Chapter 4

$P_4$-Decomposition of Triple Connected Graphs

In this chapter, we characterize complement of graphs and derived graphs which are triple connected. Also, we give necessary and sufficient conditions for the decomposition of some triple connected biregular and triregular graphs of diameter two into paths of length three.  

4.1 Introduction

The concept of connectedness plays an important role in any network. A variety of connectedness have been studied in the literature by considering the existence of a path between any two vertices. In transportation networks, this enables a traveller to have a route from one city to any other city. If a traveller can finish some work enroute in any one of the third cities, then it will minimize money, distance, time, etc. A communication network in which a

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transmitting node can send a message to two stations at one stretch will be more effective and economic. Such an optimization leads to the concept of triple connected graphs. The concept of triple connected graphs [28] was introduced in the year 2012. In this chapter, we deal about the complementary graphs and derived graphs which are triple connected. Also, we decompose biregular and triregular triple connected graphs into paths of length three.

4.2 Triple Connected Graphs

Definition 4.2.1. A graph $G$ is said to be triple connected if any three vertices lie on a path in $G$.

Example 4.2.2. All paths, cycles, complete graphs and wheels with at least four vertices are some standard examples of triple connected graphs. Petersen graph, $K_{r,s}$ where $r, s \geq 2$ and $K_{m_1, m_2, \ldots, m_k}$, $m_i \geq 1$ and $k \geq 3$ are also triple connected graphs.

Remark 4.2.3. If $G$ has a spanning subgraph which is triple connected, then $G$ is triple connected.

Theorem 4.2.4. [28] Every 2-connected graph is triple connected.

Example 4.2.5. The graph given in Figure 4.1 is not triple connected.
Theorem 4.2.6. [28] A connected graph $G$ is not triple connected if and only if there exists a $H$-cut with $\omega(G - H) \geq 3$ such that $|V(H) \cap N(C_i)| = 1$ for at least three components $C_1$, $C_2$ and $C_3$ of $G - H$.

Theorem 4.2.7. $G$ is triple connected if and only if the block graph $B(G)$ of $G$ is a path.

4.3 Complementary Graphs

In this section, we give characterization for complement of graphs to be triple connected.

Theorem 4.3.1. Let $T$ be a tree. Then $\overline{T}$ is triple connected if and only if $T \not\cong K_{1,r}$.

Proof. Let $T$ be a tree. Assume that $T \not\cong K_{1,r}$. Let $u$, $v$, $w$ be any three vertices in $V(\overline{T})$ and let $S = \{u, v, w\}$.
Case i) $\langle S \rangle = \overline{K}_3$ in $T$.

Then $\langle S \rangle = K_3$ in $\overline{T}$ and $uvw$ is a path in $\overline{T}$.

Case ii) $\langle S \rangle = K_2 \cup K_1$ in $T$.

Without loss of generality, let $u$ and $v$ be adjacent in $T$. Thus $uw, vw \in E(\overline{T})$ and hence $uvw$ is a path in $\overline{T}$.

Case iii) $\langle S \rangle = P_3$ in $T$.

Without loss of generality, let $u$ be adjacent to both $v$ and $w$ in $T$. Thus $vw \in E(\overline{T})$. Since $T \not\cong K_1, r$, there exists another vertex $x$ which is not adjacent to $u$ in $T$. Thus $xu \in E(\overline{T})$. Since $T$ is a tree, $x$ can not be adjacent to both $v$ and $w$ in $T$. Without loss of generality, assume that $x$ is not adjacent to $w$ in $T$. Then $xw \in E(\overline{T})$ and $uxwv$ is a path in $\overline{T}$.

Thus any three vertices lie on a path in $\overline{T}$. Hence $\overline{T}$ is triple connected.

Conversely, assume that $\overline{T}$ is triple connected. Assume $T \cong K_{1, r}$. This implies that $\overline{T} \cong K_1 \cup K_r$, $r \geq 2$ which is disconnected. Thus $\overline{T}$ is not triple connected, which is a contradiction. Thus $T \not\cong K_{1, r}$. \qed

Proposition 4.3.2. Let $G$ be a connected graph. Then $\overline{G}$ is disconnected with $\omega$ components if and only if $G$ contains a complete $\omega$-partite graph ($\omega \geq 2$) as a spanning subgraph.

Proof. Let $\overline{G}$ be disconnected with $\omega$ components $C_1, C_2, \ldots, C_\omega$. 
Let \( V(C_i) = V_i \) and \( |V_i| = n_i, \ i = 1, 2, \ldots, \omega. \)

**Claim** \( K_{n_1,n_2,\ldots,n_\omega} \) is a spanning subgraph of \( G. \)

It is enough if we prove that any two vertices in different partite sets are adjacent in \( G. \) Let \( u \in V_i \) and \( v \in V_j. \) Since \( u \) and \( v \) are the vertices of \( C_i \) and \( C_j \) in \( \overline{G} \) respectively, \( uv \notin E(\overline{G}). \) Hence \( uv \in E(G) \) and hence the claim.

Conversely, assume that \( G \) contains a complete \( \omega \)-partite graph as a spanning subgraph, say \( K_{n_1,n_2,\ldots,n_\omega} \) where \( \omega \) is as large as possible. Then \( V(G) \) can be partitioned into \( \omega \) subsets \( V_1, V_2, \ldots, V_\omega \) such that every two vertices in different partite sets are joined by an edge, where \( |V_i| = n_i. \) We claim that each \( \langle V_i \rangle \) is connected in \( \overline{G}. \) Assume there exists a \( j \ (1 \leq j \leq \omega) \) such that \( \langle V_j \rangle \) is disconnected with at least two components in \( \overline{G}. \) As in previous part, there exists a complete \( \omega' \)-partite graph as a spanning subgraph of \( \langle V_j \rangle \) in \( G \) where \( \omega' \geq 2. \) Then \( V_j \) can be partitioned into \( \omega' \) subsets \( V_{j_1}, V_{j_2}, \ldots, V_{j_{\omega'}} \) such that every two vertices in different partite sets \( V_{j_1}, V_{j_2}, \ldots, V_{j_{\omega'}} \) are joined by an edge in \( G. \) Then \( V_1, V_2, \ldots, V_{j-1}, V_{j+1}, \ldots, V_\omega, V_{j_1}, V_{j_2}, \ldots, V_{j_{\omega'}} \) are the partite sets of \( V \) such that any two vertices in different partite sets are adjacent in \( G. \) Hence \( G \) contains a complete \((\omega - 1) + \omega' \) partite graph as a spanning subgraph, where \((\omega - 1) + \omega' > \omega \) which is a contradiction to the choice of \( \omega. \) Hence each \( \langle V_i \rangle \) is connected in \( \overline{G}. \) Further, by
hypothesis, if \( uv \in E(G) \), then \( u \) and \( v \) belong to same partite set in \( \overline{G} \). Hence \( \overline{G} \) is disconnected with \( \omega \) components. \( \square \)

**Corollary 4.3.3.** Let \( G \) be a connected graph. Then \( \overline{G} \) is disconnected if and only if \( G \) contains a complete bipartite graph \( K_{r,s} \) \((r, s \geq 1)\) as a spanning subgraph.

**Definition 4.3.4.** A graph \( G \) satisfying proposition 4.2 is called a \( \omega \)-complement graph.

**Theorem 4.3.5.** Let \( G \) be a disconnected graph. Then \( \overline{G} \) is triple connected if and only if \( G \not\cong K_1 \cup H \) where \( H \) is a \( \omega \)-complement graph for \( \omega \geq 3 \).

**Proof.** Assume that \( \overline{G} \) is triple connected. Assume \( G \cong K_1 \cup H \). Then \( K_1 \) is a triple cut for \( G \) and hence by Theorem 4.2.6, \( G \) is not triple connected which is a contradiction. Thus \( G \not\cong K_1 \cup H \).

Conversely, assume that \( G \not\cong K_1 \cup H \). If \( \omega(G) \geq 3 \), then \( \overline{G} \) contains a complete \( \omega \)-partite graph as a spanning subgraph which is triple connected. By Remark 4.2.3, \( \overline{G} \) is triple connected. Now assume that \( \omega(G) = 2 \). If \( G \cong G_1 \cup G_2 \) such that \( |V(G_1)|, |V(G_2)| \geq 2 \), then \( \overline{G} \) contains a complete bipartite graph as a spanning subgraph which is triple connected. By Remark 4.2.3, \( \overline{G} \) is triple connected. Now let \( G \cong K_1 \cup H \), where \( H \) is a \( \omega \)-complement graph for \( \omega \leq 2 \). If \( \omega = 1 \), then \( \overline{G} = K_1 + H \) is 2-connected and hence by Theorem
4.2.4, $G$ is triple connected. If $\omega(G) = 2$, then $B(G) \cong P_2$ and hence by Theorem 4.2.7, $G$ is triple connected. \hfill \Box

**Lemma 4.3.6.** Let $G$ be a connected graph with a cut vertex $v$ and $d(v) = 3$. Then $\overline{G}$ is not triple connected if and only if the degree set of $N(v)$ is $\{1, n - 2, n - 2\}$.

**Proof.** Let $G$ be a connected graph with a cut vertex $v$ and $d(v) = 3$. Assume that $\overline{G}$ is not triple connected. Let $C$ be a component of $G - \{v\}$ where $V(C) = \{x_1, x_2, \ldots, x_{n-2}\}$. Clearly $v$ is adjacent to an end vertex $x$ and $v$ is adjacent to exactly two vertices $x_i$ and $x_j$. It is enough if we prove that $d(x_i) = n - 2$ and $d(x_j) = n - 2$. Assume this is not true, we consider two cases.

