APPENDIX : 3

ON BIANCHI TYPE VI COMOLOGICAL MODELS
1: INTRODUCTION

In recent years a number of investigations have been made into anisotropic cosmological models. Many relativists have made detailed studies of gravitational fields which can be described by space-times of various Bianchi type. In the present work, we shall discuss Bianchi type VI\textsubscript{o} space-times.

Ellis and MacCallum (1969) have discussed a class of models with expansion and shear but no rotation which is of Bianchi type VI with three Killing vectors. Tariq and Tupper (1974,1975) have discovered a Bianchi type VI\textsubscript{o} solution of Einstein-Maxwell equations. Dunn and Tupper (1976) have given a class of Bianchi type VI cosmological models with electromagnetic field. Tupper (1977) has studied conductivity in type VI\textsubscript{o} cosmologies with electromagnetic field. Dunn and Tupper (1978) have discussed tilting and viscous models in a class of Bianchi type VI\textsubscript{o} cosmologies. Banerjee, Ribeiro and Santos (1987) have presented a Bianchi type VI\textsubscript{o} viscous-fluid cosmological model. Roy and Singh (1983) have derived some Bianchi type VI\textsubscript{o} cosmological models with free gravitational field of the magnetic type. Narain (1988) has
given a new Bianchi type $\text{VI}_0$ model filled with perfect fluid. The main aim of the present work is to obtain generalizations of the perfect fluid solution of Narain (1988).

For this we consider a special class of Bianchi type $\text{VI}_0$ spacetimes described by the line element,

$$ds^2 = B^{2\alpha}[dt^2 - dx^2] - B^2[e^{2mx}dy^2 + e^{-2mx}dz^2] \ldots (1)$$

where $B$ is a function of time $t$ and $m$ and $\alpha$ are non-zero constants. We shall use the co-moving co-ordinates. Therefore the shear $\sigma$ and the expansion $\theta$ of the flow vector $v_i$ of fluid filling the universe can be determined as

$$\theta = \frac{1}{B^\alpha (a+2)} \frac{B}{B} , \quad \sigma = \frac{1}{\sqrt{3} B^\alpha (a-1)} \frac{B}{B} \ldots (2)$$

Here and in what follows an overhead dot indicates differentiation with respect to time $t$. Clearly $\frac{\sigma}{\theta} = \frac{(a-1)}{\sqrt{3}(a+2)}$ is a constant. Here it should be noted that the condition $\frac{\sigma}{\theta} = \text{constant}$ has been used by Collins, Glass and Wilkinson (1980).

2 : THE PERFECT-FLUID MATTER SOLUTION

We now show that the metric (1) is compatible with the perfect fluid distribution, i.e., it satisfies the field equations
\[ G_{ik} = -8\pi \left[ (p+\rho)\nu_i \nu_k - p g_{ik} \right] - \Lambda g_{ik} \] ... (3)

The non-diagonal components of (3) lead to \( \nu_\mu = (0,0,0,B^a) \) where \( x^1 = x, x^2 = y, x^3 = z \) and \( x^4 = t \). By a routine calculation, it is easy to verify that the remaining field equations lead to the expressions

\[ 8\pi \rho = -\Lambda - B^{-2a} \left[ m^2 - (2a+1) \frac{B^2}{B} \right] \] ... (4)

\[ 8\pi p = \Lambda + B^{-2a} \left[ \frac{2n^2}{a-1} - (a+1) \frac{B^2}{B} \right] \] ... (5)

and

\[ \frac{B}{B} + \frac{B}{B} = \frac{2n^2}{a-1} = 2n^2 \quad \text{say} \] ... (6)

where we have assumed that \( a > 1 \). The equation (6) can be easily integrated to have the solution

\[ B^2 = q^2 \cosh(2nt) \] ... (7)

where \( q \) is a constant of integration. The density and pressure are given by

\[ 8\pi \rho = -\Lambda + \frac{n^2}{q^{2a} \cosh^a(2nt)} \left[ (2a+1) \tanh^2(2nt) - (a - 1) \right] \] ... (8)
and

\[ 8\pi p = \Lambda + \frac{n^2}{q^2 \cosh^2(2nt)} \left[ (2a+1) \tanh^2(2nt) - (a + 3) \right] \]  

...(9)

The physical requirement \( \rho > p \) and \( p \geq 0 \) give the inequality

\[ \frac{2n^2}{q^2 \cosh^2(2nt)} > \Lambda \geq \frac{n^2}{q^2 \cosh^2(2nt)} \left[ (a+3) - (2a + 1) \tanh^2(2nt) \right] \]  

...(10)

When \( a = 2 \), we get \( m = n \). In this case our solution reduces to the perfect-fluid solution given by Narain (1988). The behavior of the above model is similar to the behavior of the Narain's model. Therefore we shall not enter into the details here. Thus we have a class of perfect fluid solutions with Bianchi VI\(_{0}\) symmetry depending upon the values of \( a \).

