such that $w \notin S \setminus \{v\}$. Then $w$ is a vertex of $G$ with $w \notin S$. If $w$ is adjacent to $v$ in $G$ then $w$ must be adjacent to at least $k$ other vertices of $S$ (because $w \notin E_{sk}[v,S]$). Therefore $w$ is adjacent to at least $k$ vertices of $S \setminus \{v\}$.

Since $S$ is a k-dependent set in $G$, $S \setminus \{v\}$ is also k-dependent set in $G \setminus \{v\}$. Thus, $S \setminus \{v\}$ is a k-dependent k-dominating set of $G \setminus \{v\}$. Hence (2) $\Rightarrow$ (3) is proved.

Now (3) $\Rightarrow$ (1)

Let $S$ be a minimum k-dependent k-dominating set of $G$ containing $v$ such that $S \setminus \{v\}$ is a k-dependent k-dominating set of $G \setminus \{v\}$. Then

$$i_k(G-v) \leq |S \setminus \{v\}| < |S| = i_k(G)$$

Therefore,

$$i_k(G-v) < i_k(G).$$

Hence $v \in V_{i_k}$. Thus, (3) $\Rightarrow$ (1) is proved.