**Case i)** $d(x_i), d(x_j) < n - 2$.

Clearly $x$ is adjacent to all the vertices $x_1, x_2, \ldots, x_{n-2}$ of $C$ and $v$ is adjacent to all the vertices of $\{x_1, x_2, \ldots, x_{n-2}\} - \{x_i, x_j\}$ of $C$ in $\overline{G}$. If $d(x_i), d(x_j) < n - 2$ in $G$, then $x_i$ and $x_j$ are adjacent to at least one vertex of $C$ in $\overline{G}$. If $x_i$ is adjacent to either $x_j$ or $\{x_1, x_2, \ldots, x_{n-2}\} - \{x_j\}$ in $\overline{G}$, then $\overline{G}$ is triple connected. Similarly, if $x_j$ is adjacent to either $x_i$ or $\{x_1, x_2, \ldots, x_{n-2}\} - \{x_i\}$ in $\overline{G}$, then we have at most two components which are adjacent to $x$. Thus $\overline{G}$ is triple connected, which is a contradiction.

**Case ii)** $d(x_i) < n - 2$ and $d(x_j) = n - 2$.

In this case, $x_j$ is an end vertex which is adjacent to $x$ in $\overline{G}$. 75
Since \( d(x_i) < n - 2 \) in \( G \), \( x_i \) is adjacent to at least one vertex of \( \{x_1, x_2, \ldots, x_{n-2}\} - \{x_j\} \) in \( \overline{G} \). Thus the subgraph induced by the vertices \( \{x_1, x_2, \ldots, x_{n-2}\} - \{x_j\} \cup \{v\} \) form a connected component which is adjacent to \( x \). Thus we have exactly two components which are adjacent with \( x \), which is a contradiction. Hence \( d(x_i) = n - 2 \) and \( d(x_j) = n - 2 \).

Conversely, assume that the degree set of \( N(v) \) is \( \{1, n - 2, n - 2\} \). Then in \( \overline{G} \), both \( x_i \) and \( x_j \) are end vertices that are adjacent to \( x \) and the subgraph induced by \( \{x_1, x_2, \ldots, x_{n-2}\} - \{x_i, x_j\} \cup \{v\} \) is connected with \( d(x) = n - 2 \). Thus \( \overline{G} - \{x\} \) has exactly three components such that all their vertices are adjacent to \( x \). Thus \( \{x\} \) is a triple cut and hence \( \overline{G} \) is not triple connected.

\[\square\]

**Theorem 4.3.7.** Let \( G \) be a connected but not triple connected graph with a unique vertex \( v \) of degree \( n - 2 \). Then \( \{v\} \) is a triple cut in \( G \).

**Proof.** Assume that the connected graph \( G \) is not triple connected. Then by Theorem 4.2.6, there exists a triple cut \( H \) in \( G \) such that \( G - H \) has at least three components \( C_i \) with \( |N(C_i) \cap V(H)| = 1; i = 1, 2, \ldots, l, l \geq 3 \). Let \( N(C_i) \cap V(H) = \{x_i\}, i = 1, 2, \ldots, l \) where \( x_i \) need not be distinct. Let \( \{x_1, x_2, \ldots, x_p\} \) be the set of distinct vertices of \( H \).
Claim 1) $v \in V(H)$

Assume $v \notin V(H)$, then there exists a component $C_i$ in $V - H$ such that $v \in V(C_i)$ and $v \neq x_i$. Thus we can find at least two vertices $x \in C_j$, $y \in C_k$, $i \neq j \neq k$ and $x \neq x_j$, $y \neq x_k$ such that $v$ is not adjacent with both $x$ and $y$. Then $d(v) \leq n - 3$, which is a contradiction. Hence $v \in V(H)$.

Claim 2) $v \in \{x_1, x_2, \ldots, x_p\}$

Assume $v \notin \{x_1, x_2, \ldots, x_p\}$, then we can find at least three vertices $x \in C_i$, $y \in C_j$, $z \in C_k$, $i \neq j \neq k$ and $x \neq x_i$, $y \neq x_j$, $z \neq x_k$ such that $v$ is not adjacent to $x$, $y$ and $z$. Then $d(v) \leq n - 4$, which is a contradiction. Hence $v \in \{x_1, x_2, \ldots, x_p\}$.

Claim 3) $p \leq 2$

Assume $p \geq 3$. Then $\{x_1, x_2, x_3\} \subseteq N(C_i) \cap V(H)$ and $i \geq 3$. Let $x_1 = v$. Since $N(C_1) \cap V(H) = \{x_1\} = \{v\}$, we can find at least two vertices $x \in V(C_2)$, $x \neq x_2$, $y \in V(C_3)$, $y \neq x_3$ such that $v$ is not adjacent to both $x$ and $y$. Thus $d(v) \leq n - 3$, which is a contradiction. Hence $p \leq 2$. If $p = 1$, then obviously $\{v\}$ is a triple cut in $G$.

If $p = 2$, then let $V(C_i) \cap V(H) = \{x_1, x_2\}$, $1 \leq i \leq l$, $l \geq 3$. Without loss of generality, we may assume that $x_1 = v$. Clearly by the previous argument, there exists exactly one component, say $C_j$ such that $V(C_j) \cap V(H) = \{x_2\}$ and the remaining $(l - 1)$ components are $C_1, C_2, \ldots, C_{j-1}, C_{j+1}, \ldots, C_l$ such that $V(C_i) \cap V(H) = \{x_1\} = \{v\}$.
\( \{v\}, \; i = 1, 2, \ldots, j - 1, \; j + 1. \) Also the component \( C_j \) contains exactly one vertex, which is not adjacent to \( v \) in \( G \). In particular, \( x_1 \) and \( x_2 \) are adjacent and form a cut edge of \( G \). Thus \( \{x_1\} = \{v\} \) is a triple cut of \( G \).

**Theorem 4.3.8.** Let \( G \) be a connected graph with a unique cut vertex \( v \) such that \( d(v) < n - 1 \) and \( v \) be a support with pendant vertex \( x \). Then \( \overline{G} \) is not triple connected if and only if \( G - \{v, \; x\} \) contains a complete \( k \)-partite graph \( (k \geq 3) \) as a spanning subgraph with \( V(G) - N[v] \) belonging to the same partite set.

**Proof.** Let \( G' = \langle N(v) - x \rangle \) and \( G'' = \langle V(G) - N[v] \rangle \). Assume that \( \overline{G} \) is not triple connected. Since \( v \) is a unique cut vertex of \( G \), \( d(v) \geq 3 \).

**Case i) \( d(v) = 3 \).**

Let \( V(G') = N(v) - x = \{u, \; w\} \). Then by Lemma 4.3.6, \( d(u) = n - 2, \; d(w) = n - 2 \) and hence \( u \) and \( w \) are adjacent in \( G \) and \( G' = K_{1,1} \). Thus \( \{u\} \) and \( \{w\} \) are the required bipartite sets in \( G' \). Since every vertex in \( G'' \) is adjacent to both \( u \) and \( w \), \( \{\{u\}, \; \{w\}, \; \{V(G'')\}\} \) gives the partite sets of \( G - \{v, \; x\} \) and form a complete \( k \)-partite graph \( (k \geq 3) \) as the spanning subgraph.

**Case ii) \( d(v) > 3 \).**

Since \( d(v) \neq n - 1 \) in \( G \), \( |V(G'')| \neq \phi \). Since \( v \) is adjacent to all the vertices of \( G'' \) in \( \overline{G}, \; \langle G'' \cup \{v\} \rangle \) is a connected subgraph in \( \overline{G} \).
Since $\overline{G}$ is not triple connected and $d(x) = n - 2$, by Theorem 4.3.7, 
$\{x\}$ is a triple cut of $\overline{G}$. Then there exists at least three components in $\overline{G} - \{x\}$.
Let $C_1$ be a component of $\overline{G} - \{x\}$ which contains the subgraph $\langle G'' \cup \{v\} \rangle$.
Then there exists at least two components in $\overline{G} - \{x\}$, other than $C_1$.
Let $C_2, C_3, \ldots, C_p$ be the components of $\overline{G} - \{x\}, p \geq 3$. Clearly the vertices of $C_i$ ($i = 2, 3, \ldots, p$) are in $N(v)$. But $C_1$ may or may not contain vertices of $N(v)$. Hence we distinguish two cases.

**Subcase a)** $N_G(v) \cap V(C_1) = \phi$.

Then $\langle N_G(v) \rangle$ is a disconnected subgraph of $\overline{G}$ with components $C_2, C_3, \ldots, C_p$.
Hence by Proposition 4.3.2, there exists a complete $(p - 1)$-partite graph as a spanning subgraph of $G'$. Also, since every vertex of $C_1 - \{v\} = G''$ is adjacent to every vertex in $C_i$ ($i = 1, 2, \ldots, p$), $V(G''), V(C_2), \ldots, V(C_p)$ are the partite sets of a complete $p$-partite graph as a spanning subgraph of $G - \{v, x\}$ where $V(G'')$ belongs to the same partite set.

**Subcase b)** $N_G(v) \cap V(C_1) \neq \phi$.

Let $A = V(C_1) - V(G'') - \{v\}$. Then in $G$, every vertex in $A$ is adjacent to all the vertices of $C_i$, $i = 2, 3, \ldots, p$. Clearly $N(v) = V(C_2) \cup V(C_3) \cup \ldots \cup V(C_p) \cup V(A)$. Hence by Proposition 4.3.2, $V(C_2), V(C_3), \ldots, V(C_p)$ and $V(A)$ are the partite sets of a complete $p$-partite graph, as a spanning subgraph of $G'$. In particular, every
vertex in $C_1 - \{v\}$ is adjacent to all the vertices of $C_i$, $i = 2, 3, \ldots, p$. Clearly, $G - \{v, x\} = V(C_1 - \{v\}) \cup V(C_2) \cup \ldots \cup V(C_p)$ and by Proposition 4.3.2, $\{V(C_1 - \{v\}), V(C_2), \ldots, V(C_p)\}$ are the partite sets of a complete $p$-partite graph, as a spanning subgraph of $G - \{x, v\}$ where $V(G'')$ belongs to the same partite set.

