APPENDIX : 1

RELATIVISTIC CHARGED DUST SPHERE
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1: INTRODUCTION

Several investigators have shown keen interest in obtaining exact solutions of the coupled Einstein-Maxwell equations for static spherically symmetric distributions of charged matter. These distributions constitute possible sources for a Reissner-Nordstrom metric which uniquely describes the exterior gravitational field of a spherical charged distribution of matter. Bonner [(1960),(1965)] has considered the spherical distribution of charged incoherent matter in equilibrium and has shown that the absolute value of the charge density must be equal to the matter density. He has also given an exact solution of Einstein-Maxwell equations. De and Raychaudhari (1968) have established the equality of the charge density and matter density as a direct consequence of Einstein-Maxwell equations.

Cooperstock and de la Cruz (1978) have derived an explicit solution of Einstein-Maxwell equations which represents the interior field of a uniformly charged dust sphere in equilibrium. They have assumed that the nongravitational energy density of their solution is the same as the nongravitational energy density
of the well known Schwarzschild interior solution. Cooperstock and de la Cruz's solution is a generalization of Schwarzschild interior solution, with matter density decreasing outwards. Bonner and Wickramsuriya (1975) have discussed a static interior charged dust metric with matter density increasing outward. Tikekar (1984) has obtained a class of static charged dust spheres containing a geometrical parameter $k<1$. In this class, the matter density decreases radially outwards for $k \leq 0$ and $0 \leq k \leq 0.05$, whereas for $0.05 < k < 1$, it increases outward. Here it should be noted that the choice $k = 0$ leads us to the solution of Cooperstock and de la Cruz (1978).

Finch and Skea (1989) have given a static spherically symmetric interior metric which describes a realistic stellar model. The matter distribution of their solution is a perfect fluid. In this appendix, we wish to obtain a static spherically symmetric charged dust solution which has the nongravitational energy equal to that of Finch Skea's solution.

2: EINSTEIN - MAXWELL EQUATIONS

We know that $T^t_t$ for any static spherically metric gives the nongravitational energy density. We name the co-ordinates as $x^1=r$, $x^2=\theta$, $x^3=\phi$, $x^4=t$. We assume that the value of $T^t_t$ for our solution is the same as the value of $T^t_t$ of Finch Skea's metric. Consequently we consider the line-element in the form
\[ ds^2 = y^2(x) \, dt^2 - (1+x)dr^2 - r^2(d\theta^2 + \sin^2 \theta \, d\phi) \] ...(1)

where \( x = Cr^2 \), \( C \) is an arbitrary constant and \( y \) is an undetermined function of \( x \).

We shall develop Einstein-Maxwell equations for a static spherical distribution of charged perfect fluid metric (1) as the space-time metric associated with the distribution. For a charged perfect fluid, the energy momentum tensor is given by

\[ T^i_k = (p+\rho)v^i v_k - p\delta^i_k - F^i_m F_{km} + \frac{1}{4} \delta^i_k F^{mn} \] ...(2)

Here \( p, \rho \) and \( v^i \) denote respectively the fluid pressure, matter density and the unit time-like flow vector of the fluid. \( F_{ik} \) are the components of the electromagnetic field tensor satisfying the Maxwell equations

\[ F^i_{k,n} + F^i_{kn,i} + F^i_{ni,k} = 0 \] ...(3a)

\[ (F^i_{ik} \sqrt{-g})_{,k} = 4\pi \sqrt{-g} \sigma v^i \] ...(3b)

Where \( \sigma \) denotes the charge density and comma indicates partial differentiation. Since there is only a radial electric field, the only surviving component of \( F_{ik} \) is \( F_{14} \). We use the co-moving co-ordinates. Therefore
\[ v^i = (0, 0, 0, 1/y) \] \hspace{1cm} ...(4)

We write

\[ E^2(r) = -F_{14}F^{14} \] \hspace{1cm} ...(5)

The quantity \( E \) can be interpreted as the electric field intensity. It can be easily seen that the Maxwell equations (3a) and (3b) give us

\[ F_{14} = y x^{-1} (1+x)^{1/2} \int_0^r 4\pi \sigma r^2 (1+x)^{1/2} dr \] \hspace{1cm} ...(6)

and

\[ 4\pi\sigma = r^{-2}(1+x)^{-1/2} \frac{d}{dr} (r^2 E) \] \hspace{1cm} ...(7)

The quantity

\[ Q(r) = 4\pi \int_0^r (1+x)^{1/2} \sigma r^2 dr \] \hspace{1cm} ...(8)

represents the total charge within the sphere of radius \( r \). Thus we have \( E = Q / r^2 \). The Einstein-Maxwell equations for a charged perfect fluid sphere can then be expressed in the form
where an overhead prime indicates differentiation with respect to x. Thus we have a system of three equations (9), (10) and (11) for four unknown functions $y$, $E^2$, $p$ and $\rho$.

In the next section we shall obtain an explicit solution of the above equations by assuming $p = 0$.