3: FLUID MATTER WITH ELECTROMAGNETIC FIELD

We now investigate the possibility of the metric (1) representing a solution of the field equations for the perfect fluid matter in the presence of an electromagnetic field. We take the field equations in the form

\[ G_{ik} = -8\pi E_{ik} - 8\pi \left[ (p + q)v_i v_k - p g_{ik} \right] - \Delta g_{ik} \]  

...(11)
\[ E_{ik} = -g^{ab} F_{ia} F_{kb} + \frac{1}{4} g_{ik} F_{ab} F^{ab} \quad \ldots (12) \]

\[ F_{[ik;j]} = 0 \quad \ldots (13) \]

\[ F^i_{;k} = 4\pi J^i \quad \ldots (14) \]

where \( E_{ik} \) is the electromagnetic energy tensor, \( F_{ik} \) is the electromagnetic field tensor and \( J^i \) is the current 4-vector. The flow vector \( v_i \) will be assumed to be of the form \((0,0,0,\theta)\). Dunn and Tupper (1976) have shown that for the metric of the form (1), the only non-vanishing components of \( F_{ik} \) are \( F_{23} \) and \( F_{14} \). For simplicity we take \( F_{23} = 0 \). Let \( F_{14} = E(t) \) where \( E(t) \) is an arbitrary function of \( t \). As we are interested in a generalization of Narain's solution, we assume that the metric potential \( B \) satisfies the equation

\[ \frac{B}{B} + \frac{B}{B} = 2m^2 \]

The solution of this equation is given by

\[ B^2 = q^2 \cosh(2mt) \quad \ldots (15) \]

where \( q \) is a constant of integration.

The above field equations lead to the following expressions.
\[ E^2 = \frac{m^2 (2 - a) B^{2a}}{4\pi} \quad \ldots (16) \]

\[ J^i = -m \left( \frac{2 - a}{4\pi} \right) q^{-3a} \text{Sinh}(2mt) \text{Cosh}^{-1/2} (3a+2)^2(2mt) \quad \ldots (17) \]

\[ 8\pi p = \Lambda + \frac{m^2}{B^{2a}} \left[ a - 2 - (2a + 1) \text{Sech}^2(2mt) \right] \quad \ldots (18) \]

\[ 8\pi \rho = -\Lambda + \frac{m^2}{B^{2a}} \left[ (3a - 2) - (2a + 1) \text{Sech}^2(2mt) \right] \quad \ldots (19) \]

Here we have assumed that \( a \leq 2 \). It is easy to see that the current vector \( J^i \) is space-like. The physical requirements \( \rho > p \) and \( p \geq 0 \) give the inequality

\[ \frac{m^2}{B^{2a}} \left[ 2 - a + (2a + 1) \text{Sech}^2(2mt) \right] \leq \Lambda \leq \frac{am^2}{B^{2a}} \quad \ldots (20) \]

where \( B \) is given by (15).

When \( a = 2 \), \( E^2 \) and \( J^i \) vanish. In this case we get the perfect fluid solution given by Narain (1988). Thus the solution of this section gives us an electromagnetic generalization of the Narain's solution.
4: FLUID MATTER WITH SCALAR FIELD

In this section we shall show that the metric (1) can represent a solution of the field equations for perfect fluid in the presence of zero mass scalar field. We take the field equation in the form

\[ G_{ik} = - 8 \pi \left[ (p + \rho)v_i v_k - pg_{ik} \right] - \Lambda g_{ik} - \left[ \phi, i \phi, k - \frac{1}{2} g_{ik} \phi, a \phi, a \right] \]  

...(21)

\[ \phi, i = 0 \]  

...(22)

where \( \phi \) is the zero mass scalar field and comma and semicolon indicate partial and covariant differentiation respectively.

It is not difficult to reduce the field equations (21) and (22) to the coupled system

\[ B^2 = q^2 \cosh(2nt), \quad n^2 = \frac{m^2}{a - 1} \]  

...(23)

\[ \phi = \frac{k}{nq} \tan^{-1} \left[ e^{2nt} \right] \]  

...(24)

\[ 8\pi p = \Lambda - \frac{n^2}{B^2 a} \left[ (a+3) - (2a + 1) \tanh^2(2nt) + \frac{4nk^2}{n^2 q^4} \text{Sech}^2(2nt) \right] \]  

...(25)
$$8\pi\rho = -\Lambda - \frac{n^2}{B^{2a}} \left[ (a-1) - (2a + 1) \tanh^2(2nt) + \frac{4nk^2}{n^2q^4} \text{Sech}^2(2nt) \right]$$

...(26)

where \(q\) and \(k\) are constants of integration. The model discussed in this section is a scalar field generalization of the model discussed in Section 2. Here it should be noted that the scalar field has no effect on the geometry of the solution. But the physical parameters \(p\) and \(\rho\) do depend on the scalar field.

In the case \(a = 2\), defining new time \(T\) by \(dT = q^2 \cosh(2mt)dt\) the metric becomes

$$ds^2 = dT^2 - \left[ q^4 + 4m^2T^2 \right] dx^2 \left[ q^4 + 4m^2T^2 \right]^{1/2} \left[ e^{2m\xi} dy^2 + e^{-2m\xi} dz^2 \right]$$

...(27)

The parameters \(p\) and \(\rho\) in this case are given by

$$8\pi p = \Lambda - \frac{5m^2q^4 + 4nk^2}{[q^4 + 4m^2T^2]^2}$$

...(28)

$$8\pi\rho = -\Lambda + \frac{m^2}{[q^4 + 4m^2T^2]^2} \left[ 16m^2T^2 - q^4 \left( 1 + \frac{4nk^2}{q^4m^2} \right) \right]$$

...(29)

When \(k = 0\), the scalar field disappears and we recover the solution given by Narain (1988). Therefore the metric (27)
represents a scalar field generalization of the perfect fluid solution given by Narain (1988). The physical requirements $\rho > p$ and $p \geq 0$ give rise to the inequality

$$\frac{2m^2}{[q^4 + 4m^2T^2]} > \Lambda \geq \frac{5m^2q^4 + 4nk^2}{[q^4 + 4m^2T^2]^2}...(30)$$

The physical features of this model are similar to those of the model discussed by Narain (1988). Therefore we shall not enter into the details here.

Here it should be noted that the metric (1) has been discussed by Patel and Koppar (1991) in connection with viscous-fluid world models. They have derived three Bianchi VIo viscous-fluid cosmological models in terms of the metric (1).
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