Conversely, assume that $G - \{v, x\}$ contains a complete $k$-partite graph ($k \geq 3$) as a spanning subgraph with $V(G'')$ belonging to the same partite set. In $\overline{G}$, the vertex $x$ is adjacent to all vertices except $v$. Thus $d(x) = n - 2$ in $\overline{G}$. Clearly, $\overline{G} - \{x\}$ is disconnected. Since $d(v) \neq 1$, $v$ is adjacent to at least one vertex in $\overline{G}$. Thus $\langle G'' \cup \{v\} \rangle$ is connected subgraph of $\overline{G}$. If $\langle G'' \cup \{v\} \rangle$ is maximal, then $C_1 = \langle G'' \cup \{v\} \rangle$ is a component in $G - \{x\}$. Otherwise, we can find a component $C_1$ of $\overline{G} - \{x\}$ which contains $\langle G'' \cup \{v\} \rangle$. By assumption, $V(C_1) - \{v\}$ is the required partite set, which contains $V(G'')$. Also we can find at least two partite sets other than $V(C_1) - \{v\}$. Let $V(C_2), V(C_3)$ be the partite sets of $G - \{v, x\}$. Clearly, $V(C_2), V(C_3) \subseteq V(G')$. Thus by Proposition 4.3.2, we have $\langle \overline{G} - \{v, x\} \rangle$ has at least three components $C_1 - \{v\}, C_2, C_3$. Hence $\overline{G} - \{x\}$ has at least three components $C_1, C_2, C_3$. Thus $x$ is a triple cut and hence $\overline{G}$ is not triple connected. $\square$
4.4 Derived Graphs

This section deals with the triple connectedness of a power of a graph and mycielskian of a graph.

**Remark 4.4.1.** If $\text{diam}(G) \leq 2$, then $G^2$ is complete.

**Remark 4.4.2.** If $G$ is triple connected, then $G^k$ is triple connected.

**Remark 4.4.3.** Converse of the above remark need not be true.

For example, consider the graph $G$ given in Figure 4.2 which is not triple connected. But, square of the graph given in Figure 4.2 is triple connected.

![Figure 4.2. The graphs $G$ and $G^2$](image)

**Theorem 4.4.4.** If $G$ is a connected graph, then $G^2$ is triple connected.

**Proof.** Let $G$ be a connected graph.

We claim that $G^2$ is 2-connected.

Let $v$ be any cut vertex in $G$. Then $d(v) \geq 2$. Let $u, w \in N(v)$. Then $uw \notin E(G)$. Thus $d(u, w) = 2$. Hence $uw \in E(G^2)$. Hence
\( N[v] \) is a complete subgraph in \( G^2 \).

Let \( x, y \in V(G^2) \) be such that \( x, y \neq v \).

**Case i)** \( x, y \in N(v) \).

Then \( xy \in E(G^2) \). Hence there exists a path in \( G^2 \) joining \( x \) and \( y \) which does not contain \( v \).

**Case ii)** \( x \in N(v), y \in V - N[v] \).

Since \( G \) is connected, there exists a path \( P \) joining \( x \) and \( y \) in \( G \), say \( P : x = v_1v_2\ldots v_jy \). Since \( x, v_j \in N(v), xv_j \in E(G^2) \). Hence \( P' : xv_jv_{j+1}\ldots y \) is a path in \( G^2 \) joining \( x \) and \( y \) but not containing \( v \).

**Case iii)** \( x, y \in V - N[v] \).

Since \( G \) is connected, there exists a path joining \( x \) and \( y \) in \( G \). If \( P \) does not contain \( v \), then \( G^2 \) is 2-connected. Otherwise, let \( P : x = v_1v_2\ldots v_iv_jy \) be a \( xy \)-path in \( G \). Since \( v_i, v_j \in N(v), v_iv_j \in E(G^2) \). Then \( P' : v_1v_2\ldots v_iy \) is a \( xy \)-path in \( G^2 \) which does not contain \( v \). Hence \( v \) is not a cut vertex in \( G^2 \). Hence \( G^2 \) has no cut vertices. Hence \( G^2 \) is 2-connected.

Hence in all the cases, \( G^2 \) is triple connected.

\[ \blacksquare \]

**Corollary 4.4.5.** If \( G \) is connected, then \( G^k \) is triple connected.

**Definition 4.4.6.** The **Mycielskian** of a (finite simple) graph \( G = (V, E) \) is the graph \( \mu(G) \) with vertex set the disjoint union \( V \cup V' \cup \{u\} \), where \( V' = \{x' : x \in V\} \) and
Theorem 4.4.7. If $G$ is a non-trivial connected graph, then $< V \cup V' >$ is connected in $\mu(G)$.

**Proof.** Let $G$ be a non-trivial connected graph. Let $x, y \in V \cup V'$.

If $x, y \in V$, then $x$ and $y$ are connected in $G$ and hence $x$ and $y$ are connected in $< V \cup V' >$.

If $x, y \in V'$, then $x = u'$ and $y = v'$ where $u, v \in V$. Since $G$ is connected, there exists a path $u = w_1w_2\ldots w_k = v$ in $G$. Now, $u'w_2, v'w_{k-1} \in E(< V \cup V' >)$. Thus $x = u'w_2w_3\ldots w_{k-1}v' = y$ is a path in $< V \cup V' >$. Hence $x$ and $y$ are connected in $< V \cup V' >$.

If $x \in V$ and $y \in V'$, then $y = u'$ where $u \in V$. Since $G$ is connected, there exists a path $x = w_1w_2\ldots w_k = u$ in $G$. Also, $u'w_{k-1} \in E(< V \cup V' >)$. Thus $x = w_1w_2\ldots w_{k-1}u'$ is a path in $< V \cup V' >$. Hence $x$ and $y$ are connected in $< V \cup V' >$.

Thus $< V \cup V' >$ is connected in $\mu(G)$. \qed

**Remark 4.4.8.** If $G$ is connected, then $< V \cup V' >$ need not be 2-connected in $\mu(G)$. This is illustrated in Figure 4.3.
Figure 4.3. \(< V \cup V' >\) is not 2-connected

Here \(x\) and \(y\) are cut vertices in \(< V \cup V' >\).

**Remark 4.4.9.** If \(G\) has pendent vertices \(v_1, v_2, \ldots, v_k\), then \(v'_1, v'_2, \ldots, v'_k\) are pendent vertices in \(< V \cup V' >\).

**Corollary 4.4.10.** If \(G\) has more than two pendent vertices, then \(< V \cup V' >\) is not triple connected.

**Proof.** Since \(G\) has at least three pendent vertices, by above Corollary, \(< V \cup V' >\) has at least three pendent vertices. Now, there exists no path in \(< V \cup V' >\) containing any three of these pendent vertices. Hence \(< V \cup V' >\) is not triple connected.

**Example 4.4.11.** Consider the graph \(G\) given in Figure 4.4.
There exists no path in $< V \cup V' >$ containing $x, y$ and $z$ and hence $< V \cup V' >$ is not triple connected.

**Remark 4.4.12.** If $G$ is disconnected with $\omega$-nontrivial components, then $< V \cup V' >$ is disconnected with $\omega$-nontrivial components.

**Proof.** Let $H_1, H_2, \ldots, H_\omega$ be the nontrivial components of $G$. By Theorem 4.4.7, $< V(H_i) \cup V'(H_i) >; 1 \leq i \leq \omega$ is connected. Since $H_i$ and $H_j$ are disconnected, $< V(H_i) \cup V'(H_i) >$ and $< V(H_j) \cup V'(H_j) >$ are disconnected in $< V \cup V' >; i \neq j$ and $1 \leq i \leq \omega$. Hence $< V \cup V' >$ is disconnected with $\omega$ nontrivial components $< V(H_i) \cup V'(H_i) >; 1 \leq i \leq \omega$.  

**Remark 4.4.13.** If $G$ is trivial graph, then $\mu(G)$ is disconnected graph $K_1 \cup K_2$ and hence not triple connected.
Remark 4.4.14. Let $G$ be a disconnected graph with isolated vertices. Then $\mu(G)$ is also disconnected with the same isolated vertices.

Theorem 4.4.15. If $G$ is connected, then $\mu(G)$ is 2-connected.

Proof. Let $x, y \in V(\mu(G))$.

Case i) $x, y \in V$.

Since $G$ is connected, there exists a $x - y$ path, say $P_1 : x = w_1w_2 \ldots w_k = y$ in $G$, and hence in $\mu(G)$. Now, $w'_2x, w'_{k-1}y \in E(\mu(G))$. Hence $P_2 : xw'_2uw'_{k-1}y$ is a path in $\mu(G)$. Hence $P_1$ and $P_2$ are two internally disjoint $x - y$ paths in $\mu(G)$.

Case ii) $x, y \in V'$.

Thus $x = w'$ and $y = v'$, where $w, v \in V$. Now, $P_3 : w'uv'$ is a path in $\mu(G)$. Since $G$ is connected, there exists a path $u = w_1w_2 \ldots w_k = v$ in $G$. Also, $w'w_2, v'w'_{k-1} \in E(\mu(G))$. Thus $P_4 : x = w'w_2w_3 \ldots w_{k-1}v' = y$ is a path in $\mu(G)$. Hence $P_3$ and $P_4$ are two internally disjoint $x - y$ paths in $\mu(G)$.

Case iii) $x \in V, y \in V'$.

Thus $y = v'; v \in V$. Since $G$ is connected, there exists a path $x = w_1w_2 \ldots w_k = v$ in $G$. Now, $yw_{k-1} \in E(\mu(G))$. Thus $P_5 : x = w_1w_2 \ldots w_{k-1}y$ is a $x - y$ path in $\mu(G)$. Also, $P : xw'_2uv' = y$ is a $x - y$ path in $\mu(G)$. Hence $P_5$ and $P_6$ are two internally disjoint $x - y$ paths in $\mu(G)$. 

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Case iv) \( x \in V \) and \( y = u \).

Then \( P_7 : xzx'y \) and \( P_8 : xz'y \) are two internally disjoint \( x - y \) paths in \( \mu(G) \) where \( z \in N(x) \).

Case v) \( x' \in V' \) and \( y = u \).

Then \( P_9 : x'y \) and \( P_{10} : x'zxz'y \) are two internally disjoint \( x - y \) paths in \( \mu(G) \) where \( z \in N(x) \).

Hence \( \mu(G) \) is 2-connected.

Corollary 4.4.16. If \( G \) is triple connected, then \( \mu(G) \) is also triple connected.

Proof. Assume that \( G \) is triple connected. Then \( G \) is a connected graph. By Theorem 4.4.15, \( \mu(G) \) is 2-connected. Hence by Theorem 4.2.4, \( \mu(G) \) is triple connected.

Remark 4.4.17. Converse of the above corollary need not be true. For, consider the graph \( G \) given in Figure 4.5 which is not triple connected but \( \mu(G) \) is triple connected.

Figure 4.5. The graphs \( G \) and \( \mu(G) \)
Theorem 4.4.18. Let $G$ be a nontrivial graph. Then $\mu(G)$ is triple connected if and only if $G$ has no isolated vertices and $G$ has at most two nontrivial components.

Proof. Let $G$ be a nontrivial connected graph.

Assume that $\mu(G)$ is triple connected. Assume that either $G$ has isolated vertices or $G$ has at least three nontrivial components.

If $G$ has isolated vertices, then by Remark 4.4.14, $\mu(G)$ is also disconnected with the same isolated vertices. Hence $\mu(G)$ is not triple connected, which is a contradiction.

If $G$ is disconnected with at least three nontrivial components, then by Remark 4.4.12, $< V \cup V' >$ has the same number of components, say $C_i; i \geq 3$. Let $H = \{u\}$. Then $|V(H) \cap N(C_i)| = 1$ for all $i \geq 3$. Hence by Theorem 4.2.6, $\mu(G)$ is not triple connected, which is a contradiction.

Hence $G$ has no isolated vertices and $G$ has at most two nontrivial components.

Conversely, assume that $G$ has no isolated vertices and $G$ has at most two nontrivial components.

If $G$ has exactly one component, then by Theorem 4.4.15, $\mu(G)$ is 2-connected and hence triple connected.

If $G$ has exactly two nontrivial components, say $H_1$ and $H_2$, then by Theorem 4.4.15, the subgraphs $< V(H_i) \cup V'(H_i) \cup \{u\} >$;
\[ i = 1, 2 \text{ are 2-connected. Hence } \mu(G) \text{ has exactly two blocks} \]
\[ < V(H_1) \cup V'(H_1) \cup \{u\} > \text{ and } < V(H_2) \cup V'(H_2) \cup \{u\} > \]
\[ \text{with a common vertex } u. \text{ Hence the block graph of } \mu(G) \text{ is } P_2. \text{ Hence by Theorem 4.2.7, } \mu(G) \text{ is triple connected.} \]

4.5 Some Special Graphs

In this section, we give some examples and lemmas which are used in the subsequent sections.

**Example 4.5.1.** \( K_1 + 2K_2, K_3 + \overline{K_2}, K_2 + \overline{K_4}, \overline{K_2} + (K_2 \cup \overline{K_2}), \overline{K_2} + (K_1 \cup 2K_2), \overline{K_2} + 3K_2 \) are \( P_4 \)-decomposable.

**Notation 4.5.2.** Let \( G \) be a connected graph and let \( G' \) be a subgraph of \( G \). Then \( G - G' \) denotes the induced subgraph \( \langle E(G) - E(G') \rangle \).

**Definition 4.5.3.** Let \( G \) be any graph. Then \( G \land e \) is the graph obtained by attaching a new pendant edge \( e \) with any arbitrary vertex of \( G \).

**Example 4.5.4.** The graph \( G_1 = (K_1 + P_3) \land e \) is \( P_4 \)-decomposable.

**Lemma 4.5.5.** The graph \( (K_1 + P_{3k}) \land e \) is \( P_4 \)-decomposable.

**Proof.** We prove this by induction on \( k \). When \( k = 1, (K_1 + P_{3k}) \land e \cong G_1 \), which is \( P_4 \)-decomposable. Assume that the result is true
for all integers $< k$. Let $G = (K_1 + P_{3k}) \land e$ and let $V(K_1) = \{v_0\}$. Now, $G$ contains $G_1$ as an induced subgraph. Thus $E(G - G_1) =$

$$
\begin{cases}
E[(K_1 + P_{3(k-1)}) \land e'] & e \text{ is incident with } v_0 \text{ or } v_{3r+i}; \\
E[(K_1 + P_{3r}) \land e''] & i \in \{1, 2, 3\}, \ r \in \{0, k - 1\} \\
E[(K_1 + P_{3(k-r-1)}) \land e'''] & i \in \{1, 2, 3\}; \ 1 \leq r \leq k - 2.
\end{cases}
$$

where $e' = v_3v_4$ if $ev_0 \in E(G)$ or $ev_1 \in E(G)$ or $ev_3 \in E(G)$,

$e' = v_{3k-3}v_{3k-2}$ if $ev_{3k} \in E(G),$

$e'' = v_{3r}v_{3r+1}$ and

$e''' = v_{3(k-r-1)-1}v_{3(k-r-1)}$. By induction hypothesis, each of $(K_1 + P_{3(k-1)}) \land e'$, $(K_1 + P_{3r}) \land e''$ and $(K_1 + P_{3(k-r-1)}) \land e'''$ is $P_4$-decomposable. Thus $G - G_1$ is $P_4$-decomposable and hence $G$ is $P_4$-decomposable. \qed

**Example 4.5.6.** Let $G_2$ be the graph $[K_1 + (P_1 \cup P_2)] \land \{e_1, e_2\}$ obtained by attaching two pendent edges $e_1$ and $e_2$ one to $P_1$ and the other to $P_2$ in $K_1 + (P_1 \cup P_2)$. Then $G_2$ is $P_4$-decomposable.

**Example 4.5.7.** Let $G_3$ be the graph $[K_1 + (P_2 \cup P_3)] \land e$. Then $G_3$ is $P_4$-decomposable.

**Lemma 4.5.8.** The graph $K_1 + P_{3k+2}; \ k \geq 1$ is $P_4$-decomposable.
Proof. Let $G$ be the graph $K_1 + P_{3k+2}; k \geq 1$. Let $V(G) = \{v_0, v_1, \ldots, v_{3k+2}\}$ and let $K_1 = \{v_0\}$. Consider a $P_4$, say $P = v_2v_1v_0v_{3k+2}$ in $G$. Now, $G - P \cong (K_1 + P_{3k}) \wedge e$ where $e = v_{3k+1}v_{3k+2}$. By Lemma 4.5.5, $(K_1 + P_{3k}) \wedge e$ is $P_4$-decomposable. Hence $G$ is $P_4$-decomposable.

Lemma 4.5.9. The graph $K_1 + C_{3k}; k \geq 1$ is $P_4$-decomposable.

Proof. Let $G$ be the graph $K_1 + C_{3k}; k \geq 1$. Let $V(G) = \{v_0, v_1, \ldots, v_{3k}\}$ and let $K_1 = \{v_0\}$. When $k = 1$, $K_1 + C_{3k} \cong K_4$ and hence is $P_4$-decomposable. Assume that $k \geq 1$. Consider $G' = (\{v_0, v_1, v_2, v_3, v_4\}) - \{v_0v_4\}$. Clearly, $G' \cong (K_1 + P_3) \wedge e$ and hence is $P_4$-decomposable. Now, $G - G' \cong (K_1 + P_{3(k-1)}) \wedge e'$ where $e' = v_1v_{3k}$. By Lemma 4.5.5, $G - G'$ is $P_4$-decomposable. Hence $G$ is $P_4$-decomposable.

4.6 Biregular Graphs

Armen S. Asratian et al. [2] gave sufficient conditions for the existence of a proper path factor of a simple $(3,4)$-biregular bigraph. In this section, we investigate the $P_4$-decomposition of biregular triple connected graphs.

Definition 4.6.1. A graph $G$ is called $(a,b)$-biregular graph if the degree sequence of $G$ contains exactly two values $a$ and $b$. 

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Example 4.6.2. The graph $K_{3,4}$ is $(3, 4)$-biregular graph and is $P_4$-decomposable.

Remark 4.6.3. The graph given in Figure 4.6 is a triple connected $(3, 2)$-biregular graph with $m \equiv 0(\mod 3)$ and has no $P_4$-decomposition.

Hence we concentrate on triple connected biregular graphs which are $P_4$-decomposable.

Theorem 4.6.4. Let $G$ be a triple connected $(n-1, 2)$-biregular graph of size $m \equiv 0(\mod 3)$. Then $G$ is $P_4$-decomposable.

Proof. Let $V(G) = \{v_0, v_1, \ldots, v_{n-1}\}$. We consider two cases.

Case i) $G$ is 2-connected.

Since $G$ is a $(n-1, 2)$-biregular graph, $G$ contains at most two vertices of degree $n-1$.

If $G$ has exactly one vertex of degree $n-1$, say $v_0$, then $G-v_0$ is 1-regular. Since $G$ is 2-connected, $G-v_0$ is connected. Hence $G-v_0 \cong K_2$. Hence $G \cong K_3$, which is a contradiction to the biregularity of $G$. Hence $G$ has exactly two vertices of degree $n-1$, say $v_0, v_1$ in $G$. Since $d(v_i) = 2$ for all $i \in \{2, 3, \ldots, n-1\}$, $G - \{v_0, v_1\}$ is $K_{n-2}$. 

Figure 4.6. $(3, 2)$-biregular graph
Also, since \( m \equiv 0 \pmod{3} \), \( n = 3k + 3 \); \( k \in \mathbb{N} \). Thus \( G \cong K_2 + \overline{K}_{3k+1} \) and hence

\[
E(G) = E(K_2 + \overline{K}_4) \cup E(K_{2,3}) \cup E(K_{2,3}) \ldots \cup E(K_{2,3}) \text{ \((k-1)\text{ times})}
\]

Also, \( K_{2,3} \) and \( K_2 + \overline{K}_4 \) are \( P_4 \)-decomposable. Hence \( G \) is \( P_4 \)-decomposable.

**Case ii) \( G \) is 1-connected.**

Then \( G \) has exactly one vertex of degree \( n - 1 \), say \( v_0 \). Since \( d(v_i) = 2 \) for all \( i \in \{2, 3, \ldots, n - 1\} \), \( G - v_0 \) is 1-regular. Since \( G \) is triple connected, \( G - v_0 \cong 2K_2 \) and hence \( G \cong K_1 + 2K_2 \) which is \( P_4 \)-decomposable. \(\square\)

**Theorem 4.6.5.** Let \( G \) be a triple connected \((n - 1, 3)\)-biregular graph with \( m \equiv 0 \pmod{3} \). Then \( G \) is \( P_4 \)-decomposable.