3: SOLUTION OF THE FIELD EQUATION

Charged dust spheres in equilibrium belong to the interior Papapatrou - Majumdar (1947) class and their metrics can be expressed in the form

$$ds^2 = -U^2(dx^2 + dy^2 + dz^2) + U^{-2} dt^2 \quad \ldots (12)$$

where $U = U(\bar{x}, \bar{y}, \bar{z})$. The metric (1) can be reduced to the form (12) if

$$(1+x)^{1/2} = 1 + 2x \left( y'/y \right) \quad \ldots (13)$$
the differential equation (13) can be easily integrated. Its solution can be expressed as

\[ y^2 = B^2 e^{\frac{1}{2(1+x)^2}} \left[ 2 + x + 2(1+x)^{\frac{1}{2}} \right] \]  \hspace{1cm} \ldots(14)\]

where B is an arbitrary constant of integration. The expression for \( y^2 \) is regular and positive at all points of the configuration provided \( C > 0 \).

Substituting \( p = 0 \) in (11) and using \( y^2 \) given by (14) we get

\[ E^2 = C \left\{ \frac{\sqrt{1+x}}{(1+x)^{\frac{1}{2}}} - 1 \right\} \bigg/ (1+x)\left\{ (1+x)^{\frac{1}{2}} + 1 \right\} \]  \hspace{1cm} \ldots(15)\]

The results (10) and (15) determine the material density \( \rho \). It is given by

\[ 8\pi \rho = 2C (x^2 + 3x + 3) / (1+x)^2 \left\{ (1+x)^{\frac{3}{2}} + 1 \right\} \]  \hspace{1cm} \ldots(16)\]

With the help of results (14) and (15) we have verified that the differential equation (9) is identically satisfied. The electric charge density \( \sigma \) determined from (7) and (15) yields

\[ \sigma = \pm \rho \]  \hspace{1cm} \ldots(17)\]

in accordance with De-Raychaudhari requirement. The expressions
(15) and (16) for $E^2$ and $\rho$ are regular and positive at all points of the configuration provided $C > 0$.

4: DISCUSSION

Substituting $x = 0$ in the results (15) and (16) one can obtain the values of the central electric intensity $E_0$. They are given by

$$E_0 = 0, \quad 8\pi \rho_0 = 3C \quad \cdots (18)$$

The physical requirement $\rho_0 > 0$ implies that $C > 0$. From the result (16), one can easily obtain

$$\frac{4\pi \rho'}{C} (1+x)^{2}\left\{ \frac{3}{(1+x)^2+1} \right\} = - \left[ \frac{3+x}{1+x} + \frac{3(1+x)^{1/2}(x^2 + 3x + 3)}{2\left(1+x\right)^{3/2} + 1} \right] \quad \cdots (19)$$

The result (19) shows that $\rho' < 0$. Thus $\rho$ is a decreasing function of $x$. Therefore the density $\rho$ decreases radially outwards.

We consider a situation wherein the spherical charged dust distribution extends to a finite radius $a$. The interior metric (1) with $y^2$ given by (14) should then match with the exterior Reissner-Nordstrom metric.
\[ ds^2 = \left(1 - \frac{2m}{r} + \frac{q^2}{r^2}\right)dt^2 - \left(1 - \frac{2m}{r} + \frac{q^2}{r^2}\right)dr^2 - r^2(d\theta^2 + \sin^2 \theta \, d\phi^2) \]  
\[ ... (20) \]

across the boundary \( r = a \) of the distribution. The constants \( m \) and \( q \) respectively denote the total mass and the total charge of the sphere of radius \( a \). The appropriate boundary conditions are

\[ \left( \frac{y^2}{r} \right)_{r=a} = 1 - \frac{2m}{a} + \frac{q^2}{a^2} = \frac{1}{1 + Ca^2} \]  
\[ ... (20) \]

The continuity of the electric field intensity \( E \) across the boundary \( r = a \) gives

\[ q^2 = Ca^4 \left\{ \left[1 + Ca^2\right]^{\frac{1}{2}} - 1 \right\} \bigg/ \left[1 + Ca^2\right]\left\{ \left[1 + Ca^2\right]^{\frac{1}{2}} + 1 \right\} \]  
\[ ... (21) \]

The boundary conditions (20) determine the constants \( m \) and \( B \) as

\[ m = \frac{q^2}{2a} + \frac{Ca^3}{2(1 + Ca^2)} \]  
\[ ... (22) \]

and

\[ B^2 = \frac{e^{-2(1+Ca^2)^2}}{(1+Ca^2)^2}\left\{ 2 + Ca^2 + 2(1+Ca^2)^2 \right\} \]  
\[ ... (23) \]

where \( q \) is given by (21).

Hence the space-time metric (1) with \( y^2 \) given by (14) gives a physically viable model of a charged dust sphere in equilibrium, acting as the interior source for Reissner-Nordström metric.
REFERENCES: