CHAPTER 6
SUMMARY AND CONCLUSIONS

The present chapter highlights the summary and conclusion of the research work and scope for future work.

6.1 Summary and conclusions

Derivation of cubature rules for an integration domain involves solution of many systems of non-linear equations in that domain, which is a tedious process. Due to this reason higher order cubature rules are not available for evaluating double integrals, triple integrals and higher dimensional multiple integrals. Besides, cubature rules can be derived only for specific regions, like cubature over a triangle, over a tetrahedron, over a sphere etc. For these reasons the formulation of integration rules over multidimensional regions remains an open area of research as demonstrated by many authors.

The current research started on triangles, specifically the standard triangle. It was found that more accurate results could be obtained with the use of generalized Gaussian quadrature rules (for the product of logarithmic function and polynomials in one-dimension) in the product formula derived after transforming the triangle to a square. This was a motivation to derive the general integration formula for numerical evaluation of double integrals using generalized Gaussian quadrature. Later this is extended to triple integrals and then to n-dimensional integrals. The general formula derived in the work is adept to evaluate the numerical value of

\[
\int_{a}^{b} \int_{g(x)}^{h(x)} f(x,y) dy \, dx \quad \int_{a}^{b} \int_{g_1(x)}^{h_1(x)} \int_{h_2(x,y)}^{g_2(x,y)} f(x,y,z) \, dz \, dy \, dx \quad \text{and}
\]

\[
\int_{a}^{b} \int_{g_1(x_1)}^{h_1(x_1)} \int_{g_2(x_1,x_2)}^{h_2(x_1,x_2)} \ldots \int_{g_{n-1}(x_1,x_2,\ldots,x_{n-1})}^{h_{n-1}(x_1,x_2,\ldots,x_{n-1})} f(\mathbf{x}) \, dx_n \, dx_{n-1} \ldots \, dx_1.
\]
The solver needs to know only the integration limits and the integrand to apply this formula. The integration rules that are claimed in the work are general and are based on the transformation of the regions to simpler regions. In each derivation, the domain of integration of an n-dimensional integral is transformed to an n-dimensional cube. i.e., any two-dimensional element is transformed to a two-dimensional cube which is a square, a three-dimensional region to a three-dimensional cube which is a cube, a four-dimensional region to a four-dimensional cube or a tesseract and so on. After the transformation, the integration rule is derived, in which by applying the generalized Gaussian quadrature for the product of logarithmic function and polynomials in one-dimension, one can evaluate the integration result over the domain.

During the integral tests of different functions over different domains, it is found that integration over regions bounded by circular edges should be viewed separately to get a good accuracy. As a result of this finding, different transformations are tried on such regions and finally a combination of few transformations are established, using which, new integration formulae are derived for such regions (regions like a circular disc, cylinder, cone, paraboloid, sphere and n-dimensional balls). These integration formulae are giving optimum results, i.e., more accurate results in less number of computations, for these domains.

An integration formula to integrate functions over an irregular domain (in any dimension), is also derived in the work, which can be used to integrate functions over domains like cardioids, circular and elliptic discs, spheres, ellipsoids etc. The thesis also demonstrates a method to derive nearly optimal integration rules to integrate functions over complex geometries.

The integration rules derived in this work are general, simple and direct, and any programming language or any mathematical software can be used to evaluate the integration results using these rules. Most of the numerical results obtained using these new quadrature rules are exact for at least ten significant digits. The distribution of the derived quadrature points for many regions is plotted in the thesis. For the benefit of solvers, the integration points along with weights are
listed for some commonly used elements, like standard triangle, unit circular disc, standard tetrahedron, unit cube, unit sphere, a unit tesseract, four-simplex and a unit four-ball.

It can be noted that the derived formula is proved to be optimal for the most important elements in finite element method, the triangles and the tetrahedrons. Moreover, tabulated results along with comparisons of errors and function evaluations with the results from literature are provided judiciously, in order to claim that the derived formulae gives better accuracy at minimum computational cost, for almost all domains of integrations. The proposed integration methods works well for all types of integrand functions including oscillating functions and functions with end-point singularities.

6.2 Scope for future work

The derived formulae can be used to find integrals in the areas of fluid dynamics, heat transfer, electromagnetic theory etc. and can be extended to more complicated domains which appear while solving boundary value problems using finite element methods and boundary element methods. Furthermore, generalized Gaussian quadrature can be derived over domains with infinite limits and for the wavelet functions.