**Proof.** Let \( V(G) = \{v_0, v_1, \ldots, v_{n-1}\} \). We consider two cases.

**Case i) \( G \) is 2-connected.**

Since \( G \) is a \((n - 1, 3)\)-biregular graph, \( G \) contains at most three vertices of degree \( n - 1 \).

**Subcase a) \( G \) has exactly one vertex of degree \( n - 1 \).**

Without loss of generality, let \( v_0 \) be the vertex of degree \( n - 1 \) in \( G \). Since \( d(v_i) = 3 \) for all \( i \in \{1, 2, \ldots, n - 1\} \) in \( G \), \( G - v_0 \) is 2-regular. Since \( G \) is 2-connected, \( G - v_0 \) is a cycle. Also, since \( m \equiv 0 \pmod{3} \), \( n = 3k + 1 \); \( k \in \mathbb{N} \). Thus \( G \cong K_1 + C_{3k} \); \( k \geq 1 \). By
Lemma 4.5.9, $G$ is $P_4$-decomposable.

**Subcase b)** $G$ has exactly two vertices of degree $n - 1$.

Without loss of generality, let $v_0, v_1$ be the vertices of degree $n - 1$. Since $d(v_i) = 3$ for all $i \in \{2, 3, 4, \ldots, n-1\}$ in $G$, $G - \{v_0, v_1\}$ is a 1-factor. Also, since $m \equiv 0 \pmod{3}$, $n = 6k + 4$; $k \geq 1$. Thus $G \cong K_2 + (3k+1)K_2$; $k \geq 1$.

Now, $E(G) = E(K_4) \cup E(K_2 + 3K_2) \cup \ldots \cup E(K_2 + 3K_2)$.

Also, $K_4$ and $K_2 + 3K_2$ are $P_4$-decomposable.

Hence $G$ is $P_4$-decomposable. This is illustrated in Figure 4.7.

![Figure 4.7. $(n-1, 3)$-biregular graph](image)

**Subcase c)** $G$ has exactly three vertices of degree $n - 1$.

Without loss of generality, let $v_0, v_1, v_2$ be the vertices of degree $n - 1$. Since $d(v_i) = 3$ for all $i \in \{3, 4, \ldots, n-1\}$, $G - \{v_0, v_1, v_2\}$ is $K_{n-3}$. Also, since $m \equiv 0 \pmod{3}$, $n = k + 4$; $k \geq 1$.  

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Thus $G = K_3 + \overline{K}_{k+1}; k \geq 3$.

Now,

$$E(G) = \begin{cases} 
E(\overline{K}_2 + K_3) \cup E(K_{2,3}) \cup \ldots \cup E(K_{2,3}) & , \text{k is odd} \\
E(K_4) \cup E(K_{2,3}) \cup \ldots \cup E(K_{2,3}) & , \text{k is even}
\end{cases} \times \frac{(k-1)}{2} \text{times} \times \frac{k}{2} \text{times}$$

Also, $K_{2,3}$, $K_4$ and $\overline{K}_2 + K_3$, are $P_4$-decomposable. Hence $G$ is $P_4$-decomposable.

Case ii) $G$ is 1-connected.

Thus $G$ has exactly one vertex of degree $n - 1$, which is the unique cut vertex of $G$. Without loss of generality, let $d(v_0) = n - 1$. Hence $G - v_0$ is a 2-regular disconnected graph. Since $G$ is triple connected, $G - v_0$ is the union of two cycles. Let the cycles be $C_1 : v_1v_2\ldots v_av_1$ and $C_2 : u_1u_2\ldots u_bu_1$. Now, $2a + 2b = m = 3r$.

Thus $a + b = 3k; k \geq 2$.

Subcase a) $a \equiv 0(mod 3)$.

Then $b \equiv 0(mod 3)$. Now, $C_1 + v_0$ is an $(a - 1, 3)$-biregular graph of diameter 2 and $|E(C_1 + v_0)| \equiv 0(mod 3)$. By Lemma 4.5.9, $C_1 + v_0$ is $P_4$-decomposable. Similarly, $C_2 + v_0$ is $P_4$-decomposable. Hence $G$ is $P_4$-decomposable.

Subcase b) $a \equiv 1 (mod 3)$.

Then $b \equiv 2 (mod 3)$. Hence $a = 3k + 1; k \geq 1$ and $b = 3r + 2;
$r \geq 1$. Consider $G' = \langle \{v_0, v_1, v_2, u_1, u_2, u_3\} \rangle - \{v_0v_2, v_0u_3\}$. Clearly, $G' \cong G_2$ and is $P_4$-decomposable, by Example 4.5.6.

Now, $G - G' \cong E[(K_1 + P_{3k}) \land e] \cup E[(K_1 + P_{3r}) \land e']$ where $e = v_1v_{3k+1}$, $e' = u_1u_{3r+2}$.

By Lemma 4.5.5, $(K_1 + P_{3k}) \land e$ and $(K_1 + P_{3r}) \land e'$ are $P_4$-decomposable. Hence $G$ is $P_4$-decomposable.

**Subcase c)** $a \equiv 2 (mod 3)$.

Proof of this case is similar to Subcase b.

Thus $G$ is $P_4$-decomposable. \qed

**Theorem 4.6.6.** ([15, 16]) For $n > 1$, an integer list $d$ of size $n$ is graphic if and only if $d'$ is graphic, where $d'$ is obtained from $d$ by deleting its largest element $\Delta$ and subtracting 1 from its $\Delta$ next largest elements. The only 1-element graphic sequence is $d_1 = 0$.

**Remark 4.6.7.** By above theorem, the degree sequence $(4, 4, 4, 3, 3, 3, 3)$ is graphic if and only if $(3, 3, 3, 3, 2, 2)$ is graphic. Here we write, $(4, 4, 4, 3, 3, 3, 3) \approx (3, 3, 3, 3, 2, 2)$.

**Remark 4.6.8.** There is exactly one graph with degree sequence $(4, 4, 4, 2, 2, 2)$ and is $P_4$-decomposable.

For, let $U(G) = U_1 \cup U_2$ where $U_1 = \{v_1, v_2, v_3\}$ be the vertices of degree 4 and let $U_2 = \{u_1, u_2, u_3\}$ be the vertices of degree 2 respectively. First we claim that $\langle U_2 \rangle = \phi$. Assume $\langle U_2 \rangle \neq \phi$. Then there exists at least one edge in $\langle U_2 \rangle$, say $u_1u_2$. Since $d(u_i) = 2$ for
all $i$, $1 \leq i \leq 3$ and $u_1u_2 \in E$, there can be at most 4 edges from $U_1$ to $U_2$. Hence $d(v_i) < 4$ for some $i$, $1 \leq i \leq 3$, a contradiction. Secondly, we claim that $N(u_i) \neq N(u_j)$ for all $i \neq j$, $1 \leq i, j \leq 3$. Assume $N(u_1) = N(u_2)$. Let $N(u_1) = \{v_1, v_2\}$. Then $v_3$ can be adjacent only to $u_3$ in $U_2$. Hence $d(v_3) < 4$, a contradiction. Hence $N(u_1), N(u_2)$ and $N(u_3)$ are all distinct. Since $d(v_i) = 4$ for all $v_i$, $1 \leq i \leq 3$, $\langle U_1 \rangle = K_3$. Hence $G \cong G_4$ which is given in Figure 4.8 and is $P_4$-decomposable.

Figure 4.8. The graph $G_4$

**Remark 4.6.9.** There is exactly one graph with degree sequence $(6, 6, 6, 6, 3, 3, 3, 3)$ and is $P_4$-decomposable.

For, let $U(G) = U_1 \cup U_2$ where $U_1 = \{v_1, v_2, v_3, v_4\}$ be the vertices of degree 6 and let $U_2 = \{u_1, u_2, u_3, u_4\}$ be the vertices of degree 3. First, we claim that $\langle U_2 \rangle = \phi$. Assume $u_1u_2 \in E(G)$. Since $d(u_i) = 3$ for all $u_i$; $1 \leq i \leq 4$ and $u_1u_2 \in E(G)$, there can be at most 11 edges from $U_1$ to $U_2$. Hence $d(v_i) < 6$ for some $v_i$, $1 \leq i \leq 4$, a contradiction. Secondly, we claim that $N(u_i) \cap N(u_j) \cap N(u_k) = \{v_l\}$ for some $l$, for all $1 \leq i, j, k, l \leq 4$. Assume $N(u_1) \cap N(u_2) \cap N(u_3) = \{v_1, v_2\}$. Since $d(v_3) = 6$, $v_3$ must be adjacent to at least 3 vertices.
of $U_2$. Now, $d(u_i) = 3$ for at least two $u_i; 1 \leq i \leq 3$. Hence $v_4$ can be adjacent to at most two vertices in $U_2$. Hence $d(v_4) < 6$, a contradiction. Hence each $v_i; 1 \leq i \leq 4$ is adjacent to exactly three vertices of $U_2$. Since $d(v_i) = 6$ for all $i ; 1 \leq i \leq 4$, $\langle U_1 \rangle = K_4$. Hence $G \cong G_5$ which is given in Figure 4.9 and is $P_4$-decomposable.

![Figure 4.9. The graph $G_5$](image)

**Remark 4.6.10.** There is exactly one graph with degree sequence $(6, 6, 6, 6, 6, 3, 3)$ and is $P_4$-decomposable.

For, let $U(G) = U_1 \cup U_2$ where $U_1 = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ be the vertices of degree 6 and let $U_2 = \{u_1, u_2\}$ be the vertices of degree 2 in $G$. First, we claim that $\langle U_2 \rangle = \phi$. Assume $u_1u_2 \in E(G)$. Since $d(u_1) = 3, d(u_2) = 3$ and $u_1u_2 \in E(G)$, there can be at most 5 edges from $U_1$ to $U_2$. Hence $d(v_i) < 6$ for some $i ; 1 \leq i \leq 6$, a contradiction. Secondly, we claim that $N(u_1) \cap N(u_2) = \phi$. Assume $N(u_1) \cap N(u_2) = \{v_1\}$. Hence there can be at most four edges from $U_1$ to $U_2$. Hence $d(v_i) < 5$ for some $i ; 2 \leq i \leq 6$, a contradiction. Thus $N(u_1) \cap N(u_2) = \phi$. Hence the graph is $G \cong G_6$ which is given in Figure 4.10 and is $P_4$-decomposable.
Figure 4.10. The graph $G_6$

**Theorem 4.6.11.** Let $G$ be a triple connected $(n - 2, 2)$-biregular graph with $m \equiv 0 \pmod{3}$. Then $G$ is $P_4$-decomposable.

**Proof.** Let $G$ be a triple connected $(n - 2, 2)$-biregular graph with $m \equiv 0 \pmod{3}$. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$. Now, we claim that $G$ has exactly 2 or 3 vertices of degree $n - 2$.

Assume that $G$ has exactly 1 vertex of degree $n - 2$. Let $v_1$ be the only vertex of degree $n - 2$. Let $v_2 \notin N(v_1)$. Since $d(v_2) = 2$, $v_2v_3, v_2v_4 \in E(G)$ where $v_3, v_4 \in N(v_1)$. Also, since $d(v_i) = 2$ for all $v_i \in V - \{v_0\}$, $G - \{v_1, v_2, v_3, v_4\}$ is 1-regular. Since $G$ is triple connected, $G - \{v_1, v_2, v_3, v_4\} \cong K_2$. Hence $G \cong [K_1 + (P_2 \cup P_3)] - e$ where $e$ is the edge whose ends are of degree 5 and 3. Hence $m = 7$, which is a contradiction.

Assume that $G$ has exactly 4 vertices of degree $n - 2$. Then the degree sequence is $(n - 2, n - 2, n - 2, n - 2, 2, 2, 2, \ldots, 2) \approx (n - 3, n - 3, n - 3, 2, 1, \ldots, 1) \approx (n - 4, n - 4, 1, 1, 0, \ldots, 0)$. This implies that $n - 4 \leq 2$. If $n - 4 = 2$, then the degree sequence is $(4, 4, 4, 4, 2, 2)$. Hence $m = 10$, which is a contradiction. If $n - 4 = 1$,
then the degree sequence is \((3, 3, 3, 3, 2)\). Hence \(m = 7\), which is a contradiction.

Assume that \(G\) has 5 or more vertices of degree \(n - 2\). Then the number of vertices of degree 2 is exactly 1 (otherwise, it will contain at least 1 vertex of degree \(n - 3\)). Let \(v_n\) be the vertex of degree 2. Let \(v_1, v_2 \in N(v_n)\). Then \(v_i v_j \in E(G)\) for all \(i \neq j; i \in \{3, 4, \ldots, n - 1\}, j \in \{1, 2, \ldots n - 1\}\). Since \(d(v_1) = d(v_2) = n - 2\), \(v_1 v_2 \notin E(G)\). Hence \(m = \frac{(n-1)(n-2)}{2} + 1\).

If \(n \equiv 0(\text{mod } 3)\), then \(n = 3k; \ k \geq 2\).
Now, \(m = \frac{(n-1)(n-2)}{2} + 1 = \frac{n^2-3n+4}{2} = \frac{(3k)^2-3(3k)+4}{2} = 3\left(\frac{k(3k-3)}{2}\right) + 2\).
Hence \(m \not\equiv 0(\text{mod } 3)\), a contradiction.

If \(n \equiv 1(\text{mod } 3)\), then \(n = 3k + 1; \ k \geq 2\). Now, \(m = \frac{n^2-3n+4}{2} = \frac{(3k+1)^2-3(3k+1)+4}{2} = 3\left(\frac{k(3k-1)}{2}\right) + 1\).
Hence \(m \not\equiv 0(\text{mod } 3)\), a contradiction.

If \(n \equiv 2(\text{mod } 3)\), then \(n = 3k + 2; \ k \geq 2\). Now, \(m = \frac{n^2-3n+4}{2} = \frac{(3k+2)^2-3(3k+2)+4}{2} = 3\left(\frac{k(3k+1)}{2}\right) + 1\).
Hence \(m \not\equiv 0(\text{mod } 3)\), a contradiction.

Hence \(G\) has exactly 2 or 3 vertices of degree \(n - 2\).

**Case i)** \(G\) has exactly 3 vertices of degree \(n - 2\).

Then the degree sequence is \((n - 2, n - 2, n - 2, 2, 2, \ldots, 2)\)
\(\approx (n - 3, n - 3, 2, 1, \ldots, 1) \approx (n - 4, 1, 1, 0, \ldots 0)\). Thus \(n - 4 = 2\) and the degree sequence is \((4, 4, 4, 2, 2, 2)\). By Remark 4.6.8, the only
graph with this degree sequence is $G_4$ and is $P_4$-decomposable.

**Case ii)** $G$ has exactly 2 vertices of degree $n - 2$.

Let $v_1$ and $v_2$ be the vertices of degree $n - 2$.

**Subcase a)** $\{\{v_1, v_2\}\} \cong \overline{K_2}$.

Since $v_1v_2 \notin E(G)$, both $v_1$ and $v_2$ are adjacent to all the vertices in $V - \{v_1, v_2\}$. Since $d(v_i) = 2$ for all $v_i \in V - \{v_1, v_2\}$, $G - \{v_1, v_2\}$ is totally disconnected. Thus $n = 3r + 2; \ r \in \mathbb{N}$. Hence $G \cong K_{2,3r}$. By Theorem 1.2.8, $K_{2,3r}$ is $P_4$-decomposable. Hence $G$ is $P_4$-decomposable.

**Subcase b)** $\{\{v_1, v_2\}\} \cong K_2$.

Let $x \notin N(v_1) \text{ and } y \notin N(v_2)$.

If $x = y$, then $G$ is disconnected, a contradiction.

If $x \neq y$, then let $x = v_3, \ y = v_4$. Then $v_1$ and $v_2$ are adjacent to all the vertices in $\{v_5, v_6, \ldots, v_n\}$. Since $d(v_i) = 2$ for all $i \in \{3,4,\ldots,n\}$, $G - \{v_1, v_2, v_3, v_4\}$ is totally disconnected and $v_3v_4 \in E(G)$. Consider the graph $H$ in Figure 4.11 which is $P_4$-decomposable.

Since $n = 3r + 2; \ r \in \mathbb{N}$, $G$ is the graph in Figure 4.11

Figure 4.11. The graphs $H$ and $G$
and $E(G) = E(G') \cup E(K_{2,3(r-1)})$. By Theorem 1.2.8, $K_{2,3(r-1)}$ is $P_4$-decomposable. Hence $G$ is $P_4$-decomposable.

**Theorem 4.6.12.** Let $G$ be a triple connected $(n - 2, 3)$-biregular graph with $m \equiv 0(\mod 3)$. Then $G$ is $P_4$-decomposable.

**Proof.** Let $G$ be a triple connected $(n - 2, 3)$-biregular graph with $m \equiv 0(\mod 3)$. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$. Then we claim that $G$ has exactly 2 or 4 or 6 vertices of degree $n - 2$. Otherwise, we have the following contradictions.

If $G$ has a single vertex of degree $n - 2$, then
\[
\sum_{v \in V(G)} d(v) = n - 2 + 3(n - 1) = 4n - 7, \text{ odd.}
\]

If $G$ has three vertices of degree $n - 2$, then
\[
\sum_{v \in V(G)} d(v) = 3(n - 2) + 3(n - 3) = 6n - 15, \text{ odd.}
\]

If $G$ has five vertices of degree $n - 2$, then
\[
\sum_{v \in V(G)} d(v) = 5(n - 2) + 3(n - 5) = 8n - 25, \text{ odd.}
\]

If $G$ has 7 or more vertices of degree $n - 2$, then $G$ has exactly one vertex of degree 3. If $n$ is even, then $n = 2k$; $k \geq 4$. Hence the degree sequence is $(2k - 2, 2k - 2, \ldots, 2k - 2, 3)$, which is not graphic. If $n$ is odd, then $n = 2k - 1$. Hence the degree sequence is $(2k - 3, 2k - 3, \ldots, 2k - 3, 3)$, which is not graphic.
Hence we consider the following three cases:

**Case i)** \( G \) has two vertices of degree \( n - 2 \).

Let \( v_1, v_2 \) be the vertices of degree \( n - 2 \) in \( G \).

**Subcase a)** \( \{\{v_1, v_2\}\} \cong K_2 \).

Then both \( v_1 \) and \( v_2 \) are adjacent to all other vertices. Since \( d(v_i) = 3 \) for all \( i \in \{3, 4, \ldots, n\} \), \( G - \{v_1, v_2\} \) is 1-regular. Thus \( n = 6r + 2 \); \( r \in \mathbb{N} \) and \( G \) is the graph in Figure 4.12.

Thus \( E(G) = E(G') \cup E(G') \cup \ldots \cup E(G') \) where \( G' \) is the graph in Figure 4.12.

![Figure 4.12. The graphs G and G'](image)

Also, \( G' \) is \( P_4 \)-decomposable. Hence \( G \) is \( P_4 \)-decomposable.

**Subcase b)** \( \{\{v_1, v_2\}\} \cong K_2 \).

Let \( x \notin N(v_1) \) and \( y \notin N(v_2) \).

If \( x = y \), then let \( x = v_3 \). Since \( d(v_3) = 3 \), \( v_3 \) is adjacent to three vertices in \( V - \{v_1, v_2, v_3\} \), say \( v_4, v_5, v_6 \). Also, since \( d(v_i) = 3 \) for all \( i \in \{7, 8, \ldots, n\} \), \( G - \{v_1, v_2, \ldots, v_6\} \) is 1-regular. Hence \( G \) is the graph in Figure 4.13.

Since \( n = 6r + 2 \); \( r \in \mathbb{N} \), \( E(G) = E(G'') \cup E(G') \cup \ldots \cup E(G') \) where
$G''$ is the graph in Figure 4.13.

Also, $G'$ and $G''$ are $P_4$-decomposable. Hence $G$ is $P_4$-decomposable.

If $x \neq y$, then let $x = v_3$ and $y = v_4$.

If $v_3v_4 \in E(G)$, then $v_3$ and $v_4$ will be adjacent to two distinct vertices in $V - \{v_1, v_2, v_3, v_4\}$, say $v_5, v_6$ respectively (otherwise, $d(v_i) = 4$ if $v_3v_i, v_4v_i \in E(G)$, which is impossible). Hence $G$ is the graph in Figure 4.14.

Since $n = 6r + 2; \ r \in \mathbb{N}, \ E(G) = E(G''') \cup E(G') \cup \ldots \cup E(G')_{2(r-1)}$ times

where $G'''$ is the graph in Figure 4.14.

Also, $G'''$ is $P_4$-decomposable. Hence $G$ is $P_4$-decomposable.

If $v_3v_4 \notin E(G)$, then $v_3$ will be adjacent to two distinct vertices
in $V - \{v_1, v_2, v_3, v_4\}$, say $v_5, v_6$ and $v_4$ will be adjacent to two distinct vertices in $V - \{v_1, v_2, \ldots, v_6\}$, say $v_7, v_8$ (otherwise, $d(v_i) > 4$, which is impossible). Hence $G$ is the graph in Figure 4.15.

Since $n = 6r + 2; r \in \mathbb{N}$, $E(G) = E(G''') \cup E(G') \cup \ldots \cup E(G')$ where $G'''$ is the graph in Figure 4.15.

![Figure 4.15. The graphs $G$ and $G'''$](image)

Also, $G'''$ is $P_4$-decomposable. Hence $G$ is $P_4$-decomposable.

**Case ii)** $G$ has exactly four vertices of degree $n - 2$.

Then the degree sequence is $(n - 2, n - 2, n - 2, n - 2, 3, \ldots, 3) \approx (n - 5, 1, 1, 1, 0, \ldots, 0)$. Thus $n - 5 \leq 3$. If $n - 5 = 2$ or $1$, then either the degree sequence is non-graphic or $G$ is disconnected, which is a contradiction. If $n - 5 = 3$, then $n = 8$ and the degree sequence is $(6, 6, 6, 6, 3, 3, 3, 3)$. By Remark 4.6.9, the only graph with this degree sequence is $G_5$ and is $P_4$-decomposable.

**Case iii)** $G$ has exactly 6 vertices of degree $n - 2$.

Then the degree sequence is $(n - 2, n - 2, n - 2, n - 2, n - 2, n - 2, 3, \ldots, 3) \approx (n - 5, n - 5, n - 5, 1, 1, 1, 0, \ldots, 0)$. Thus $n - 5 \leq 3$. If $n - 5 = 2$, then the degree sequence is $(5, 5, 5, 5, 5, 3) \approx
(4,4,4,4,4,3), which is not graphic, a contradiction. If \( n - 5 = 1 \), then \( G \) is not biregular, a contradiction. If \( n - 5 = 3 \), then \( n = 8 \) and the degree sequence is \((6,6,6,6,6,3,3)\). By Remark 4.6.10, the only graph with this degree sequence is \( G_6 \) and is \( P_4 \)-decomposable.

\[ \square \]

### 4.7 Triregular Graphs

In 2010, Shasha Li et al. [31] studied about triregular graphs whose energy exceeds the number of vertices. In this section, we investigate the \( P_4 \)-decomposition of triregular triple connected graphs.

**Definition 4.7.1.** A graph \( G \) is called \((a, b, c)\)-triregular graph if the degree sequence of \( G \) contains exactly three values \( a, b \) and \( c \).

**Example 4.7.2.** The graph given in Figure 4.16 is a triple connected \((4,3,2)\)-triregular graph with \( m \equiv 0 \text{ (mod 3)} \) and has no \( P_4 \)-decomposition.

![Figure 4.16](image-url)

Figure 4.16. \((4,3,2)\)-triregular graph
Hence we concentrate on triple connected triregular graphs which are $P_4$-decomposable.

**Theorem 4.7.3.** Let $G$ be a triple connected $(n - 1, n - 3, 2)$-triregular graph of size $m \equiv 0 \pmod{3}$. Then $G$ is $P_4$-decomposable.

**Proof.** Let $G$ be a triple connected $(n-1, n-3, 2)$-triregular graph. Let $V(G) = \{v_0, v_1, \ldots, v_{n-1}\}$. Since $G$ is $(n - 1, n - 3, 2)$-triregular graph, it has exactly one vertex of degree $n - 1$ and one vertex of degree $n - 3$, say $v_0, v_1$ respectively. Since $d(v_1) = n - 3$ and $d(v_i) = 2$ for all $2 \leq i \leq n - 1$ in $G$, in $G - \{v_0, v_1\}$, $d(v_i) = 0$ for all $v_i \in N(v_1)$ and $d(v_i) = 1$ for the remaining two vertices $v_i \notin N(v_1)$.

Thus $G - v_0 \cong K_{1,n-4} \cup K_2$. Hence $G \cong K_1 + (K_{1,n-4} \cup K_2)$. Since $m \equiv 0 \pmod{3}$, $n = 3k + 5; \ k \in \mathbb{N}$. Thus $G \cong K_1 + (K_{1,3k+1} \cup K_2)$.

Now, $E(G) = E(2K_2 + K_1) \cup E(K_{2,3}) \cup \ldots \cup E(K_{2,3})$. $k$ times

Also, $K_1 + 2K_2$ and $K_{2,3}$ are $P_4$-decomposable.

Hence $G$ is $P_4$-decomposable. \qed

**Remark 4.7.4.** There does not exist a triple connected $(n - 1, n - 3, 1)$-triregular graph, since the vertex of degree $n - 1$ is a $H$-cut of $G$.

**Remark 4.7.5.** There does not exist a triple connected $(n - 1, n - 5, 2)$-triregular graph.
Theorem 4.7.6. For $n > 5$, let $G$ be a triple connected $(n-1, 3, 1)$-triregular graph of size $m \equiv 0 \pmod{3}$. Then $G$ is $P_4$-decomposable.

Proof. Let $V(G) = \{v_0, v_1, \ldots, v_{n-1}\}$ and let $d(v_0) = n - 1$. Since $G$ is triple connected, $G - v_0 \cong K_1 \cup C_k$ where $C_k = v_2v_3\ldots v_{n-1}v_2$.

Since $m \equiv 0 \pmod{3}$, $n = 3k + 3$; $k \in \mathbb{N}$. Consider a $P_4$ in $G$, say $P = v_1v_0v_2v_3$. Then,

$$G - P \cong \begin{cases} (K_1 + P_{3k}) \land e, & n \geq 9; \ e = v_2v_{3k+2} \\ G_1, & n = 6 \end{cases}$$

By Lemma 4.5.5, $(K_1 + P_{3k}) \land e$ and $G_1$ are $P_4$-decomposable and hence $G$ is $P_4$-decomposable. \qed

Theorem 4.7.7. Let $G$ be a triple connected $(n-1, 3, 2)$-triregular graph of size $m \equiv 0 \pmod{3}$. Then $G$ is $P_4$-decomposable.

Proof. Let $V(G) = \{v_0, v_1, \ldots, v_{n-1}\}$.

Case i) $G$ has exactly one vertex of degree $n - 1$.

Let $v_0$ be the unique vertex of degree $n - 1$ in $G$. Now, in $G - v_0$, $d(v_i) = 2$ or 1 for all $i \in \{1, 2, \ldots, n - 1\}$. Thus $G - v_0$ is a path of length greater than two or union of paths and cycles. Since $G$ is triple connected, $G - v_0$ has at most two components.

If $G - v_0$ has exactly one component, then it is a path on $n - 1$ vertices. That is, $G - v_0 \cong P_{n-1}$. Since $m \equiv 0 \pmod{3}$, $n = \ldots$
$3k + 3, \ k \in \mathbb{N}.$ Thus $G \cong K_1 + P_{3k+2}$. By Lemma 4.5.8, $G$ is $P_4$-decomposable.

If $G - v_0$ has two components, then they can not be cycles or paths of length two. Hence $G - v_0$ is the union of a path and a cycle. They are dealt in the following subcases.

**Subcase a)** $G - v_0 \cong P_2 \cup C_{n-3}$.

Thus $G \cong K_1 + (P_2 \cup C_{n-3})$. Since $m \equiv 0(\text{mod } 3), \ n = 3k + 3; \ k \in \mathbb{N}$. Now,

$$E(G) = \begin{cases} E[(K_1 + (P_2 \cup P_3)) \land e] \cup E[(K_1 + P_{3(k-1)}) \land e'], & n \geq 9 \\ E[(K_1 + (P_2 \cup P_3)) \land e], & n = 6 \end{cases}$$

where $e = v_{3k-3}v_{3k-2}, \ e' = v_1v_{3k}$ are pendent edges attached to $K_1 + (P_2 \cup P_3)$ and $K_1 + P_{3(k-1)}$ respectively.

By Lemma 4.5.5 and Example 4.5.7, $(K_1 + P_{3(k-1)}) \land e'$ and $[K_1 + (P_2 \cup P_3)] \land e$ are $P_4$-decomposable and hence $G$ is $P_4$-decomposable. This is illustrated in Figure 4.17.

![Figure 4.17. (n - 1, 3, 2)-triregular graph](image)

**Subcase b)** $G - v_0 \cong P_k \cup C_{n-k-1}; \ k \geq 3$.

Thus $G \cong (P_k \cup C_{n-k-1})$. Since $m \equiv 0(\text{mod } 3), \ n = 3r+6; \ r \in \mathbb{N}$.
If \( k \equiv 0(\text{mod } 3) \), then \( k = 3s \); \( s \in \mathbb{N} \). Also, \( n-k-1 \equiv 2(\text{mod } 3) \). Thus \( n-k-1 = 3t + 2 \); \( t \in \mathbb{N} \).

Now, \( E(G) = \)

\[
\begin{align*}
E(G_3) &\cup E[(K_1 + P_{3r}) \land e_1] & &, k = 3 \\
E(G_1) &\cup E(G_3) \cup E[(K_1 + P_{3(r-1)}) \land e_2] & &, n-k-1 = 5 \\
E(G_2) &\cup E[(K_1 + P_{3t}) \land e_3] \cup E(K_1 + P_{3(s-1)+2}) & &\text{otherwise}
\end{align*}
\]

where \( e_1 = e_3 = w_1w_{3t+2} \) and \( e_2 = u_3u_4 \) are pendent edges attached to \( K_1 + P_{3r}, K_1 + P_{3(r-1)}, K_1 + P_{3t} \) respectively.

By Examples 4.5.6, 4.5.7, Lemmas 4.5.5 and 4.5.8, each of \( (K_1 + P_{3r}) \land e_1, (K_1 + P_{3(r-1)}) \land e_2, (K_1 + P_{3t}) \land e_3, K_1 + P_{3(s-1)+2}, G_1, G_2 \) and \( G_3 \) is \( P_4 \)-decomposable. Hence \( G \) is \( P_4 \)-decomposable.

If \( k \equiv 1(\text{mod } 3) \), then \( k = 3s + 1 \); \( s \in \mathbb{N} \). Also, \( n-k-1 \equiv 1(\text{mod } 3) \). Thus \( n-k-1 = 3t + 1 \); \( t \in \mathbb{N} \). Rename the vertices of the path as \( u_1, u_2, \ldots, u_{3s+1} \) and the vertices of the cycle as \( w_1, w_2, \ldots, w_{3t+1} \).

Then \( E(G) = \)

\[
\begin{align*}
E(G_1) &\cup E(G') & &, k = 4 \\
E(G_1) &\cup E(P) \cup E[(K_1 + P_{3s}) \land e_1] & &, n-k-1 = 4 \\
E(P) &\cup E[(K_1 + P_{3s}) \land e_2] \cup E[(K_1 + P_{3t}) \land e_3] & &\text{otherwise}
\end{align*}
\]

where \( e_1 = u_1u_2, e_2 = u_3u_{3s+1} \) and \( e_3 = w_1w_{3t+1} \) are pendent edges attached to \( (K_1 + P_{3s}), (K_1 + P_{3s}) \) and \( (K_1 + P_{3t}) \) respectively,
\[ P = w_2w_1v_0u_1, \quad P' = w_{3t}w_{3t+1}v_0u_{3s+1} \] and \( G' \) is a triple connected \((n-1,3,1)\)-triregular graph and \(|E(G')| \equiv 0\pmod{3}\).

By Lemma 4.5.5 and Theorem 4.7.6, each of \(G_1, (K_1 + P_{3r}) \land e_1, (K_1 + P_{3s}) \land e_2, (K_1 + P_{3t}) \land e_3\) and \(G'\) is \(P_4\)-decomposable. Hence \(G\) is \(P_4\)-decomposable.

If \(k \equiv 2\pmod{3}\), then \(k = 3s + 2; \ s \in \mathbb{N}\). Also, \(n - k - 1 \equiv 0\pmod{3}\). Then \(n - k - 1 = 3t; \ t \in \mathbb{N}\). Thus \(E(G) = E(K_1 + P_{3s+2}) \cup E(K_1 + C_{3t})\).

By Lemma 4.5.8 and 4.5.9, both \(K_1 + P_{3s+2}\) and \(K_1 + C_{3t}\) are \(P_4\)-decomposable. Hence \(G\) is \(P_4\)-decomposable.

**Subcase c)** \(G - v_0 \cong P_2 \cup P_{n-3}\).

Thus \(G \cong K_1 + (P_2 \cup P_{n-3})\). Since \(m \equiv 0\pmod{3}\), \(n = 3k+5; \ r \in \mathbb{N}\). Thus \(G \cong K_1 + (P_2 \cup P_{3k+2})\). Thus \(E(G) = E(K_1 + 2K_2) \cup E[(K_1 + P_{3k}) \land e]\).

By Example 4.5.1 and Lemma 4.5.5, both \(K_1 + 2K_2\) and \((K_1 + P_{3k}) \land e\) are \(P_4\)-decomposable. Hence \(G\) is \(P_4\)-decomposable.

**Subcase d)** \(G - v_0 \cong P_r \cup P_s; \ r \geq 3; \ s \geq 3\).

Since \(m \equiv 0\pmod{3}\), \(n = 3r + 5; \ r \in \mathbb{N}\). Thus \(G \cong K_1 + (P_k \cup P_{n-k-1})\).

If \(k \equiv 0\pmod{3}\), then \(k = 3s; \ s \in \mathbb{N}\). Also, \(n - k - 1 \equiv 1\pmod{3}\). Then \(n - k - 1 = 3t + 1; \ t \in \mathbb{N}\). Rename the vertices of the path as \(u_1, u_2, \ldots, u_{3s}\) and \(w_1, w_2, \ldots, w_{3t+1}\). Consider a \(P_4\), say...
\[ P = w_1v_0u_1w_2 \text{ in } G. \text{ Then} \]
\[ E(G) = \begin{cases} 
E(G_3) \cup E(K_1 + P_{3(r-1)+2}), & k = 3 \text{ or } n - k - 1 = 3 \\
E(P) \cup E[(K_1 + P_{3t}) \wedge e_1] \cup E[(K_1 + P_{3(s-1)+2})] & \text{otherwise}
\end{cases} \]
where \( e_1 = w_1w_2 \) is a pendent edge attached to \( K_1 + P_{3t} \).

By Example 4.5.7, Lemmas 4.5.5 and 4.5.8, each of \( G_3, K_1 + P_{3(r-1)+2}, K_1 + P_{3(s-1)+2} \) and \( (K_1 + P_{3t}) \wedge e_1 \) is \( P_4 \)-decomposable. Hence \( G \) is \( P_4 \)-decomposable.

If \( k \equiv 1(\text{mod } 3) \), then the case is similar as above.

If \( k \equiv 2(\text{mod } 3) \), then \( k = 3s + 2; \ s \in \mathbb{N} \). Also, \( n - k - 1 \equiv 2(\text{mod } 3) \). Then \( n - k - 1 = 3t + 2; \ s, t \in \mathbb{N} \). Thus \( E(G) = E(K_1 + P_{3s+2}) \cup E(K_1 + P_{3t+2}) \).

By Lemma 4.5.8, both \( (K_1 + P_{3s+2}) \) and \( (K_1 + P_{3t+2}) \) are \( P_4 \)-decomposable. Hence \( G \) is \( P_4 \)-decomposable.

**Case ii)** \( G \) has two vertices of degree \( n - 1 \).

Without loss of generality, let \( v_0, v_1 \) be the vertices of degree \( n - 1 \). Since \( d(v_i) = 0 \) or 1 for all \( i \in \{1, 2, \ldots, n - 1\} \), \( G - \{v_0, v_1\} \cong rK_2 \cup \overline{K_{n-2-2r}}; \ r \geq 1. \)

\[ \text{Since } m \equiv 0(\text{mod } 3), \ n = \frac{3k - r + 3}{2}; \ r, k \in \mathbb{N}. \quad (4.1) \]

**Subcase a)** \( r \equiv 0(\text{mod } 3) \).

Then \( r = 3s; \ s \in \mathbb{N} \). By Equation (4.1), \( n - 2r - 2 \equiv 1(\text{mod } 3) \).

Thus \( n - 2r - 2 = 3t + 1; \ t \in \mathbb{N} \).
Then $E(G) = E(K_2 + \overline{K}_{3t+1}) \cup E(K_2 + 3K_2) \cup \ldots \cup E(\overline{K}_2 + 3K_2)$. s times

By Theorem 4.6.4 and Example 4.5.1, both $K_2 + \overline{K}_{3t+1}$ and $\overline{K}_2 + 3K_2$ are $P_4$-decomposable.

Hence $G$ is $P_4$-decomposable.

**Subcase b)** $r \equiv 1 \,(mod\, 3)$.

Then $r = 3s + 1; \, s \in \mathbb{N}$. By Equation (4.1), $n - 2r - 2 \equiv 0 \,(mod\, 3)$. Thus $n - 2r - 2 = 3t; \, t \in \mathbb{N}$. Then

$E(G) = E(\overline{K}_2 + (K_2 \cup \overline{K}_2)) \cup E(K_2 + \overline{K}_{3(t-1)+1}) \cup E(\overline{K}_2 + 3K_2) \cup \ldots \cup E(\overline{K}_2 + 3K_2). s times

By Example 4.5.1 and Theorem 4.6.4, each of $\overline{K}_2 + (K_2 \cup \overline{K}_2)$, $\overline{K}_2 + 3K_2$ and $K_2 + \overline{K}_{3(t-1)+1}$ is $P_4$-decomposable.

Hence $G$ is $P_4$-decomposable.

**Subcase c)** $r \equiv 2 \,(mod\, 3)$.

Then $r = 3s + 2; \, s \in \mathbb{N}$. By Equation (4.1), $n - 2r - 2 \equiv 2 \,(mod\, 3)$. Thus $n - 2r - 2 = 3t + 2; \, t \in \mathbb{N}$. Then

$E(G) = E(K_2 + \overline{K}_{3t+1}) \cup E(\overline{K}_2 + (K_1 \cup 2K_2)) \cup E(\overline{K}_2 + 3K_2) \cup \ldots \cup E(\overline{K}_2 + 3K_2). s times

By Example 4.5.1 and Theorem 4.6.4, each of $\overline{K}_2 + (K_1 \cup 2K_2)$, $\overline{K}_2 + 3K_2$ and $K_2 + \overline{K}_{3t+1}$ is $P_4$-decomposable.

Hence $G$ is $P_4$-decomposable. □
4.8 Open Problems

1. Characterize triple connected (n-1, 4)-biregular graphs which are $P_4$-decomposable.

2. Characterize triple connected (n-2, 4)-biregular graphs which are $P_4$-decomposable.

3. Characterize triple connected (n-1, n-2, n-3)-triregular graphs which are $P_4$-decomposable.

4. Characterize triple connected (n-1, n-3, n-4)-triregular graphs which are $P_4$-decomposable.

5. Characterize triple connected (n-2, n-3, n-4)-triregular graphs which are $P_4$-decomposable.